Applied Analysis (APPM 5440): Section exam 1

8:30am – 9:50am, Sep. 21, 2009. Closed books.

Problem 1: In what follows, (X, d_X) and (Y, d_Y) are metric spaces.

(a) Define what it means for a function $f: X \to Y$ to be continuous.

(b) Define what it means for a subset Ω of X to be compact.

(c) Define what a completion of (X, d_X) is.

(d) Let Ω be a subset of X. Define the closure of Ω.

Solution:

(a) For instance: f is continuous if $f^{-1}(G)$ is open whenever G is open.

(b) Ω is compact if every open cover has a finite subcover.

Many provided the answer that Ω is compact if it is totally bounded and complete. One could define compactness this way, so the answer gave full credit, but it is non-standard.

(c) A metric space (Y, d_Y) is a completion of (X, d_X) if:

(1) There is an isometry $i: X \to Y$.
(2) $i(X)$ is dense in Y.
(3) (Y, d_Y) is complete.

Answering that you "add the limit points" is too imprecise, and does not result in full credit.

(d) The closure is the intersection of all closed sets that contain Ω. Alternatively, you could define the closure as the set of all limit points of sequences in Ω.

The answer "the smallest closed set that contains Ω" gave full credit but is not great since you would then have to prove that such a smallest set in fact exists.

Problem 2: Let $\mathbb{Q}$ denote the set of rational numbers. On $\mathbb{Q}$, we define the discrete metric

$$d(x,y) = \begin{cases} 0, & \text{when } x = y, \\ 1, & \text{when } x \neq y. \end{cases}$$

(a) What subsets of $\mathbb{Q}$ are open in $(\mathbb{Q}, d)$? Prove your claim.

(b) Specify which sequences in $(\mathbb{Q}, d)$ are <u>convergent</u>. *No motivation required.*

(c) Set $\Omega = \{q \in \mathbb{Q} : \; q^2 < 2\}$. What is the <u>closure</u> of Ω in $(\mathbb{Q}, d)$? *No motivation required.*

(d) Set $\Omega = \{q \in \mathbb{Q} : \; q^2 < 2\}$. What is the <u>completion</u> of (Ω, d)? *No motivation required.*

Solution:

(a) All subsets are open. To prove this, let Ω be an arbitrary subset of Ω. Let $x \in \Omega$. We need to prove that for some ε, there is an ε-ball centered at x that is entirely contained in Ω. Set $\varepsilon = 1/2$. Then $B_\varepsilon(x) = \{x\}$ which is clearly contained in Ω.

(b) A sequence $(x_n)_{n=1}^\infty$ is convergent iff it is constant beyond a certain point. In other words:

$$(x_n)_{n=1}^\infty \text{ is convergent} \quad \Leftrightarrow \quad \exists N \text{ such that } x_n = x_N \text{ when } n \geq N.$$

The "$\Leftarrow$" should be obvious. To prove "$\Rightarrow$", suppose that (x_n) is Cauchy. Then there exists N such that $d(x_n, x_N) < 1/2$ when $n \geq N$. Note that if $d(x_n, x_N) < 1/2$, then $x_n = x_N$.

(c) Since every subset in $(\mathbb{Q}, d)$ is open, every subset is also closed. Therefore, $\overline{\Omega} = \Omega$.

(d) Since only sequences that become constant are Cauchy, (Ω, d) is complete in its own right. Therefore, the completion of (Ω, d) is (Ω, d).

Problem 3: Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. Set $y_n = x_{2n}$. Which of the following two statements must necessarily be true:

$$\begin{aligned} &\text{(a)} \qquad \limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n, \\ &\text{(b)} \qquad \limsup_{n\to\infty} y_n \leq \limsup_{n\to\infty} x_n. \end{aligned}$$

Motivate your answers carefully. State the definition of "limsup" that you use and make sure that your argument follows directly from this definition.

Solution: Definition of "limsup": $\boxed{\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup\{x_k\,:\, k \geq n\}\right)}$

(a) This is not necessarily true. For a counterexample, set $x_n = (-1)^{n+1}$ so that $x_n = 1$ if n is odd and $x_n = -1$ if n is even. We have

$$\limsup_{n\to\infty} x_n = 1.$$

On the other hand, $y_n = x_{2n} = (-1)^{2n+1} = -1$, so

$$\limsup_{n\to\infty} y_n = -1.$$

(b) This is true. We have

$$\begin{aligned} \limsup_{n\to\infty} y_n = \lim_{n\to\infty} \sup\{x_{2k}\,:\, k \geq n\} &\leq \lim_{n\to\infty} \sup\{x_{2k},\, x_{2k+1}\,:\, k \geq n\} \\ &= \lim_{n\to\infty} \sup\{x_k\,:\, k \geq 2n\} = \limsup_{n\to\infty} x_n. \end{aligned}$$

The inequality in the second step holds since we enlarge the set over which the sup is taken.

Problem 4: Let $(X, ||\cdot||)$ be a normed linear space. Suppose that every sequence $(x_n)_{n=1}^{\infty}$ in X such that $||x_m - x_n|| \leq 1/N$ whenever $m, n \geq N$ converges to a point in X. Is $(X, ||\cdot||)$ necessarily complete? Prove this if you answer yes, and give a counterexample if you answer no.

Solution: Yes, the set must be complete, as we will prove.

Let $(y_n)_{n=1}^{\infty}$ be an arbitrary Cauchy sequence.

From (y_n), we pick a rapidly convergent subsequence as follows:

- Pick n_1 such that $m, n \geq n_1 \quad \Rightarrow \quad ||y_m - y_n|| \leq 1$.
- Pick for $j = 2, 3, \dots$ an integer $n_j > n_{j-1}$ such that $m, n \geq n_j \quad \Rightarrow \quad ||y_m - y_n|| \leq 1/j$.

The sequence $(y_{n_j})_{j=1}^{\infty}$ satisfies $i, j \geq J \quad \Rightarrow \quad ||y_{n_i} - y_{n_j}|| \leq 1/J$, so our assumption on $(X, ||\cdot||)$ implies that there exists a point $y \in X$ such that $y_{n_j} \to y$ as $j \to \infty$.

It only remains to prove that $y_n \to y$ as $n \to \infty$. Pick $\varepsilon > 0$. Since $(y_n)_{n=1}^{\infty}$ is Cauchy, there exists an N such that

$$m, n \geq N \quad \Rightarrow \quad ||y_m - y_n|| < \varepsilon/2.$$

Now pick j_0 such that $n_{j_0} \geq N$ and $d(y_{n_{j_0}}, y) < \varepsilon/2$. Then if $n \geq N$, we have

$$d(y_n, y) \leq d(y_n, y_{n_{j_0}}) + d(y_{n_{j_0}}, y) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Problem 5: Let (X,d) be a compact metric space. Let $C_{\mathrm{b}}(X)$ denote the set of all bounded real-valued continuous functions on X, equipped with the uniform norm,

$$||f||_{\mathrm{u}} = \sup_{x\in X} |f(x)|.$$

Prove that $C_{\mathrm{b}}(X)$ is complete.

Solution: The assumption that X is compact is a red herring — this property is not required for the statement to be true.

Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in (X,d). We will construct a limit function, and then prove that it is bounded, that it is indeed the limit of the sequence in the uniform norm, and finally that it is continuous.

Step 1 — construct the limit point f: Fix $x \in X$. Since $|f_n(x) - f_m(x)| \leq ||f_n - f_m||$ and $(f_n)_{n=1}^{\infty}$ is Cauchy, the sequence $(f_n(x))_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, the sequence is convergent, we can therefore define a function f via

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Step 2 — prove that f is bounded: We have

$$\sup_{x\in X} |f(x)| = \sup_{x\in X} \left(\lim_{n\to\infty} |f_n(x)|\right) \leq \liminf_{n\to\infty} \left(\sup_{x\in X} |f_n(x)|\right) = \liminf_{n} ||f_n|| < \infty,$$

where in the last step we used that (f_n) is Cauchy, and therefore bounded.

Step 3 — prove that $f_n \to f$ uniformly: Fix $\varepsilon > 0$. Pick N such that $||f_m - f_n|| < \varepsilon/2$ when $m, n \geq N$. Then for $n \geq N$, we have

$$\begin{aligned} ||f_n - f|| = \sup_{x\in X} |f_n(x) - f(x)| &= \sup_{x\in X} \left(\lim_{m\to\infty} |f_n(x) - f_m(x)|\right) \\ &\leq \liminf_{m\to\infty} \left(\sup_{x\in X} |f_n(x) - f_m(x)|\right) = \liminf_{m\to\infty} ||f_n - f_m|| \leq \varepsilon/2 < \varepsilon. \end{aligned}$$

Step 4 — prove that f is continuous: This follows directly from the fact that each f_n is continuous and $f_n \to f$ uniformly (since uniform convergence preserves continuity).

Steps 2 and 4 prove that $f \in C_{\mathrm{b}}(X)$, and step 3 proves that f is the limit point of (f_n). The proof is therefore complete.