

Notes on Chapter 5 – Banach Spaces

Most important topics:

You are expected to know these definitions and results well.

- Definition of a Banach space. The spaces $\ell^p(\mathbb{N})$, $C_b(I)$, $C_b^k(I)$.
- The space $\mathcal{B}(X, Y)$. The operator norm (equations (5.2) and (5.3) are important). Strong convergence. Norm convergence implies strong convergence.
- For a linear operator: Continuity \Leftrightarrow Boundedness. (Thm. 5.18)
- Equivalent norms.
- Properties of the kernel and the range of a linear operator. The kernel of a continuous operator is (topologically) closed.
- Simplifications in finite-dimensional spaces (all linear operators are bounded, all norm topologies are equivalent, *etc*).
- Theorem 5.37 ($\|ST\| \leq \|S\| \|T\|$).
- Definition of a compact operator. Prop. 5.43.
- Definition of the (topological) dual X^* of a normed linear space X .
- In a Banach space X , the definition of weak convergence is that $x_n \rightharpoonup x$ if and only if $\varphi(x_n) \rightarrow \varphi(x) \forall \varphi \in X^*$. Norm convergence implies weak convergence.

Important topics:

Study these concepts once you master the core concepts above. They are no less important than the topics above, but you may find them more challenging. (These topics are not off-bounds for exam questions.)

- Isomorphisms between Banach spaces.
- The open mapping theorem (statement only, not the proof).
- Coercive operators have closed range (Prop. 5.30).
- The Hahn-Banach theorem. The linear functionals separate points in X . The elements of X separate points in X^* (so that the weak-* topology on X^* is Hausdorff).

Important topics that you will not be quizzed on:

Excellent reading for the Christmas break!

- Extension of a bounded linear operator defined on a dense set (Thm 5.19).
- Definition of the exponential of an operator. Differential equations in Banach spaces. Semi-groups.
- Lax equivalence.
- Weak-* convergence in X^* . Alaoglu's theorem. Isometric embedding of X into X^{**} . (Note that Alaoglu's theorem for reflexive Banach spaces is part of the prelim syllabus!)

What type of questions to expect:

On the midterm and on the final, some questions will be of the type below. (There may of course be other types of questions as well, but these are archetypical ones.)

- Consider the Banach space X defined by Consider the vectors $x_n \in X$ defined by Does the sequence (x_n) converge weakly? If so, to what? Does x_n converge in norm? If so, to what?
- Consider the Banach spaces X and Y defined by Consider the sequence of operators $T_n \in \mathcal{B}(X, Y)$ defined by Prove that (T_n) converges strongly but not in norm. What is the strong limit?
- Consider the Banach space X defined by ..., and the operator $T \in \mathcal{B}(X)$ defined by Prove that T is compact.
- Consider the Banach space X defined by Give an example of a sequence in X that is weakly convergent, but not norm convergent.
- Consider the Banach space X defined by The vectors $\{e_n\}_{n=1}^{\infty}$ where $e_n = \dots$ form a basis for X . Consider the operator $T \in \mathcal{B}(X)$ defined by What is the matrix of T in the basis (e_n) ?
- Consider the Banach space X defined by Consider the operator $T \in \mathcal{B}(X)$ defined by Prove that T maps weakly convergent sequences to strongly convergent sequences.
- Consider the Banach space X defined by Consider the operator $T \in \mathcal{B}(X)$ defined by Prove that the range of T is not topologically closed.