

**Homework set 5 solutions — APPM5440 — Fall 2012**

**2.7:** Set  $I = [0, 1]$ , and  $\Omega = \{f \in C(I) : \text{Lip}(f) \leq 1, \int f = 0\}$ .

We will use the Arzelà-Ascoli theorem, of course.

The Lipschitz condition implies that  $\Omega$  is equicontinuous. (To prove this, fix any  $\varepsilon > 0$ . Set  $\delta = \varepsilon$ . Then for any  $f \in \Omega$ , and  $|x - y| < \delta$ , we have  $|f(x) - f(y)| \leq \text{Lip}(f) |x - y| \leq |x - y| < \varepsilon$ .)

To prove that  $\Omega$  is bounded, note that if  $\int f = 0$ , and  $f$  is continuous, then there must exist an  $x_0 \in I$  such that  $f(x_0) = 0$ . Then for any  $x \in I$  and any  $f \in \Omega$ , we have  $|f(x)| = |f(x) - f(x_0)| \leq \text{Lip}(f) |x - x_0| \leq |x - x_0| \leq 1$ . So  $\|f\|_{\infty} \leq 1$ .

Finally we need to prove that  $\Omega$  is closed. Let  $(f_n)$  be a Cauchy sequence in  $\Omega$ . Since  $C(I)$  is complete, there exists an  $f \in C(I)$  such that  $f_n \rightarrow f$  uniformly. We need to prove that  $f \in \Omega$ . Since  $f_n \rightarrow f$  uniformly, we know both that  $\text{Lip}(f) \leq \limsup_{n \rightarrow \infty} \text{Lip}(f_n) \leq 1$ , and that  $\int f = \lim_{n \rightarrow \infty} \int f_n = 0$ . This proves that  $f \in \Omega$ .

**2.8:** We will explicitly construct a dense countable subset  $\Omega$  of  $C([a, b])$ . Without loss of generality, we can assume that  $a = 0$  and that  $b = 1$ .

For  $n = 1, 2, \dots$ , and for  $j = 0, 1, 2, \dots, n$ , set  $x_j^{(n)} = j/n$ . Let  $\Omega_n$  denote the subset of  $C(I)$  of functions that (1) are linear on each interval  $[x_{j-1}^{(n)}, x_j^{(n)}]$ , and (2) take on rational values for each  $x_j^{(n)}$ . Since each function in  $\Omega_n$  is uniquely defined by its values on the  $x_j^{(n)}$ 's, we can identify  $\Omega_n$  by  $\mathbb{Q}^{n+1}$ . Hence  $\Omega_n$  is countable.

Set  $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ . Since each  $\Omega_n$  is countable,  $\Omega$  is countable.

It remains to prove that  $\Omega$  is dense in  $C(I)$ . Fix any  $f \in C(I)$ , and any  $\varepsilon > 0$ . Since  $I$  is compact,  $f$  is uniformly continuous on  $I$  so there exists a  $\delta > 0$  such that  $|x - y| < \delta$  implies that  $|f(x) - f(y)| < \varepsilon/2$ . Pick an  $n$  such that  $1/n < \delta$ , and pick a  $\varphi \in \Omega_n$  such that  $|\varphi(x_j^{(n)}) - f(x_j^{(n)})| < \varepsilon/2$  for  $j = 0, 1, 2, \dots, n$ . We will prove that  $\|\varphi - f\|_u < \varepsilon$ : Fix an  $x \in I$ . Then pick  $j \in \{1, 2, \dots, n\}$  so that  $x \in [x_{j-1}^{(n)}, x_j^{(n)}]$ . Since  $\varphi$  is linear in this interval, there is a number  $\alpha \in [0, 1]$  such that

$$\varphi(x) = \alpha \varphi(x_{j-1}^{(n)}) + (1 - \alpha) \varphi(x_j^{(n)}).$$

Now

$$\begin{aligned} (1) \quad |f(x) - \varphi(x)| &= |\alpha f(x) + (1 - \alpha) f(x) - \alpha \varphi(x_{j-1}^{(n)}) - (1 - \alpha) \varphi(x_j^{(n)})| \\ &\leq \alpha |f(x) - \varphi(x_{j-1}^{(n)})| + (1 - \alpha) |f(x) - \varphi(x_j^{(n)})|. \end{aligned}$$

Since  $|f(x) - f(x_{j-1}^{(n)})| \leq \varepsilon/2$  (by the uniform continuity) and since  $|f(x_{j-1}^{(n)}) - \varphi(x_{j-1}^{(n)})| < \varepsilon/2$  (by the choice of  $\varphi$ ), we have

$$(2) \quad |f(x) - \varphi(x_{j-1}^{(n)})| \leq |f(x) - f(x_{j-1}^{(n)})| + |f(x_{j-1}^{(n)}) - \varphi(x_{j-1}^{(n)})| < \varepsilon.$$

Analogously,

$$(3) \quad |f(x) - \varphi(x_j^{(n)})| \leq |f(x) - f(x_j^{(n)})| + |f(x_j^{(n)}) - \varphi(x_j^{(n)})| < \varepsilon.$$

Together, (1), (2), and (3) imply that  $|f(x) - \varphi(x)| < \varepsilon$ .

**2.9:** (a) Suppose that  $w(x) > 0$  for  $x \in (0, 1)$ . Then  $\|\cdot\|_w$  is a norm since:

(i)  $\|\lambda f\|_w = \sup_x w(x)|\lambda f(x)| = |\lambda| \sup_x w(x)|f(x)| = |\lambda| \|f\|_w$ .

(ii)  $\|f+g\|_w = \sup_x w(x)|f(x)+g(x)| \leq \sup_x w(x)(|f(x)|+|g(x)|) \leq \sup_x w(x)|f(x)| + \sup_x w(x)|g(x)| = \|f\|_w + \|g\|_w$ .

(iii) If  $f = 0$ , then clearly  $\|f\|_w = 0$ . Conversely, if  $f \neq 0$ , then  $f(x_0) \neq 0$  for some  $x_0 \in (0, 1)$ . Then  $\|f\|_w \geq w(x_0)|f(x_0)| > 0$ .

(b) Assume that  $w(x) > 0$  for  $x \in [0, 1] =: I$ . Set  $m = \inf_{x \in I} w(x)$  and  $M = \sup_{x \in I} w(x)$ . Since  $I$  is compact and  $w$  is continuous,  $w$  attains both its inf and its sup, and therefore  $m > 0$  and  $M < \infty$ . Then

$$\|f\|_u = \sup_{x \in I} |f(x)| \geq \sup_{x \in I} \frac{w(x)}{M} |f(x)| = \frac{1}{M} \|f\|_w.$$

and

$$\|f\|_u = \sup_{x \in I} |f(x)| \leq \sup_{x \in I} \frac{w(x)}{m} |f(x)| = \frac{1}{m} \|f\|_w.$$

It follows that

$$\frac{1}{M} \|f\|_w \leq \|f\|_u \leq \frac{1}{m} \|f\|_w.$$

(c) Set  $\| \|f\| \| = \sup_{x \in I} |x f(x)|$ . We will prove that  $\| \| \cdot \| \|$  is not equivalent to the uniform norm. Set for  $n = 1, 2, \dots$

$$f_n(x) = \begin{cases} 1 - nx & x \in [0, 1/n], \\ 0 & x \in (1/n, 1]. \end{cases}$$

Then  $\|f_n\|_u = 1$  for all  $n$ , while  $\| \|f_n\| \| = \sup_x x |f_n(x)| \leq 1/n$ . This proves that there cannot be a finite  $M$  such that  $\| \|f\| \| \leq M \|f\|_u$  for all  $f$ .

(d) We will prove that the set  $C(I)$  equipped with the norm  $\| \| \cdot \| \|$  is not a Banach space by constructing a Cauchy sequence with no limit point in  $C(I)$ . For  $n = 1, 2, \dots$ , define  $f_n \in C(I)$  by

$$f_n(x) = \begin{cases} x^{-1/2} & x \in (1/n, 1], \\ \sqrt{n} & x \in [0, 1/n]. \end{cases}$$

Fix a positive integer  $N$ . Then, if  $m, n \geq N$ , we have

$$\begin{aligned} \| \|f_n - f_m\| \| &= \sup_{x \in [0, 1/N]} x |f_n(x) - f_m(x)| \\ &\leq \sup_{x \in [0, 1/N]} (x |f_n(x)| + x |f_m(x)|) \\ &\leq \sup_{x \in (0, 1/N)} (x \cdot x^{-1/2} + x \cdot x^{-1/2}) = 2N^{-1/2}. \end{aligned}$$

Consequently,  $(f_n)_{n=1}^\infty$  is a Cauchy sequence.

Now suppose that  $(f_n)$  converges with respect to the  $\|\cdot\|_w$  norm to some function  $f \in C(I)$ . Then for any  $x \in I$  we have,

$$\|f - f_n\| \geq x |f(x) - f_n(x)|$$

Take the limit as  $n \rightarrow \infty$  to get

$$0 \geq x |f(x) - x^{-1/2}|.$$

It follows that  $f(x) = x^{-1/2}$  whenever  $x \neq 0$ , and consequently  $\|f\|_u = \infty$ . In other words,  $f \notin C(I)$ . (We do not know anything about  $f(0)$ , but that is OK.)

**Problem 1:**

(d) The set  $\Omega$  can contain a single function, for instance  $f(x) = \sin(1/x)$ . Note that for any fixed  $x$ , you can find a  $\kappa$  (say  $\kappa = x/2$ ) such that  $f'(x)$  is bounded on  $[x - \kappa, x + \kappa]$ . But this does not imply that  $f$  is uniformly continuous.

(e) Fix  $a \in (0, 1)$  and set  $f_n(x) = n(x - a)^2$ . Then  $f'_n(a) = 0$  for all  $n$ . But  $\{f_n\}_{n=1}^{\infty}$  is not equicontinuous at  $a$ . (Prove this!)

(f) Note of the conditions imply that  $\Omega$  is bounded. You could for instance have the sequence of constant functions  $f_n(x) = n$ . Then  $f'_n(x) = 0$  for all  $n$  and all  $x$  so all conditions are satisfied. But  $\Omega = \{f_n\}_{n=1}^{\infty}$  is not a bounded set (with respect to the uniform norm).