Hints for homework set 12 — APPM5440 — Fall 2012

Problem 1: Let X denote the linear space of polynomials of degree 2 or less on $I = [0,\,1]$. For $f \in X$, set $||f|| = \sup_{x\in I} |f(x)|$. For $f \in X$, define

$$\varphi_1(f) = \int_0^1 f(x)\,dx, \quad \varphi_2(f) = f(0), \quad \varphi_3(f) = f'(1/2), \quad \varphi_4(f) = f'(1/3).$$

Prove that $\varphi_j \in X^*$ for $j = 1,2,3,4$. Prove that $\{\varphi_1,\,\varphi_2,\,\varphi_3\}$ forms a basis for X^*. Prove that $\{\varphi_1, \varphi_2, \varphi_4\}$ does not form a basis for X^*.

Solution: You can easily prove that all the φ_j's are linear.

To prove that each φ_j is continuous, simply invoke the theorem that any linear map on a finite dimensional space is continuous. For fun, let's work it out explicitly for $j = 1, 2$:

$$|\varphi_1(f)| \le \int_0^1 |f(x)|\,dx \le \int_0^1 ||f||\,dx = ||f||.$$
$$|\varphi_2(f)| = |f(0)| \le ||f||.$$

Proving directly that φ_3 and φ_4 are bounded takes a little more work.

To prove that $\{\varphi_1,\,\varphi_2,\,\varphi_3\}$ forms a basis, we first observe that since X has dimension three, we know that X^* also has dimension three. So all we need to prove is that the set is linearly independent. Suppose

$$c_1\,\varphi_1 + c_2\,\varphi_2 + c_3\,\varphi_3 = 0.$$

In other words, for every $f \in X$, we must have $c_1\,\varphi_1(f) + c_2\,\varphi_2(f) + c_3\,\varphi_3(f) = 0$. By plugging in $f = 1$, $f = x$, $f = x^2$, we get three equations for c_1, c_2, and c_3,

$$\begin{aligned} f = 1 \quad &\Rightarrow \quad c_1 + c_2 = 0,\\ f = x \quad &\Rightarrow \quad (1/2)c_1 + c_3 = 0,\\ f = x^2 \quad &\Rightarrow \quad (1/3)c_1 + c_3 = 0. \end{aligned}$$

It is easy to show that the only solution is $c_1 = c_2 = c_3 = 0$.

To prove that $\{\varphi_1,\,\varphi_2,\,\varphi_4\}$ does not form a basis, we will prove that they are linearly dependent. Suppose

$$c_1\,\varphi_1 + c_2\,\varphi_2 + c_4\,\varphi_4 = 0.$$

By plugging in $f = 1$, $f = x$, $f = x^2$, we get three equations for c_1, c_2, and c_4,

$$\begin{aligned} f = 1 \quad &\Rightarrow \quad c_1 + c_2 = 0,\\ f = x \quad &\Rightarrow \quad (1/2)c_1 + c_4 = 0,\\ f = x^2 \quad &\Rightarrow \quad (1/3)c_1 + (2/3)c_4 = 0. \end{aligned}$$

This system has infinitely many solutions. For any $t \in \mathbb{R}$, the triple $\{c_1 = t,\, c_2 = -t,\, c_4 = -t/2\}$ is a solution. In particular, for $t = 1$ we find that

$$\varphi_1 - \varphi_2 - \frac{1}{2}\varphi_4 = 0.$$

Problem 2: Let $X = \ell^2$. Recall from class that every $\varphi \in X^*$ is of the form $\varphi(x) = \sum x_n\, y_n$ for some $y \in X$. Set $D = \{x \in \ell^2 : ||x|| = 1\}$. Prove that the weak closure of D is the closed unit ball in ℓ^2. (Hint: To prove that the closed unit ball is contained in the weak closure of D, you can for any element x such that $||x|| < 1$ explicitly construct a sequence $(x^{(n)})_{n=1}^{\infty} \subset D$ that weakly converges to x, such that $||x^{(n)}|| = 1$.)

Set $Y = \ell^3$. What is Y^*? Prove that the weak closure of the surface of the unit ball in ℓ^3 is the closed unit ball in ℓ^3.

Solution: Fix $x \in X$ such that $||x|| < 1$. For $n = 1, 2, 3, \ldots$, define α_n via

$$\alpha_n = \sqrt{1 - \sum_{j \neq n} x_j^2}.$$

Since $||x|| < 1$, we know that $0 < \alpha_n \leq 1$ for every n. Then define $x^{(n)}$ as the sequence obtained by swapping x_n for α_n. In other words

$$x^{(n)} = (x_1,\, x_2,\, \ldots,\, x_{n-2},\, x_{n-1},\, \alpha_n,\, x_{n+1},\, x_{n+2},\, \ldots).$$

We find that

$$||x^{(n)}||^2 = \alpha_n^2 + \sum_{j \neq n} x_j^2 = 1$$

so $x^{(n)} \in D$ for every n.

It remains to show that $x^{(n)} \rightharpoonup x$. Fix $y \in X$. Then

$$(x - x_n, y) = (x_n - \alpha_n)\, y_n.$$

Since $||y_n|| \to 0$ and $|x_n - \alpha_n| \leq |x_n| + |\alpha_n| \leq 2$, we find that

$$\lim_{n \to \infty} (x - x_n, y) = 0.$$

In treating the set Y, we first recall that the dual of ℓ^p for $p \in (1, \infty)$ is the set ℓ^q where $1/p + 1/q = 1$. When $p = 3$, we find $q = 3/2$. In other words, any $\varphi \in Y^3$ takes the form

$$\varphi(x) = \sum_{n=1}^{\infty} x_n\, y_n,$$

where $y \in \ell^{3/2}$. The density argument now follows pretty much the same lines as it did for $X = \ell^2$. Given x such that $||x|| < 1$, pick α_n so that

$$\alpha_n = \left(1 - \sum_{j \neq n} |x_j|^3\right)^{1/3},$$

set

$$x^{(n)} = (x_1,\, x_2,\, \ldots,\, x_{n-2},\, x_{n-1},\, \alpha_n,\, x_{n+1},\, x_{n+2},\, \ldots),$$

show that $||x^{(n)}|| = 1$, and then that $x^{(n)} \rightharpoonup x$.

Problem 3: Let X be a normed linear space, let M be a closed subspace, and let $\hat{x}$ be an element not contained in M. Set

$$d = \text{dist}(M, \hat{x}) = \inf_{y \in M} ||y - \hat{x}||.$$

Prove that $d > 0$. Prove that there exists an element $\varphi \in X^*$ such that $\varphi(\hat{x}) = 1$, $\varphi(y) = 0$ for $y \in M$, and $||\varphi|| = 1/d$.

Solution: First we prove that $d > 0$. Suppose M is a closed linear subspace, and that x is a point such that $\text{dist}(M, x) = 0$. Then there are $x_n \in M$ such that $\lim ||x_n - x|| = 0$. Since M is closed and $x_n \to x$, we must have $x \in M$. Since $\hat{x} \notin M$, it follows that $d > 0$.

Set $Z = \text{Span}(M, \hat{x})$.

Prove that any $z \in Z$ can be written $z = y + \alpha\,\hat{x}$ for a unique $\alpha \in \mathbb{R}$ and a unique vector $y \in M$. (This is not hard.)

Define for $z \in Z$ the functional ψ via $\psi(z) = \alpha$, where α is the unique number such that $z = y + \alpha\hat{x}$. Then $\psi(\hat{x}) = 1$ and $\psi(y) = 0$ for every $y \in M$.

We will now prove that the norm of ψ viewed as a functional on Z equals $1/d$. To this end, set

$$C = \sup_{z \in Z,\ z \neq 0} \frac{|\varphi(z)|}{||z||}.$$

We then need to prove that $C = 1/d$. First observe that for any $z \in Z \backslash M$ we have

$$||z|| = ||y + \alpha\hat{x}|| = |\alpha|\,||\frac{1}{\alpha}y + \hat{x}|| \geq |\alpha|\,d.$$

(Observe that $||\frac{1}{\alpha}y + \hat{x}|| \geq d$ since $(1/\alpha)y \in M$ and the distance between any element in M and $\hat{x}$ is at least d.) It follows that

$$|\varphi(z)| = |\alpha| \leq \frac{||z||}{d}.$$

This shows that $C \leq 1/d$. To prove the opposite inequality, pick $y_n \in M$ such that

$$\lim_{n \to \infty} ||\hat{x} - y_n|| = d.$$

Set $z_n = \hat{x} - y_n$. Then

$$C \geq \lim_{n \to \infty} \frac{|\varphi(z_n)|}{||z_n||} = \lim_{n \to \infty} \frac{1}{||z_n||} = \frac{1}{d}.$$

Finally, invoke the Hahn-Banach to assert the existence of an extension of ψ to all of X satisfying all requirements.

Problem 4: Let X be a normed linear space with a linear subspace M. Prove that the weak closure of M equals the closure of M in the norm topology.

Solution: Since the norm closure of any set is contained in the weak closure, all we need to prove is that any point *not* in the norm closure is also not in the weak closure.

Suppose $\hat{x} \notin \bar{M}$. From Problem 3, we know that there exists a functional $\varphi \in X^*$ such that $\varphi(\hat{x} - y) = 1$ for any vector $y \in \bar{M}$. Since M is a subset of $\bar{M}$, this shows that there can be no sequence in M that converges weakly to $\hat{x}$.