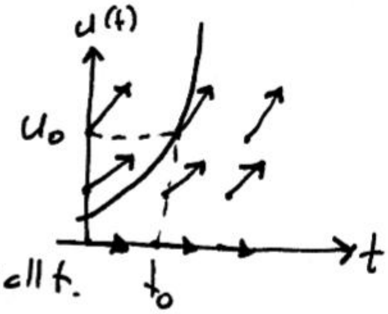
ODE's

An archetypical ODE takes the following form:
Given a function $f = f(t,u)$, and an initial value u_0, find the function $u = u(t)$ such that

$$\begin{cases} \dot{u}(t) = f(t, u(t)) \\ u(t_0) = u_0 \end{cases}$$

<u>Example</u> $\begin{cases} \dot{u} = u \\ u(t_0) = u_0 \end{cases} \Rightarrow u(t) = u_0 e^{t-t_0}$



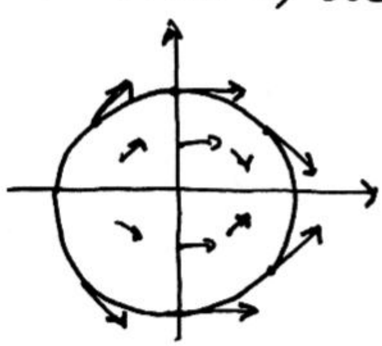
The solution is unique and exists for all t.

<u>Example</u> $\begin{cases} \dot{u} = -t/u \\ u(0) = R \end{cases} \Leftrightarrow \frac{du}{dt} = -\frac{t}{u} \Leftrightarrow u\,du = -t\,dt \Rightarrow \frac{1}{2}u^2 = -\frac{1}{2}t^2 + C$

$$\Rightarrow u^2 + t^2 = 2C$$

To determine C, use initial cond^n: $u(0)^2 + 0^2 = 2C$

$R^2 + 0^2 = 2C \Rightarrow C = \frac{1}{2}R^2$



So $u(t) = \text{sign}(R)\sqrt{R^2 - t^2}$

If $R \neq 0$, there exists a unique solution on the interval $(-R, R)$.

<u>Example</u> $\begin{cases} \dot{u} = u^2 \\ u(0) = u_0 \end{cases} \Rightarrow \frac{du}{u^2} = dt \Rightarrow -\frac{1}{u} = t + C \Rightarrow u = -\frac{1}{t+C} \overset{\text{Use initial cond}^n}{\Rightarrow} u = \frac{u_0}{1 - u_0 t}$

Solution exists and is unique, but it is defined only when $t < 1/u_0$

<u>Example</u> $\begin{cases} \dot{u} = \sqrt{|u|} \\ u(0) = 0 \end{cases}$

$u(t) = 0$ is a sol^n

$u(t) = \begin{cases} 0 & \text{for } t \le a \\ \frac{1}{4}(t-a)^2 & \text{for } t > a \end{cases}$ is also a sol^n for any $a \in (0, \infty)$

Solution exists for all t but it is not unique.

We will prove that as long as f is continuous, an ODE <u>always</u> has a C^1-sol^n on some interval $(t_0-\varepsilon, t_0+\varepsilon)$.

For uniqueness and global existence, stronger cond^{ns} will be required.

<u>The Forward Euler method</u> ← This is a (bad) numerical method and a (good) tool in analysis.

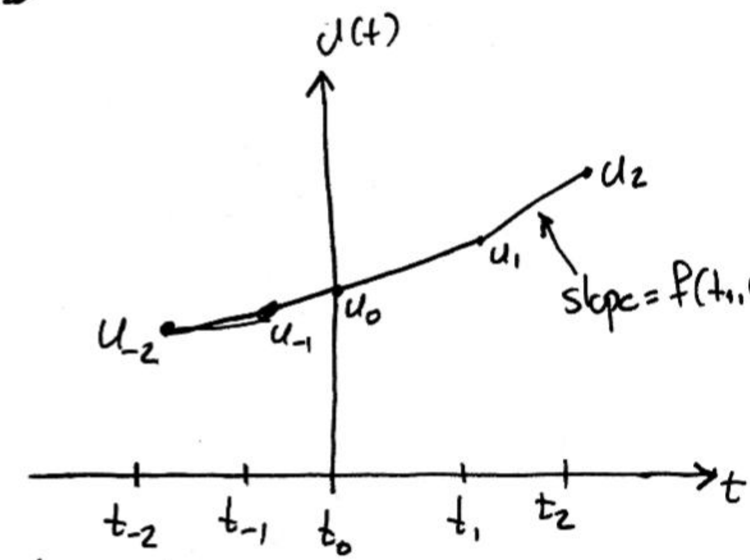
Consider the ODE $\begin{cases} \dot{u}(t) = f(t, u(t)) \\ u(t_0) = u_0 \end{cases}$

Fix a "step-size" ε and set $t_k = t_0 + \varepsilon k$ for $k \in \mathbb{Z}$.

Define for $k \in \mathbb{Z}$, u_k by

for $k>0$ $\begin{cases} u_1 = u_0 + \varepsilon f(t_0, u_0) \\ u_2 = u_1 + \varepsilon f(t_1, u_1) \\ \quad \vdots \\ u_k = u_{k-1} + \varepsilon f(t_{k-1}, u_{k-1}) \end{cases}$

for $k<0$ $\begin{cases} u_{-1} = u_0 - \varepsilon f(t_0, u_0) \\ u_{-2} = u_1 - \varepsilon f(t_1, u_1) \\ \quad \vdots \\ u_k = u_{k+1} - \varepsilon f(t_{k+1}, u_{k+1}) \end{cases}$



Let the piecewise linear functions that interpolates the points (t_k, u_k) be called $u_\varepsilon(t)$.

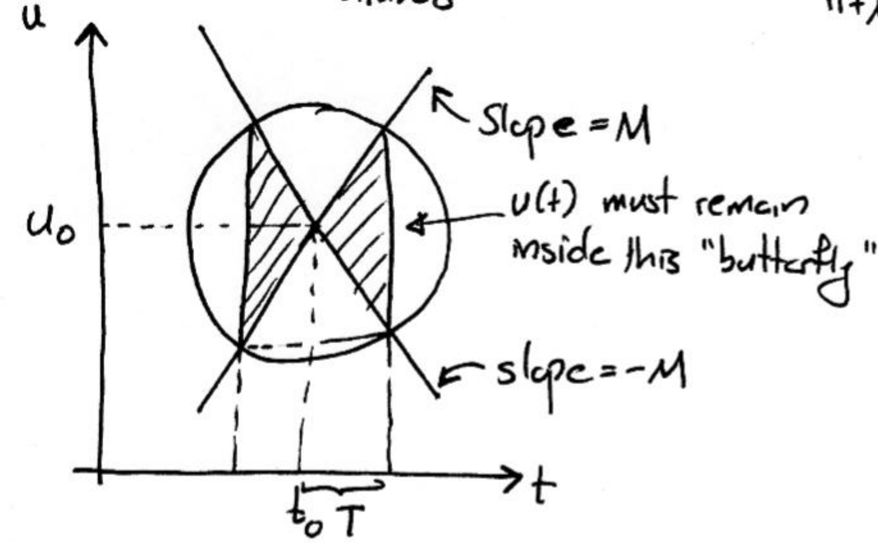
<u>Theorem</u> (The Peano existence theorem) Let $f = f(t,u)$ be a function that is continuous in some neighborhood of a point (t_0, u_0). Then there exists a $T>0$ such that the ODE

(IVP) $\begin{cases} \dot{u}(t) = f(t, u(t)) \\ u(t_0) = u_0 \end{cases}$

has a continuously differentiable solution on $[t_0 - T, t_0 + T]$.

<u>Proof</u> Pick an $R>0$ such that f is continuous on the ball $B=\{(t,u) : (t-t_0)^2+(u-u_0)^2 \leq R^2\}$.

Set $M = \sup_{(t,u)\in B} |f(t,u)|$ and $T = \frac{R}{\sqrt{1+M^2}}$



For any $\varepsilon>0$, let u_ε denote the approximate sol^n given by the forwards Euler method. For $t \in \underbrace{[t_0-T, t_0+T]}_{=I}$ we have ~~Lip(f) ≤ M and~~ $\text{Lip}(u_\varepsilon) \leq M$ and $\|u_\varepsilon\| \leq |u_0| + TM$

The A-A thm assures us that the set $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ is precompact.

Therefore, $\exists$ $u \in C(I)$ and a subseq $(u_{\varepsilon_n})_{n=1}^\infty$ s.t. $u_{\varepsilon_n} \to u$ as $n\to\infty$
uniform convergence.

We can pick the seq so that $\varepsilon_n < \frac{1}{n}$ and $\|u_{\varepsilon_n} - u\| < 1/n$

We will prove that u solves (IVP) and that $u \in C^1(I)$.

Set $u_n = u_{\varepsilon_n}$ and note that u_n satisfies

$$(1) \quad u_n(t) = u_0 + \int_{t_0}^{t} \dot{u}_n(s)ds = u_0 + \int_{t_0}^{t} \underbrace{f(s,u_n(s))}_{=\varphi_n(s)}ds + \int_{t_0}^{t} \underbrace{(\dot{u}_n(s) - f(s,u_n(s)))}_{=\psi_n(s)}ds$$

<u>Correction</u>: We never proved that $\varepsilon_n \to 0$ as $n\to\infty$.
To sidestep this difficulty, consider instead of the set $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$, the set $\Omega = \{u_\varepsilon : \varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$
Then Ω is still pre-compact $\Rightarrow$ $\exists$ a Cauchy subseq $(u_{\varepsilon_n})_{n=1}^\infty \subset \Omega$. For this sequence, we <u>must</u> have $\varepsilon_n \to 0$.

Claim 1 $\varphi_n(s) \to f(s,u(s))$ uniformly on I

Claim 2 $\psi_n(s) \to 0$ uniformly on I

Claim 3 If $g_n \to g$ uniformly, then $\int_{t_0}^{t} g_n(s)ds \to \int_{t_0}^{t} g(s)ds$. ← Proved in homework.

Applying the three claims to eq^n (1) we find that

$$u(t) = u_0 + \int_{t_0}^{t} f(s,u(s))ds.$$

This proves both that $u \in C^1(I)$ and that u solves (IVP).

Proof of Claim 1: Pick $\varepsilon > 0$.

Since f is uniformly cont on B, $\exists\, \delta > 0$ such that

$$|y - y'| < \delta \Rightarrow |f(s,y) - f(s,y')| < \varepsilon.$$

Pick N s.t. $n \geq N \Rightarrow \|u_n - u\| < \delta$.

Then for $n \geq N$, $|\varphi_n(s) - f(s,u(s))| = |f(s,u_n(s)) - f(s,u(s))| < \varepsilon$.

Proof of Claim 2: Pick $\varepsilon > 0$.

$$\psi_n(s) = \dot{u}_n(s) - f(s,u_n(s)) \overset{\text{say } t_k \leq s < t_{k+1} \text{ where } t_k = t_0 + \varepsilon_n k}{=} f(t_k, u_n(t_k)) - f(s,u(s))$$

Now $|t_k - s| \leq \varepsilon_n < 1/n$

$$|u_n(t_k) - u(s)| < 2M\varepsilon_n < 2M/n$$

f is uniformly cont $\Rightarrow \exists\, \delta > 0$ s.t. $|x - x'| < \delta$ & $|y - y'| < \delta \Rightarrow |f(x,y) - f(x',y')| < \varepsilon$.

Pick N s.t. $\frac{1}{N} < \delta$ & $\frac{2M}{N} < \delta$. Then, if $n \geq N$

$$|\psi_n(s)| = |f(t_k, u_n(t_k)) - f(s,u_n(s))| < \varepsilon.$$

THE GRÖNWALL INEQUALITY

Lemma (Grönwall ineq.) Suppose that $u, \varphi \in C([0,T])$, and that $u(t) \geq 0$ & $\varphi(t) \geq 0$. Moreover, assume that for some $u_0 \geq 0$, we have

$$u(t) \leq u_0 + \int_0^t \varphi(s) u(s) ds \quad \forall t \in [0,T]. \qquad (1)$$

Then

$$u(t) \leq u_0 \exp\left(\int_0^t \varphi(s) ds \right) \qquad (2)$$

Note 1 If $u \in C^1$ & $\dot{u} \leq \varphi u$, then $u(t) - u(0) \leq \int_0^t \varphi(s) u(s) ds \iff (1)$

Note 2 If $u_0 = 0$, then $u(t) = 0$. $\forall t \in [0,T]$.

Note 3 If equality holds in (1), then

$$\dot{u} = \varphi u \Rightarrow \frac{du}{u} = \varphi dt \Rightarrow \log u = \int_0^t \varphi(s) ds + C'$$

$$\Rightarrow u(t) = e^{C'} \exp\left(\int_0^t \varphi(s) ds \right) = u_0 \exp\left(\int_0^t \varphi(s) ds \right)$$

Proof Assume first that $u_0 > 0$. Set

$$U(t) = u_0 + \int_0^t u(s) \varphi(s) ds \Rightarrow u(t) \leq U(t) \Rightarrow$$

$$\Rightarrow \dot{U}(t) = \varphi(t) u(t) \leq \varphi(t) U(t) \Rightarrow$$

$$\Rightarrow \frac{\dot{U}(t)}{U(t)} \leq \varphi(t) \Rightarrow \frac{d}{dt} \log U(t) \leq \varphi$$

since $u(t) \leq U(t)$

$$\Rightarrow \log U(t) - \log \underbrace{U(0)}_{= u_0} \leq \int_0^t \varphi(s) ds \Rightarrow U(t) \leq u_0 \exp\left(\int_0^t \varphi(s) ds \right) \Rightarrow (2)$$

Assume next that $u_0=0$. Then for any $\varepsilon>0$ we have

$$u(t) \le \int_0^t \varphi(s)u(s)ds \le \varepsilon + \int_0^t \varphi(s)u(s)ds. \qquad (*)$$

We proved that $(*) \Rightarrow u(t) \le \varepsilon \exp\left(\int_0^t \varphi(s)ds\right)$

Since ε was arbitrary, this shows that $u(t)=0$.

<u>Thm</u> Consider the initial value problem

$$(IVP) \begin{cases} \dot{u}(t) = f(t, u(t)) \\ u(t_0) = u_0 \end{cases}$$

Assume that f is continuous on $R=[t_0-T, t_0+T] \times [u_0-L, u_0+L]$

Set $M = \sup_{(t,u)\in R} |f(t,u)|$

Set $\delta = \min\left(T, \frac{L}{M}\right)$

Set $I = [t_0-\delta, t_0+\delta]$

(1) If u solves (IVP), then

$|u(t)-u_0| \le L$ for $t \in I$.

(2) If there exists a finite number C' such that

$$|f(t,u)-f(t,v)| \le C'|u-v| \qquad \forall \begin{array}{l} t\in I \\ u,v \in [u_0-L, u_0+L] \end{array}$$

then ~~the~~ (IVP) has a unique soln in $C^1(I)$.

Proof (1) Let u be a sol^n.

Set $\tau = \sup\{\eta : 0 \le \eta \le \delta \;\&\; |u(t) - u(t_0)| \le L \text{ when } |t - t_0| \le \eta\}$

Then claim (1) is that $\tau = \delta$.

Assume that $\tau < \delta$ (we will show that this will lead to a contradiction).

Then $|u(t_0 + \tau) - u(t_0)| = \left|\int_{t_0}^{t_0+\tau} f(s, u(s))\,ds\right| \le M\tau < M\delta \le L$. (a)

But since u is continuous, we must also have $|u(t_0+\tau) - u(t_0)| = L$. (b)

(a) & (b) contradict each other.

Proof of (2): That a C^1 sol^n exists is guaranteed by the Peano thm.

Now suppose that both u & v solve (IVP).

Then $u(t) - v(t) = (u(t) - u_0) - (v(t) - u_0) =$

$$= \int_{t_0}^{t} \dot{u}(s)\,ds - \int_{t_0}^{t} \dot{v}(s)\,ds =$$

$$= \int_{t_0}^{t} f(s, u(s))\,ds - \int_{t_0}^{t} f(s, v(s))\,ds$$

Set $w(t) = |u(t) - v(t)|$. Then

$$w(t) \le \int_{t_0}^{t} |f(s, u(s)) - f(s, v(s))|\,ds \le$$

Use Lipschitz continuity.

$$\le \int_{t_0}^{t} C'|u(s) - v(s)|\,ds = \int_{t_0}^{t} C' w(s)\,ds$$

Apply the Grönwall inequality with $\varphi = C'$ and $u_0 = 0$ to see that $w(t)$ is identically zero.