<u>Corollary</u> Let X be a NLS

(a) If $x \in X$, $\exists \varphi \in X^*$ s.t. $\|\varphi\|=1$ & $\varphi(x)=\|x\|$

(b) $\|x\| = \sup_{\|\varphi\|=1} |\varphi(x)|$

(c) The bdd linear functionals separate point.
(in other words, given $x, y \in X$ s.t. $x \neq y$
$\exists \varphi \in X^*$ s.t. $\varphi(x) \neq \varphi(y)$.)

(d) ~~For~~ $x \in X$, define an element $F_x \in X^{**}$ by setting
$F_x(\varphi) = \varphi(x)$ (note that $F_x : X^* \to \mathbb{R}$).
The map $x \mapsto F_x$ is a linear isometry $X \to X^{**}$

<u>Proof</u>: (a) Set $Y = \text{span}(x)$ & $\varphi(\lambda x) = \lambda \|x\|$

(b) Set $A = \sup_{\|\varphi\|=1} |\varphi(x)|$.
Then $A \leq \sup_{\|\varphi\|=1} \|\varphi\| \, \|x\| = \|x\|$.
Conversely let $\hat{\varphi}_x$ be the functional s.t. $\|\hat{\varphi}_x\|=1$ & $\hat{\varphi}_x(x)=\|x\|$
Then $A \geq |\hat{\varphi}(x)| = | \, \|x\| \, | = \|x\|$.

(c) If $x \neq y$, $\exists \varphi \in X^*$ s.t. $\varphi(x-y) = \|x-y\| \neq 0$.
Then $\varphi(x) \neq \varphi(y)$

(d) It is obvious that $F_x \in X^{**}$ since $|F_x(\varphi)| \leq \|\varphi\| \, \|x\|$. $(*)$
It is simple to prove that the map $x \mapsto F_x$ is linear.
It remains to prove isometry.
$(*) \Rightarrow \|F_x\| \leq \|x\|$
Conversely, let φ_x be the element from (a) s.t.
$\|\varphi_x\|=1$ & $\varphi_x(x)=\|x\|$. Then
$\|F_x\| = \sup_{\varphi \in X^*} |\varphi(x)| \geq |\varphi_x(x)| = \|x\|$.

Let $\mathbb{X}$ be a NLS.

Set $\hat{\mathbb{X}} = \{f_x : x \in \mathbb{X}\}$.

Then (i) $\hat{\mathbb{X}} \subseteq \mathbb{X}^{**}$ ← a Banach space

(ii) $\hat{\mathbb{X}}$ is isomorphic to $\mathbb{X}$.

Thus, the topological closure $\bar{\hat{\mathbb{X}}}$ of $\hat{\mathbb{X}}$ in $\mathbb{X}^{**}$ is isomorphic to the completion $\bar{\mathbb{X}}$ of $\mathbb{X}$.

This is an alternative way of defining the ~~closu~~ completion of $\mathbb{X}$.

(Does not work for general metric spaces.)

WEAK CONVERGENCE

Def^n Let $\mathbb{X}$ be a Banach space.

We say that x_n converges weakly to x, $x_n \rightharpoonup x$, if $\varphi(x_n) \to \varphi(x)$ $\forall \varphi \in \mathbb{X}^*$.

(The weak topology is the smallest top in which all $\varphi \in \mathbb{X}^*$ are continuous. A subbase is given by the sets
$$B_\varepsilon^\varphi(x_0) = \{y \in \mathbb{X} : |\varphi(x) - \varphi(y)| < \varepsilon\} \qquad \varepsilon > 0,\ x \in \mathbb{X},\ \varphi \in \mathbb{X}^*)$$

Example $\mathbb{X} = \ell^2(\mathbb{N})$ $e^{(n)}$ canonical basis.

Then $e^{(n)} \xrightarrow{w} 0$.

To prove this, pick $\varphi \in \mathbb{X}^*$.

We know $\exists y \in \mathbb{X}$ s.t. $\varphi(x) = \langle x, y \rangle$ $\forall x$.

Then $\varphi(e^{(n)}) = \langle e^{(n)}, y \rangle = y_n \to 0$ as $n \to \infty$.

~~A~~ Funny consequence: Set $S=\{x\in X : \|x\|=1\}$.

Then $e^{(n)}\in S \quad \forall n$, $e^{(n)} \rightharpoonup 0$, so

$0\in \overline{S}^{w}=\{x\in X : \exists x_n\in S \text{ s.t. } x_n \xrightarrow{w} x\}$

↑ Closure of S in weak topology.

In fact: $\overline{S}^{w}=\{x\in X : \|x\|\leq 1\}$

So the surface of the unit ball is dense in the unit ball!

(Homework!)

Thm If $x_n \xrightarrow{w} x$, then $\|x\|\leq \liminf\limits_{n\to\infty}\|x_n\|$

Proof $\|x\| = \sup\limits_{\|\varphi\|=1}|\varphi(x)| = \sup\limits_{\|\varphi\|=1}\lim\limits_{n\to\infty}|\varphi(x_n)| \leq \liminf\limits_{n\to\infty}\overbrace{\sup\limits_{\|\varphi\|=1}|\varphi(x_n)|}^{=\|x_n\|} = \liminf\|x_n\|$

Discuss the fact that B is compact; how weak topologies are useful in existence proofs, etc.

Prop^n Let X be a ~~NLS~~ Banach space, let $\Omega\subseteq X$, and set

$\overline{\Omega}=\{x\in X : \exists x_n\in\Omega \text{ s.t. } x_n\to x\}$

$\overline{\Omega}^{w}=\{x\in X : \exists x_n\in\Omega \text{ s.t. } x_n\rightharpoonup x\}$

(i) $\overline{\Omega}\subseteq\overline{\Omega}^{w}$ always.

(ii) If Ω is a linear space, then $\overline{\Omega}=\overline{\Omega}^{w}$

Proof: (i) Pick $x\in\overline{\Omega}$. Then $\exists x_n\in\Omega$ s.t. $x_n\to x$.

But then $x_n\rightharpoonup x$ so $x\in\overline{\Omega}^{w}$

(ii) Homework

Now let us consider the space X^*. We already know two types of conv:

$\varphi_n \to \varphi$ in norm if $\|\varphi_n - \varphi\| \to 0$ as $n \to \infty$

$\varphi_n \xrightarrow{w} \varphi$ weakly if $F(\varphi_n) \to F(\varphi)$ as $n \to \infty$ $\forall F \in X^{**}$

We define a third mode of convergence:

$\varphi_n \xrightarrow{*} \varphi$ weak-$*$ if $\varphi_n(x) \to \varphi(x)$ as $n \to \infty$ $\forall x \in X$

$\Leftrightarrow$ $F_x(\varphi_n) \to F_x(\varphi)$ as $n \to \infty$ $\forall x \in X$.

Since $X \subset X^{**}$, weak-$*$ is a weaker topology than the weak one.

If X is reflexive, $X = X^{**}$; then weak & weak-$*$ topologies are the same.

Hahn-Banach $\Rightarrow$ weak-$*$ top is always Hausdorff.

The weak-$*$ topology is very useful due to the following fact.

Set $S^* = \{\varphi \in X^* : \|\varphi\| \le 1\}$. It is very desirable that S^* be compact

S^* is compact in the ~~strong~~ norm top $\Leftrightarrow$ X is finite-dimensional

S^* is compact in the weak top $\Leftrightarrow$ X is reflexive ~~$\Leftrightarrow S = \{x \in X : \|x\| \le 1\}$ is compact in the weak top~~

S^* is compact in the weak-$*$ top $\leftarrow$ Always!

Thm Let X be a NLS.

Alaoglu's thm ~~The set $S^* = \{\varphi \in X^*$ s.t.~~

Let S^* denote the closed unit ball in X^*.

Then S^* is a compact Hausdorff space in the weak-$*$ topology.

Thm Let X be a Banach space and let S denote its closed unit ball.

Then S is compact in the weak top $\Leftrightarrow$ X is reflexive.