Applied Analysis (APPM 5440): Section exam 2
8:30am – 9:50am, Oct. 28, 2009. Closed books.

Problem 1: (24 points) For each of the statements below, state whether it is TRUE or FALSE. ("TRUE" of course means "necessarily true".) No motivation required.

(a) Define for $n = 1, 2, 3, \dots$ the function $f_n : \mathbb{R} \to \mathbb{R} : x \mapsto e^{-(x-n)^2}$. The sequence $(f_n)_{n=1}^{\infty}$ converges pointwise to zero.

(b) With f_n defined as in (a), the sequence $(f_n)_{n=1}^{\infty}$ converges uniformly to zero.

(c) With f_n defined as in (a), the set $\Omega = \{f_n\}_{n=1}^{\infty}$ is equicontinuous.

(d) With f_n defined as in (a), the set $\{f_n\}_{n=1}^{\infty}$ is pre-compact in $C_{\mathrm{b}}(\mathbb{R})$.

(e) Let $(g_n)_{n=1}^{\infty}$ be a sequence of real-valued functions on the set $I = [0, 1]$ that converges pointwise to a function g. Suppose further that the set $\{g_n\}_{n=1}^{\infty}$ is equicontinuous. Then g is continuous.

(f) Suppose that $(h_n)_{n=1}^{\infty}$ is a sequence of functions $h_n : \mathbb{R} \to \mathbb{R}$ that converges uniformly to zero. Then $\int_{-\infty}^{\infty} h_n(t)\,dt \to 0$.

(g) The set of continuously differentiable functions on the interval $I = [0, 1]$ is (topologically) closed in $C_{\mathrm{b}}(I)$.

(h) The set $\Omega = \{f \in C_{\mathrm{b}}(I) : ||f||_{\mathrm{u}} \leq 2 : \mathrm{Lip}(f) \leq 3\}$ is compact in $C_{\mathrm{b}}(I)$.

Problem 2: (26 points) Set $A = \{f \in C_{\mathrm{b}}(\mathbb{R}) : \lim_{t\to\infty} |f(t)| = \lim_{t\to-\infty} |f(t)| = 0\}$.

(a) Prove that A is closed in $C_{\mathrm{b}}(\mathbb{R})$.

(b) Prove that A is the closure of the set of compactly supported functions in $C_{\mathrm{b}}(\mathbb{R})$.

(c) Is the set A equipped with the uniform norm a Banach space? Motivate your answer briefly.

(d) Set $B = \{f \in C_{\mathrm{b}}(\mathbb{R}) : \sup_{t\in\mathbb{R}} e^{|t|}\,|f(t)| < \infty\}$. Prove that B is not closed in $C_{\mathrm{b}}(\mathbb{R})$.

Problem 3: (25 points) State the Arzelà-Ascoli theorem. (No proof necessary.) Set $I = [0, 1]$ and let $k : I^2 \to \mathbb{R}$ be continuous. Define on $C_{\mathrm{b}}(I)$ the integral operator

$$[A\,u](x) = \int_0^1 k(x, y)\,u(y)\,dy.$$

Let $(u_n)_{n=1}^{\infty}$ be a bounded sequence in $C_{\mathrm{b}}(I)$. Prove that $(A\,u_n)_{n=1}^{\infty}$ has a uniformly convergent subsequence.

Problem 4: (25 points) State and prove the contraction mapping theorem.