Homework set 2 — APPM5440, Fall 2009

Problem 1.10: Prove that

$$\limsup_{n\to\infty} \inf_\alpha x_{n,\alpha} \leq \inf_\alpha \limsup_{n\to\infty} x_{n,\alpha}, \tag{1}$$

and that

$$\sup_\alpha \liminf_{n\to\infty} x_{n,\alpha} \leq \liminf_{n\to\infty} \sup_\alpha x_{n,\alpha}, \tag{2}$$

Solution: Set $y_n = \inf_\alpha x_{n,\alpha}$. Then clearly

$$y_n \leq x_{n,\alpha}, \qquad \forall\alpha.$$

Take the limsup of both sides:

$$\limsup y_n \leq \limsup x_{n,\alpha}, \qquad \forall\alpha.$$

Finally take the infimum over α, nothing that $\limsup y_n$ does not depend on α:

$$\limsup y_n \leq \inf_\alpha \limsup x_{n,\alpha}.$$

This relation is (1).

To prove (2), analogously set $z_n = \sup_\alpha x_{n,\alpha}$. Then $x_{n,\alpha} \leq z_n$ for all α. Take the liminf to get $\liminf x_{n,\alpha} \leq \liminf z_n$, and finally take the sup over α to get (2).

Problem 2: Suppose that $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are Cauchy sequences in a metric space (X, d). Prove that the sequence $(d(x_n, y_n))_{n=1}^\infty$ converges.

Solution: Set $\alpha_m = d(x_m, y_m)$. Since $\mathbb{R}$ is complete, all we need to prove is that (α_m) is a Cauchy sequence.

Fix any two natural integers m and n. Via two applications of the triangle inequality, we obtain

$$d(x_m, y_m) \leq d(x_m, x_n) + d(x_n, y_m) \leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m).$$

It follows that

$$d(x_m, y_m) - d(x_n, y_n) \leq d(x_m, x_n) + d(y_n, y_m). \tag{3}$$

An analogous argument shows that

$$d(x_n, y_n) - d(x_m, y_m) \leq d(x_m, x_n) + d(y_n, y_m). \tag{4}$$

Together, (3) and (4) imply that

$$|d(x_m, y_m) - d(x_n, y_n)| \leq d(x_m, x_n) + d(y_m, y_n). \tag{5}$$

Fix $\varepsilon > 0$. Since (x_n) and (y_n) are Cauchy, there exist N_1 and N_2 such that

$$m, n \geq N_1 \qquad \Rightarrow \qquad d(x_m, x_n) < \varepsilon/2, \tag{6}$$

$$m, n \geq N_2 \qquad \Rightarrow \qquad d(y_m, y_n) < \varepsilon/2. \tag{7}$$

Set $N = \max(N_1, N_2)$. Then (5), (6), (7) imply that

$$m, n \geq N \qquad \Rightarrow \qquad |\alpha_m - \alpha_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$