Homework set 5 — APPM5440 Fall 2012

From the textbook: 2.7, 2.8, 2.9.

Problem 1: Set $I = (0,\,1)$ and let $(f_n)_{n=1}^{\infty}$ be a sequence of continuously differentiable functions on I. Set $\Omega = \{f_n\,:\, 1 \leq n < \infty\}$.

(a) For a given n, suppose that

$$\sup_{x\in I} |f_n'(x)| < \infty.$$

Prove that then f_n is uniformly continuous.

(b) Suppose that

$$\sup_{x\in I}\ \sup_{1\leq n<\infty} |f_n'(x)| < \infty.$$

Prove that then Ω is uniformly equicontinuous.

(c) Suppose that for every $x \in I$, there exists a $\kappa > 0$ such that

$$\sup_{1\leq n<\infty}\ \sup_{y\in B_\kappa(x)} |f_n'(y)| < \infty.$$

Prove that then Ω is equicontinuous.

(d) Give an example of a set Ω of functions satisfying the condition in (c) that is not uniformly equicontinuous.

(e) Suppose that for a given $x \in I$, it is the case that

$$\sup_{1\leq n<\infty} |f_n'(x)| < \infty.$$

Prove that Ω is not necessarily equicontinuous at x.

(f) Which, if any, of the examples listed in (a) – (e) represent a bounded set Ω?