Solution for 2.4: Let's consider $X = [-1, 1]$ instead. Then set $f(x) = |x|$, and

$$f_n(x) = \frac{1 + n\,x^2}{\sqrt{n + n^2 x^2}}.$$

Then $f_n \to f$ uniformly, $f_n \in C^\infty(X)$, and f is not differentiable. (To justify the shift we made initially, simply note that if we define $g_n \in C([0,\,1])$ by $g_n(y) = f_n(2y - 1)$, then g_n is an answer to the original problem.)

Solution for 2.5: Set $I = [a, b]$. Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $C^1(I)$. Since

$$||f_n - f_m||_{\rm u} \le ||f_n - f_m||_{C^1},$$

the sequence (f_n) is Cauchy in $C(I)$. Since $C(I)$ is complete, there exists a function $f \in C(I)$ such that $f_n \to f$ uniformly.

Next set $g_n = f_n'$. Then

$$||g_n - g_m||_{\rm u} = ||f_n' - f_m'||_{\rm u} \le ||f_n - f_m||_{C^1},$$

so (g_n) is Cauchy in $C(I)$. Therefore, there exists a function $g \in C(I)$ such that $g_n \to g$ uniformly.

It remains to prove that $f \in C^1(I)$, and that $f_n \to f$ in $C^1(I)$. Fix any $x \in I$, and any $h \in \mathbb{R}$ such that $x + h \in I$. Then

$$\begin{aligned}\frac{1}{h}\big(f(x+h) - f(x)\big) &= \lim_{n\to\infty} \frac{1}{h}\big(f_n(x+h) - f_n(x)\big)\\ &= \lim_{n\to\infty} \frac{1}{h}\int_0^h f_n'(x+t)\,dt\\ &= \lim_{n\to\infty} \frac{1}{h}\int_0^h g_n(x+t)\,dt.\end{aligned}$$

Now recall that uniform convergence on a finite interval implies convergence of integrals. Since $g_n \to g$ uniformly, we find that

$$\frac{1}{h}\big(f(x+h) - f(x)\big) = \frac{1}{h}\int_0^h g(x+t)\,dt.$$

Since g is continuous, the limit as $h \to 0$ exists, and so

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\big(f(x+h) - f(x)\big) = \lim_{h\to 0} \frac{1}{h}\int_0^h g(x+t)\,dt = g(x).$$

This proves that $f \in C^1(I)$. To prove that $f_n \to f$ in $C^1(I)$, we note that

$$||f - f_n||_{C^1} = ||f - f_n||_{\rm u} + ||f' - f_n'||_{\rm u} = ||f - f_n||_{\rm u} + ||g - g_n||_{\rm u}.$$

By the construction of f and g, it follows that $||f - f_n||_{C^1(I)} \to 0$.