Homework set 1 — APPM5440, Fall 2009

Problem 2c: Set $I = [0, 1]$ and consider the set X consisting of all continuous functions on I, with the norm

$$||f|| = \int_0^1 |f(x)|\,dx.$$

Prove that the space X is not complete.

Solution: A straight-forward way of proving this is to construct a Cauchy-sequence that does not have a limit point in X. One example is

$$f_n(x) = \begin{cases} -1 & x < 1/2 - 1/n, \\ n(x - 1/2) & 1/2 - 1/n \le x \le 1/2 + 1/n, \\ 1 & x > 1/2 + 1/n. \end{cases}$$

We first prove that (f_n) is Cauchy. Note that for any m, n, and x, we have $|f_n(x) - f_m(x)| \le 1$. When $m, n \ge N$, we further have $f_n(x) - f_m(x) = 0$ outside the interval $[1/2 - 1/N, 1/2 + 1/N]$, so

$$||f_n - f_m|| = \int_{1/2-1/N}^{1/2+1/N} |f_n(x) - f_m(x)|\,dx \le \int_{1/2-1/N}^{1/2+1/N} 1\,dx = 2/N.$$

We next prove that (f_n) cannot converge to any element in X. Pick an arbitrary $\varphi \in X$. Assume temporarily that $\varphi(1/2) \ge 0$. Since φ is continuous, there exists a $\delta > 0$ such that $\varphi(x) \ge -1/2$ for $x \in B_\delta(1/2)$. Pick an integer $N > 2/\delta$. Then, for $n \ge N$, we have $f_n(x) = -1$ when $x \in [1/2 - \delta, 1/2 - \delta/2]$, and so

$$||f_n - \varphi|| \ge \int_{1/2-\delta}^{1/2-\delta/2} |f_n(x) - \varphi(x)|\,dx \ge \int_{1/2-\delta}^{1/2-\delta/2} 1/2\,dx = \delta/4.$$

If on the other hand $\varphi(1/2) < 0$, then pick $\delta > 0$ such that $\varphi(x) \le 1/2$ on $[1/2, 1/2 + \delta]$ and proceed analogously. □

Remark 1: Note that you cannot solve a problem like the one above by constructing a Cauchy sequence (f_n) in X, point to a non-continuous function f, and claim that since f_n "converges to f", X cannot be complete. Note that the metric is *not even defined* for functions outside of X.

Remark 2: Can you somehow add the limit points of Cauchy sequences in X and obtain a complete space $\tilde{X}$? The answer is yes, you can do that for any metric space; the resulting space $\tilde{X}$ is called the "completion" of X and is (in a certain sense) unique. For the present example, $\tilde{X}$ is the set of all (Lebesgue measurable) real-valued functions on I for which

$$\int_0^1 |f(x)|\,dx < \infty,$$

where the integral is what is called a "Lebesgue" integral. This space is denoted $L^1(I)$. Strictly speaking, an element of $L^1(I)$ is an equivalence class of functions that differ only on a set of Lebesgue measure zero. This roughly means that two functions f and g are considered identical if

$$\int_0^1 |f(x) - g(x)|\,dx = 0.$$