THE CONTRACTION MAPPING PRINCIPLE

Def^n Let X be a metric space, and f a map $f: X \to X$.
We say that f is a CONTRACTION (MAPPING) if there exists an α s.t. $0 < \alpha < 1$ and

$$d(f(x), f(y)) < \alpha\, d(x,y) \quad \forall x, y \in X.$$

Alt def^n: $\exists\, \alpha \in (0,1)$ s.t. $\forall x \in X,\ r \in \mathbb{R}_+ :\ f(B_r(x)) \subseteq B_{\alpha r}(f(x))$.

Note: Obviously, every contraction is continuous.
(In fact, uniformly continuous.)

Def^n Let X be any set and f a map $f: X \to X$.
We say that x_0 is a fixed point of f if $f(x_0) = x_0$.

Thm Suppose that X is a complete metric space and that the map $f: X \to X$ is a contraction. Then f has a unique fixed point in X.

Proof First we prove existence.
Let x_0 be an arbitrarily chosen point in X.
Set $x_1 = f(x_0)$, $x_2 = f(f(x_0))$, ..., $x_n = f(x_{n-1})$.
We will prove that $(x_n)_{n=1}^{\infty}$ is a Cauchy seq in X.
Pick positive integers m & n s.t. $m < n$.

Proof cont^d

Then $d(x_n, x_m) = d(f(x_{n-1}), f(x_{m-1})) \leq$

$$\leq \alpha\, d(x_{n-1}, x_{m-1}) = d(f(x_{n-2}), f(x_{m-2})) \leq$$
$$\leq \alpha^2 d(x_{n-2}, x_{m-2}) \leq \cdots \leq \alpha^m d(x_{n-m}, x_0)$$
$$\leq \alpha^m \big(d(x_{n-m}, x_{n-m-1}) + d(x_{n-m-1}, x_{n-m-2}) + \cdots + d(x_1, x_0)\big)$$
$$\leq \alpha^m \big(\alpha^{n-m-1} d(x_1, x_0) + \alpha^{n-m-2} d(x_1, x_0) + \cdots + d(x_1, x_0)\big)$$
$$= \alpha^m \frac{1-\alpha^{n-m}}{1-\alpha} d(x_1, x_0) \leq \frac{\alpha^m}{1-\alpha} d(x_1, x_0) \to 0 \text{ as } m \to \infty.$$

Since X is complete & $(x_n)_{n=1}^{\infty}$ is Cauchy $\exists x$ s.t. $x_n \to x$.

f is cont $\Rightarrow$ $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n-1} = x$ so x is a fixed point.

It remains to prove uniqueness. Suppose that $x = f(x)$ & $y = f(y)$.

Then $d(x,y) = d(f(x), f(y)) \leq \alpha d(x,y) \Rightarrow d(x,y) = 0 \Rightarrow x = y$.

Thm Suppose that I is an interval in $\mathbb{R}$ and that $t_0 \in I$.
Suppose that $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function s.t.

$$|f(t,u) - f(t,v)| \leq C|u-v| \quad \forall u, v \in \mathbb{R}^n \ t \in I$$

for some finite number C. Then the eq^n

$$\begin{cases} \dot{u}(t) = f(t, u(t)) & t \in I \\ u(t_0) = u_0 \end{cases}$$

has for any $u_0 \in \mathbb{R}^n$ a unique sol^n on I.

Proof: Fix a $t_1 \in I$ and ~~*~~ consider the eq^n

$$(*) \begin{cases} \dot{u}(t) = f(t, u(t)) \\ u(t_1) = u_1 \end{cases}$$

Note that u solves $(*)$ iff $\quad u(t) = u_1 + \int_{t_1}^{t} f(s, u(s))\,ds \quad (**)$

~~Define F on C(I) by setting~~

Proof cont'd

Now fix a $\delta>0$ and consider the map

$$F: C([t_1, t_1+\delta]) \to C([t_1, t_1+\delta]) : u \mapsto [Fu](t) = u_1 + \int_{t_1}^{t} f(s,u(s))\,ds.$$

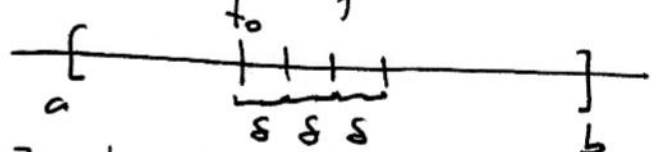
We will prove that if δ is small enough, then F is a contraction:

$$\|F(u)-F(v)\| = \sup_{t_1\le t\le t_1+\delta} \left| \int_{t_1}^{t} f(s,u(s))\,ds - \int_{t_1}^{t} f(s,v(s))\,ds \right| \le$$

$$\le \sup \int_{t_1}^{t} |f(s,u(s)) - f(s,v(s))|\,ds \le C\,\delta\, \|u-v\|$$

We see that if $\delta < 1/C$, then F is a contraction on $C([t_1,t_1+\delta])$.
This implies that $(**)$, and thus $(*)$, has a unique sol^n on $[t_1, t_1+\delta]$.

Suppose that $I=[a,b]$.



By splitting the interval $[t_0,b]$ into pieces of length δ, and repeating the existence proof, we prove existence on $[t_0,b]$.
By "going backwards", we similarly prove existence on $[a,t_0]$.

Remark: The restriction to first-order ODE's is non-essential, since any higher order ODE can be rewritten to a first order one.

Example:
$$(1)\begin{cases} \ddot{u} = f(t,u,\dot{u}) \\ \dot{u}(t_0) = a \\ u(t_0) = b \end{cases}$$

Set $v_1 = u$ $\quad v_2 = \dot{u}$. Then (1) is equivalent to

$$(2)\begin{cases} \dot{v} = \begin{bmatrix} \cancel{f(t,v_1,v_2)}\ v_2 \\ f(t,v_1,v_2) \end{bmatrix} \\ v(t_0) = \begin{bmatrix} a \\ b \end{bmatrix} \end{cases}$$

Remark The theorem applies to all linear ODE's

$$\begin{cases} \dot{u}(t) = A(t)u(t) + b(t) \\ u(t_0) = u_0 \end{cases}$$

as long as $C' = \sup_{t\in I} \sup_{\|\varphi\|=1} |A(t)\varphi|$ is finite.

In this case $|f(t,u) - f(t,v)| = |A(t)(u-v)| \le C'|u-v|$.

A potential drawback of the thm is that is required the function $f(t,u)$ to be globally Lipschitz in u.

What if we only know that

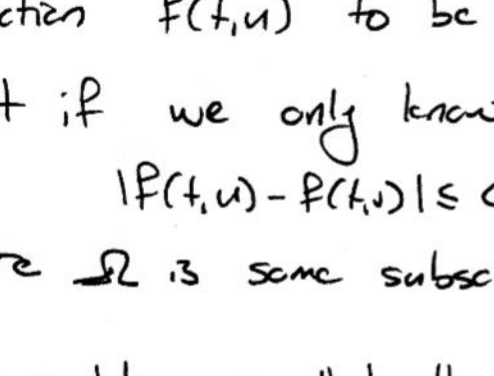
$$|f(t,u) - f(t,v)| \le C'|u-v| \qquad \forall u,v \in \Omega$$

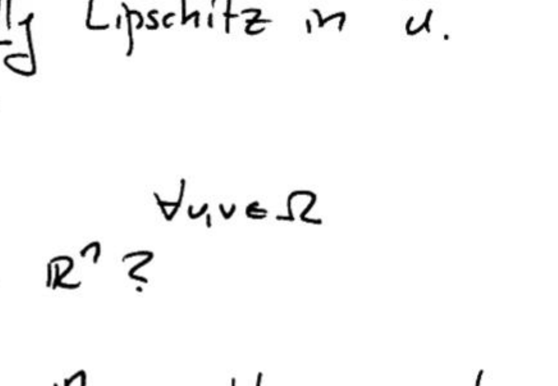
where Ω is some subset of $\mathbb{R}^n$?

The problem is that then the sol^n may blow up and escape Ω.

In such cases, one can always prove local existence on some interval $[t_0-\delta, t_0+\delta]$.

Example Say $I = [-T,T]$ & $\Omega = B_R(u_0) = \{u \in \mathbb{R}^n : |u-u_0| \le R\}$.





Set $M = \sup_{(t,u)\in I\times\Omega} |f(t,u)|$ & $\delta = \min\left(\frac{R}{M}, T\right)$

Then u cannot escape Ω in time δ and so we can prove existence & uniqueness on $[-\delta,\delta]$.

<u>Example</u> $f(t,u)=u^2$ & $u_0>0$

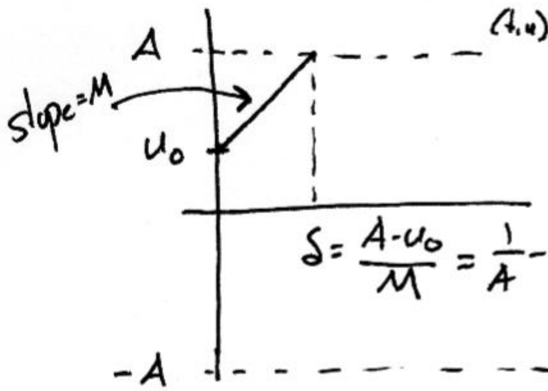
$$(\text{IVP})\begin{cases} \dot{u}=u^2 \\ u(0)=u_0 \end{cases}$$

This f is <u>not</u> globally Lipschitz.

However, for any finite A, it is Lipschitz on $\Omega=[-A,A]$.

$$|f(t,u)-f(t,v)|=|u^2-v^2|=|(u-v)(u+v)|\leq 2A\,|u-v|.$$

We have $M=\sup\limits_{(t,u)\in\mathbb{R}\times\Omega}|f(t,u)|=A^2$



$$\delta=\frac{A-u_0}{M}=\frac{1}{A}-\frac{u_0}{A^2}$$

$$\boxed{\begin{aligned} \varphi(A)&=\frac{1}{A}-\frac{u_0}{A^2} \\ \varphi'(A)&=-\frac{1}{A^2}+\frac{2u_0}{A^3} \\ \varphi'(A)&=0 \quad\Rightarrow A=2u_0 \end{aligned}}$$

To maximize δ, we set $A=2u_0$, $\Rightarrow \delta=\frac{1}{2u_0}-\frac{1}{4u_0}=\frac{1}{4u_0}$

(The exact sol^n is $u(t)=\frac{u_0}{1-u_0t}$ so we get blow-up at $t=1/u_0$.)

Sometimes problem-specific information allows us to do better.

<u>Example</u> Consider a particle moving in a potential field φ. At time t, let $u(t)\in\mathbb{R}^n$ denote the particle location. Then at time t, the particle is subjected to the conservative force $F(u)=-\nabla\varphi(u)$.

Newton $\Rightarrow m\ddot{u}(t)=-\nabla\varphi(u)$.

Set $\begin{aligned} p(t)&=m\dot{u}(t) \\ q(t)&=u(t) \end{aligned}$ $\Rightarrow \begin{cases} \dot{p}(t)=m\ddot{u}(t)=-\nabla_q\varphi(q) \\ \dot{q}(t)=\dot{u}=\frac{p}{m}=\nabla_p\frac{|p|^2}{2m} \end{cases}$

Set $H(p,q)=\frac{|p|^2}{2m}+\varphi(q)$, then we can write the ODE

$$\begin{cases} \dot{p} = -\nabla_q H \\ \dot{q} = \nabla_p H \end{cases} \Leftarrow \text{Hamiltonian ODE.}$$

Note that physically, the Hamiltonian

$H(u) = H(p,q) = \frac{|p|^2}{2m} + \varphi(q) = \frac{1}{2} m |\dot{u}(t)|^2 + \varphi(u(t))$ is the total energy.

$$\frac{d}{dt} H(u) = \frac{d}{dt} H(p,q) = \dot{p} H_p + \dot{q} H_q = -\nabla_q H \cdot \nabla_p H + \nabla_p H \cdot \nabla_q H = 0.$$

so (p,q) stays on the set $\{(p,q) \in \mathbb{R}^{2n} : H(p,q) = \underset{\uparrow \text{ energy at } t_0}{E_0}\} =: \Omega_0$

If the map $f(t, \begin{bmatrix} p \\ q \end{bmatrix}) = \begin{bmatrix} -\nabla_q H \\ \nabla_p H \end{bmatrix} = \begin{bmatrix} -\nabla\varphi(q) \\ \frac{p}{m} \end{bmatrix}$

is Lipschitz in some neighborhood of Ω_0, then global existence is assured. (For this, a sufficient condition is that $D^2\varphi$ is bdd.)

More generally, consider

$$\begin{cases} \dot{u} = -\nabla V(u) \\ u(t_0) = u_0 \end{cases}$$

Then $\frac{d}{dt} V(u(t)) = \dot{u} \cdot \nabla V(u) = -\nabla V(u) \cdot \nabla V(u) = -|\nabla V(u)|^2 \le 0$

Thus the sol^n stays inside the set $\Omega = \{u : V(u) \le V(u_0)\}$.

If this is a closed set, and if $f(t,u) = -\nabla V(u)$ is uniformly Lipschitz on $I \times \Omega$, then existence & uniqueness are assured on I.