Applied Analysis (APPM 5440): Midterm 3 – Solutions
5.30pm – 6.50pm, Dec. 4, 2006. Closed books.

Problem 1: No motivation required. 2p each:

(a) Let $(X, \mathcal{T})$ denote a topological space. Specify the axioms that $\mathcal{T}$ must satisfy.

(b) Let $(X, \mathcal{T})$ denote a topological space. Define what it means for $\mathcal{T}$ to be Hausdorff.

(c) Let $(X, \mathcal{T})$ denote a topological space, let $(x_n)_{n=1}^{\infty}$ denote a sequence in X, and let x denote an element of X. Define what it means for x_n to converge to x. ($\mathcal{T}$ is not necessarily metrizable.)

Solution: Check textbook.

Problem 2: Consider the set $X = \{a, b, c\}$, and the collection of subsets $\mathcal{T} = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. Is $\mathcal{T}$ a metrizable topology? List the compact subsets of X. Give an example of a function $f : X \to \mathbb{R}$ that is continuous, and one example of a function $g : X \to \mathbb{R}$ that is not. Justify your answers briefly. (6p)

Solution: $\mathcal{T}$ is a topology, but it is not metrizable. To prove this, we assume that there exists a metric d that generates $\mathcal{T}$. Set $\varepsilon = \min(d(b, a), d(b, c))$. Then $\{b\} = B_{\varepsilon/2}(b)$ so $\{b\}$ should be an open set. However, $\{b\} \notin \mathcal{T}$.

Every subset of X is compact (since $\mathcal{T}$ is finite, every open cover of any subset is itself finite). Thus the compact sets are

$$\emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{a, b\},\ \{a, c\},\ \{b, c\},\ \{a, b, c\}.$$

The function f defined by $f(x) = 1$ for $x = a, b, c$ is continuous. To prove this, let G be an open subset of $\mathbb{R}$. If $1 \in G$, then $f^{-1}(G) = X$ which is an open set. If $1 \notin G$, then $f^{-1}(G) = \emptyset$ which is also open.

The function g defined by

$$g(a) = 0, \quad g(b) = 0, \quad g(c) = 1$$

is not continuous. To prove this, consider the open set $G = (1/2,\ 3/2)$ in $\mathbb{R}$. Then $g^{-1}(G) = \{c\}$ which is not an open set in $\mathcal{T}$.

Problem 3: Let X denote the set of all continuous functions on the interval $I = [-\pi,\, \pi]$. Equip X with the norm

$$||f|| = \int_{-\pi}^{\pi} |f(y)|\, dy.$$

Consider the operator $T \in \mathcal{B}(X)$ that is defined by

$$[Tf](x) = \int_{0}^{\pi} \sin(x)\, y^2\, f(y)\, dy.$$

Calculate the norm of T in $\mathcal{B}(X)$. (4p total: 2p for the correct answer α, and 1p each for the proofs that $\alpha \leq ||T||$ and that $\alpha \geq ||T||$.)

Solution: We have

$$\begin{aligned} ||Tf|| = \int_{-\pi}^{\pi} \left| \int_0^{\pi} \sin(x)\, y^2\, f(y)\, dy \right| dx = \int_{-\pi}^{\pi} |\sin(x)|\, dx \left| \int_0^{\pi} y^2\, f(y)\, dy \right| \\ = 4 \left| \int_0^{\pi} y^2\, f(y)\, dy \right| \leq 4 \left(\sup_{y \in I} y^2 \right) \int_0^{\pi} |f(y)|\, dy \leq 4\, \pi^2\, ||f||. \end{aligned}$$

It follows that $||T|| \leq 4\, \pi^2$.

To prove that $||T|| \geq 4\, \pi^2$, pick[1] non-negative functions $f_n \in X$ such that $||f_n|| = 1$ and $\mathrm{supp}(f) \subseteq [\pi - 1/n, \pi]$. Then

$$\begin{aligned} ||T|| = \sup_{||f||=1} ||Tf|| \geq \sup_n ||Tf_n|| = \sup_n \int_{-\pi}^{\pi} |\sin(x)|\, dx \int_0^{\pi} y^2\, f_n(y)\, dy \\ = \sup_n 4 \int_{\pi-1/n}^{\pi} y^2\, f_n(y)\, dy \geq \sup_n 4 \left(\inf_{y \in [\pi - 1/n, \pi]} y^2 \right) \int_{\pi-1/n}^{\pi} f_n(y)\, dy \\ = \sup_n 4\, (\pi - 1/n)^2 = 4\, \pi^2. \end{aligned}$$

[1] In your solutions, drawing a picture of such a sequence is fine. An explicit formula is not required, but if you insist on one, consider

$$f_n(x) = \begin{cases} 0 & x \in [-\pi,\, \pi - 1/n], \\ 2\, n^2\, (x - (\pi - 1/n)) & x \in (\pi - 1/n,\, \pi]. \end{cases}$$

Problem 4: Let X be a Banach space with a compact subset K. Suppose that $(x_n)_{n=1}^{\infty}$ is a sequence of elements in K that converges weakly to some element $x \in K$. Is it necessarily the case that the sequence also converges in norm to x? Either prove that this is the case, or give a counter-example. (4p)

Solution: The answer is yes. Suppose that the sequence $(x_n)_{n=1}^{\infty}$ satisfies the assumptions of the problem, but does not converge in norm to x. Then there exists an $\varepsilon > 0$, and a subsequence $(x_{n_j})_{j=1}^{\infty}$ such that

$$||x - x_{n_j}|| \geq \varepsilon, \quad \text{for } j = 1, 2, 3, \ldots \tag{1}$$

However, since (x_{n_j}) is a sequence in a compact set, it has a subsequence $(x_{n_{j_k}})_{k=1}^{\infty}$ that converges in norm. Since $x_{n_{j_k}} \rightharpoonup x$, this element must be x, which is impossible in view of (1).

Problem 5: Consider the Banach space $X = l^2(\mathbb{N})$, and the operator $T \in \mathcal{B}(X)$ defined by

$$Tx = (\tfrac{1}{1}\,x_1,\ \tfrac{1}{2}\,x_2,\ \tfrac{1}{3}\,x_3,\ \dots).$$

Prove that $\mathrm{ran}(T)$ is not topologically closed. (4p)

Solution: We know that a one-to-one operator has closed range if and only if it is coercive. We will prove that T is one-to-one, but not coercive.

To see that T is one-to-one, simply note that if $Tx = 0$, then clearly x must be zero.

Next we prove that T is not coercive. Let $e^{(n)}$ denote the canonical basis vectors,

$$\begin{aligned} e^{(1)} &= (1,\,0,\,0,\,0,\,\dots), \\ e^{(2)} &= (0,\,1,\,0,\,0,\,\dots), \\ e^{(3)} &= (0,\,0,\,1,\,0,\,\dots), \\ &\ \vdots \end{aligned}$$

We have

$$||T\,e^{(n)}|| = ||\frac{1}{n}e^{(n)}|| = \frac{1}{n}||e^{(n)}||$$

so there can exist no $c > 0$ such that $||Tx|| \geq c||x||$ for all x.

Alternative solution: We will prove that the element

$$y = (\tfrac{1}{1},\ \tfrac{1}{2},\ \tfrac{1}{3},\dots) \in X$$

belongs to $\overline{\mathrm{ran}(T)}$, but not to $\mathrm{ran}(T)$. This proves that $\mathrm{ran}(T)$ is not closed.

To prove that $y \in \overline{\mathrm{ran}(T)}$, consider the elements $x^{(n)} \in X$ defined by

$$\begin{aligned} x^{(1)} &= (1,\,0,\,0,\,0,\,\dots), \\ x^{(2)} &= (1,\,1,\,0,\,0,\,\dots), \\ x^{(3)} &= (1,\,1,\,1,\,0,\,\dots), \\ &\ \vdots \end{aligned}$$

Set $y^{(n)} = Tx^{(n)}$ so that $y^{(n)} \in \mathrm{ran}(T)$. Since $y^{(n)} \to y$, it follows that $y \in \overline{\mathrm{ran}(T)}$.

To prove that $y \notin \mathrm{ran}(T)$, note that if $Tx = y$, then $x = (1,\,1,\,1,\,\dots)$ which is not an element of X.