Homework set 3 — APPM5440, Fall 2012

From the textbook: 1.17, 1.18, 1.20, 1.22, 1.27. (Understanding exercise 1.22 perfectly is necessary to take in the proof that every metric space has a completion.)

Problem 1: We define a subset Ω of $\mathbb{R}$ via

$$\Omega = \{0\} \cup \left(\bigcup_{n=1}^{\infty} \left[\tfrac{1}{n+1/2},\, \tfrac{1}{n} \right] \right).$$

Prove that Ω is compact.

Problem 2: Consider our recurring example of the metric space $\mathbb{Q}$ (with the standard metric), and its subset $\Omega = \{q \in \mathbb{Q} :\ q^2 < 2\}$.

(a) Prove the Ω is both open and closed in $\mathbb{Q}$.

(b) Ω is bounded. Does the claim in (a) imply that Ω is compact? If yes, then motivate, if not, then decide whether Ω is in fact compact.

Problem 3: Let X be an infinite set equipped with the discrete metric. Decide which subsets of X (if any) are compact.

Problem 4: Consider the metric space $\mathbb{R}$ with the usual metric.

(a) Construct an open cover of $\Omega_1 = (0, 1]$ that does not have a finite subcover.

(b) Construct an open cover of $\Omega_2 = [0, \infty)$ that does not have a finite subcover.

(c) Construct a real-valued continuous function f on Ω_1 that is not uniformly continuous. Demonstrate that for your choice of f, there exists an $\varepsilon > 0$ such that for any $\delta > 0$, there are numbers $x_n, y_n \in \Omega_1$ such that $d(x_n, y_n) \leq 1/n$ and $d(f(x_n, y_n)) > \varepsilon$. Is it possible to construct such a function that is bounded?