**Homework set 2 — APPM5440, Fall 2012**

**Problem 1.8:** Let $(x_n)$ be a sequence in $\mathbb{R}$, let $C$ denote the set of cluster points of $(x_n)$. Set $M = \sup C$, $y_k = \sup\{x_n : n \geq k\}$ and recall that

$$\limsup_{n\to\infty} x_n \lim_{k\to\infty} y_k.$$

Show that $C$ is closed: We will prove that $C$ is *complete*. (Since $\mathbb{R}$ is complete, $C$ is closed iff it is complete.) Let $(c_j)$ be a Cauchy sequence in $C$, and set $c = \lim c_n$. We need to prove that $c \in C$. Set $n_0 = 0$. Then for $n = 1, 2, 3, \dots$, pick an $n_j$ such that $n_{j-1} < n_j$ and $|c_j - x_{n_j}| < 1/j$ (this is possible since each $c_j$ is the limit of a subsequence of $(x_n)$). Then

$$\limsup_{j\to\infty} |c - x_{n_j}| \leq \limsup_{j\to\infty}\big(|c - c_j| + |c_j - x_{n_j}|\big) \leq \limsup_{j\to\infty}\big(|c - c_j| + 1/j\big) = 0.$$

So $x_{n_j} \to c$, and therefor $c \in C$.

Show that $\limsup x_n \geq \max C$: Since $M \in C$, we know there exists a sequence $(x_{n_j})$ such that $x_{n_j} \to M$. Then for any $k$, we have

$$y_k = \sup\{x_n : n \geq k\} \geq \sup\{x_{n_j} : n_j \geq k\} \geq M.$$

Now take the limsup as $k \to \infty$ to get the desired inequality.

Show that $\limsup x_n \leq \max C$: Pick any $\varepsilon > 0$. We know that for some $k$, we have

$$y_k \leq M + \varepsilon$$

(since if this were not true, then $(x_n)$ would have at least one cluster point larger than $M$). Take the limsup as $k \to \infty$ to show that $\limsup x_n \leq M + \varepsilon$. Since $\varepsilon$ is arbitrary, we are done.

**Problem 1.10:** Prove that

$$\limsup_{n\to\infty} \inf_{\alpha} x_{n,\alpha} \leq \inf_{\alpha} \limsup_{n\to\infty} x_{n,\alpha}, \tag{1}$$

and that

$$\sup_{\alpha} \liminf_{n\to\infty} x_{n,\alpha} \leq \liminf_{n\to\infty} \sup_{\alpha} x_{n,\alpha}, \tag{2}$$

*Solution:* Set $y_n = \inf_\alpha x_{n,\alpha}$. Then clearly

$$y_n \leq x_{n,\alpha}, \qquad \forall\alpha.$$

Take the limsup of both sides:

$$\limsup y_n \leq \limsup x_{n,\alpha}, \qquad \forall\alpha.$$

Finally take the infimum over $\alpha$, nothing that $\limsup y_n$ does not depend on $\alpha$:

$$\limsup y_n \leq \inf_{\alpha} \limsup x_{n,\alpha}.$$

This relation is (1).

To prove (2), analogously set $z_n = \sup_\alpha x_{n,\alpha}$. Then $x_{n,\alpha} \leq z_n$ for all $\alpha$. Take the liminf to get $\liminf x_{n,\alpha} \leq \liminf z_n$, and finally take the sup over $\alpha$ to get (2).

**Problem 2:** Suppose that $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are Cauchy sequences in a metric space $(X, d)$. Prove that the sequence $(d(x_n, y_n))_{n=1}^{\infty}$ converges.

*Solution:* Set $\alpha_m = d(x_m, y_m)$. Since $\mathbb{R}$ is complete, all we need to prove is that $(\alpha_m)$ is a Cauchy sequence.

Fix any two natural integers $m$ and $n$. Via two applications of the triangle inequality, we obtain

$$d(x_m, y_m) \leq d(x_m, x_n) + d(x_n, y_m) \leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m).$$

It follows that

$$d(x_m, y_m) - d(x_n, y_n) \leq d(x_m, x_n) + d(y_n, y_m). \tag{3}$$

An analogous argument shows that

$$d(x_n, y_n) - d(x_m, y_m) \leq d(x_m, x_n) + d(y_n, y_m). \tag{4}$$

Together, (3) and (4) imply that

$$|d(x_m, y_m) - d(x_n, y_n)| \leq d(x_m, x_n) + d(y_m, y_n). \tag{5}$$

Fix $\varepsilon > 0$. Since $(x_n)$ and $(y_n)$ are Cauchy, there exist $N_1$ and $N_2$ such that

$$m, n \geq N_1 \qquad \Rightarrow \qquad d(x_m, x_n) < \varepsilon/2, \tag{6}$$

$$m, n \geq N_2 \qquad \Rightarrow \qquad d(y_m, y_n) < \varepsilon/2. \tag{7}$$

Set $N = \max(N_1, N_2)$. Then (5), (6), (7) imply that

$$m, n \geq N \qquad \Rightarrow \qquad |\alpha_m - \alpha_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$