Applied Analysis (APPM 5440): Section exam 2 — solutions
8:30am – 9:50am, Oct. 28, 2009. Closed books.

Problem 1: (24 points) For each of the statements below, state whether it is TRUE or FALSE. ("TRUE" of course means "necessarily true".) No motivation required.

(a) Define for $n = 1, 2, 3, \dots$ the function $f_n : \mathbb{R} \to \mathbb{R} : x \mapsto e^{-(x-n)^2}$. The sequence $(f_n)_{n=1}^\infty$ converges pointwise to zero.

(b) With f_n defined as in (a), the sequence $(f_n)_{n=1}^\infty$ converges uniformly to zero.

(c) With f_n defined as in (a), the set $\Omega = \{f_n\}_{n=1}^\infty$ is equicontinuous.

(d) With f_n defined as in (a), the set $\{f_n\}_{n=1}^\infty$ is pre-compact in $C_\mathrm{b}(\mathbb{R})$.

(e) Let $(g_n)_{n=1}^\infty$ be a sequence of real-valued functions on the set $I = [0,\,1]$ that converges pointwise to a function g. Suppose further that the set $\{g_n\}_{n=1}^\infty$ is equicontinuous. Then g is continuous.

(f) Suppose that $(h_n)_{n=1}^\infty$ is a sequence of functions $h_n : \mathbb{R} \to \mathbb{R}$ that converges uniformly to zero. Then $\int_{-\infty}^{\infty} h_n(t)\, dt \to 0$.

(g) The set of continuously differentiable functions on the interval $I = [0,\,1]$ is (topologically) closed in $C_\mathrm{b}(I)$.

(h) The set $\Omega = \{f \in C_\mathrm{b}(I) : \ ||f||_\mathrm{u} \leq 2 : \ \mathrm{Lip}(f) \leq 3\}$ is compact in $C_\mathrm{b}(I)$.

Solution:

(a) TRUE. (For any fixed x we have $\lim_{n\to\infty} e^{-(x-n)^2} = 0$.)

(b) FALSE. (In fact $||f_n||_\mathrm{u} = 1$ for all n.)

(c) TRUE. (Uniformly equicontinuous in fact since $||f_n'||_\mathrm{u} = \sqrt{2}\, e^{-1/2}$.)

(d) FALSE. (For instance, it is clear that no subsequence of (f_n) can converge uniformly. Note that since $\mathbb{R}$ is not compact, the A-A theorem does not apply.)

(e) TRUE. (Exercise 2.12.)

(f) FALSE. (Consider for instance $h_n(x) = 1/n$.)

(g) FALSE.

(h) TRUE. (A-A theorem and the fact that if $\mathrm{Lip}(f_n) \leq C$ and $||f_n - f||_\mathrm{u} \to 0$, then $\mathrm{Lip}(f) \leq C$.)

Problem 2: (26 points) Set $A = \{f \in C_{\rm b}(\mathbb{R}) : \lim_{t\to\infty} |f(t)| = \lim_{t\to-\infty} |f(t)| = 0\}$.

(a) Prove that A is closed in $C_{\rm b}(\mathbb{R})$.

(b) Prove that A is the closure of the set of compactly supported functions in $C_{\rm b}(\mathbb{R})$.

(c) Is the set A equipped with the uniform norm a Banach space? Motivate your answer briefly.

(d) Set $B = \{f \in C_{\rm b}(\mathbb{R}) : \sup_{t\in\mathbb{R}} e^{|t|}\,|f(t)| < \infty\}$. Prove that B is not closed in $C_{\rm b}(\mathbb{R})$.

Solution:

(a) We show that the complement of A is open. Pick $f \in A^{\rm c}$. There exists an $\varepsilon > 0$ and a sequence of points (t_n) such that $|t_n| \to \infty$ and $|f(t_n)| \geq \varepsilon$. We will show that $B_{\varepsilon/2}(f) \subseteq A^{\rm c}$. To this end, suppose $g \in B_{\varepsilon/2}(f)$. Then $|f(t_n) - g(t_n)| \leq \varepsilon/2$, so $|g(t_n)| \geq \varepsilon/2$. This proves that $g \in A^{\rm c}$.

(b) Since we show in (a) that A is closed, it is sufficient to prove that $C_{\rm c}$ is dense in A. To this end, pick $f \in A$, and an arbitrary $\varepsilon > 0$. We will construct a function $g \in C_{\rm c}$ such that $||f - g||_{\rm u} \leq \varepsilon$. Since $f \in A$, there is an R such that $|f(t)| \leq \varepsilon$ whenever $|t| \geq R$. Set

$$\varphi_R(t) = \begin{cases} 1 & |t| \leq R, \\ R - |t| & R < |t| < R+1, \\ 0 & R+1 \leq |t|. \end{cases} \tag{1}$$

Set $g(t) = \varphi_R(t)\, f(t)$. Then $||f - g|| \leq \varepsilon$.

(c) Yes; it is a topologically closed subspace of a complete space, and hence complete itself.

(d) We will construct a convergent sequence of functions in B that does not have a limit in B. For instance, set $f(x) = 1/(1+t^2)$, and set $f_n(t) = \varphi_n(t)\, f(t)$, where φ_n is defined as in (1). Then $||f_n - f||_{\rm u} \to 0$, each $f_n \in C_{\rm c} \subseteq B$, but $f \notin B$.

Problem 3: (25 points) State the Arzelà-Ascoli theorem. (No proof necessary.) Set $I = [0, 1]$ and let $k : I^2 \to \mathbb{R}$ be continuous. Define on $C_b(I)$ the integral operator

$$[A\, u](x) = \int_0^1 k(x,y)\, u(y)\, dy.$$

Let $(u_n)_{n=1}^{\infty}$ be a bounded sequence in $C_b(I)$. Prove that $(A\, u_n)_{n=1}^{\infty}$ has a uniformly convergent subsequence.

Solution: The statement of the theorem is given in the text book and the class notes.

We will show that the set $\Omega = \{A\, u_n\}$ is pre-compact. According to the A-A theorem, we will have done so once we have shown that Ω is bounded and equicontinuous.

Set $C = \sup_n ||u_n||_{\rm u}$.

Proof that Ω is bounded: Set $M = \sup_{(x,y)\in I^2} |k(x,y)|$. Since k is a continuous function on a compact set, $M < \infty$. It follows that

$$|[A\, u_n](x)| = \left|\int_0^1 k(x,y)\, u_n(y)\, dy\right| \leq \int_0^1 |k(x,y)|\, |u_n(y)|\, dy \leq M\, C,$$

Take the sup over x to obtain $||A\, u_n||_{\rm u} \leq M\, C$.

Proof that Ω is (uniformly) equicontinuous: Fix $\varepsilon > 0$. Since k is a continuous function on a compact set, it is uniformly continuous. In consequence, there is a $\delta > 0$ such that $|k(x,y) - k(x',y)| < \varepsilon/C$ whenever $|x - x'| < \delta$. Now suppose that $|x - x'| < \delta$. Then

$$\begin{aligned} |[A\, u_n](x) - [A\, u_n](x')| &= \left|\int_0^1 \big(k(x,y) - k(x',y)\big)\, u_n(y)\, dy\right| \\ &\leq \int_0^1 \big|k(x,y) - k(x',y)\big|\, |u_n(y)|\, dy \leq \int_0^1 \frac{\varepsilon}{C}\, C\, dy = \varepsilon. \end{aligned}$$

Problem 4: (25 points) State and prove the contraction mapping theorem.

Solution: See text book or class notes.