APPM5440 — Applied Analysis: Section exam 3

17:15 – 18:30, Dec. 4, 2012. Closed books.

WRITE YOUR NAME: ____

Please fill out your answers to problems 1, 2, 3 directly on the problem sheet, if possible.

Write your answer to problem 4 either on the exam, or on a separate sheet.

Problem 1: (28p) No motivation requireds — please just write the answers.

(a) Let X be a set, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Suppose that $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$ (so that $\mathcal{T}_1 \subseteq \mathcal{T}_2$). Mark the true statements in the table below:

	Check if true:
If K is compact in $(X, \mathcal{T}_1)$, then K is compact in $(X, \mathcal{T}_2)$.	
Suppose $f : \mathbb{R} \to (X, \mathcal{T}_1)$ is continuous, then $f : \mathbb{R} \to (X, \mathcal{T}_2)$ is continuous.	
If $x_n \to x$ in $(X, \mathcal{T}_1)$, then $x_n \to x$ in $(X, \mathcal{T}_2)$.	

(b) For which values of p is the Banach space ℓ^p reflexive? ________________

(c) State the open mapping theorem:

(d) Set $X = \ell^2$ and $S = \{x \in \ell^2 : \ ||x|| = 1\}$. What is the closure of S in the weak topology?

Problem 2: (28p) In this problem, you are given four pairs of linear spaces X and Y, and you are given some information about a map $T \in \mathcal{B}(X, Y)$.

(a) $X = Y = \ell^2$. $T(x_1,\, x_2,\, x_3,\, \dots) = (\frac{1}{1}x_1,\, \frac{1}{2}x_2,\, \frac{1}{3}x_3,\, \dots)$.
(b) $X = \ell^2$. Y is a normed linear space. We know that $||Tx|| = 2||x||$ for every $x \in X$.
(c) $X = Y = \ell^2$. We know that $||Tx|| \geq 2||x||$ for every $x \in X$.
(d) $X = \ell^2$. $Y = \mathbb{R}^n$. We know that $||Tx|| \leq 2||x||$ for every $x \in X$.

Fill out the table below. Write "yes" if the statement is necessarily true. Write "no" if it is necessarily false. Leave blank if there is not enough information. (No motivations required.)

	T is necessarily compact	$\text{ran}(T)$ is necessarily closed
(a)		
(b)		
(c)		
(d)		

Turn the page!

Problem 3: (28p) In this problem, you are given four maps φ from different normed linear spaces X to $\mathbb{R}$. State which ones are elements of X^*. Motivate each answer briefly (in the space provided whenever possible).

(a) $I = [0,1]$. $X = C(I)$ with standard norm. $\varphi(f) = \max\limits_{x \in I} f(x)$.

(b) X is the space of polynomials of degree at most 2 and $||f|| = \sup\limits_{x \in [0,1]} |f(x)|$. $\varphi(f) = f'(0)$.

(c) $I = [0,1]$. $X = C^1(I)$ with norm $||f|| = \sup\limits_{x \in I} |f(x)|$. $\varphi(f) = f'(0)$.

(d) $I = [0,1]$. $X = C^1(I)$ with standard norm. $\varphi(f) = \sum\limits_{n=1}^{\infty} \frac{(-1)^n f(1/n)}{n}$.

Problem 4: (16p) Let X denote a finite set, and let $\mathcal{T}$ be a metrizable topology on X. Prove that $\mathcal{T}$ is the discrete topology on X. (Write your solution in the space below, or on a separate sheet.)