Homework set 12 — APPM5440, Fall 2006

This homework set has two components. The first set of questions relate to the material covered in class Nov. 13 – Nov. 17. Most of this material is drawn from the "core" part of the syllabus (the exception are the questions on exponentials of operators, but these are good basic exercises). The second set of questions concerns the material covered in the Nov. 27 and Nov. 29 classes. Most of this material is not part of the core curriculum, so do not attempt those questions unless you feel comfortable with the first half.

Part 1:

From the textbook: 5.10, 5.11, 5.12, 5.13, 5.17. (Hint for question 5.10: Use the Ascoli-Arzelà theorem.)

Problem 1: For $n = 1, 2, 3, \dots$, we define the operator T_n on $X = l^2(\mathbb{N})$ by

$$T_n(x_1, x_2, x_3, \dots) = \frac{1}{\sqrt{n}}(x_1, x_2, \dots, x_n, 0, 0, \dots).$$

Prove that $T_n \in \mathcal{B}(X)$. Does T_n converge to anything in norm? Strongly?

Problem 2: Let T be a compact operator that is one-to-one on an infinite dimensional Banach space X. Prove that $\text{ran}(T)$ is not closed. (Hint: Prove that T cannot be coercive.)

Part 2:

Problem 3: Let X denote the linear space of polynomials of degree 2 or less on $I = [0, 1]$. For $f \in X$, set $||f|| = \sup_{x \in I} |f(x)|$. For $f \in X$, define

$$\varphi_1(f) = \int_0^1 f(x)\,dx, \quad \varphi_2(f) = f(0), \quad \varphi_3(f) = f'(1/2), \quad \varphi_4(f) = f'(1/3).$$

Prove that $\varphi_j \in X^*$ for $j = 1, 2, 3, 4$. Prove that $\{\varphi_1,\ \varphi_2,\ \varphi_3\}$ forms a basis for X^*. Prove that $\{\varphi_1, \varphi_2, \varphi_4\}$ does not form a basis for X^*.

Problem 4: Consider the space $X = \mathbb{R}^2$ equipped with the Euclidean norm. Explicitly construct the set X^*. Set

$$\mathcal{S} = \{B_\varepsilon^\varphi(x_0) :\ \varphi \in X^*,\ \varepsilon > 0,\ x_0 \in X\},$$

where

$$B_\varepsilon^\varphi(x_0) = \varphi^{-1}(B_\varepsilon(\varphi(x_0))).$$

Give a geometric description of each set $B_\varepsilon^\varphi(x_0)$. Prove that the topology generated by the sub-basis $\mathcal{S}$ in X^* equals the norm topology on $\mathbb{R}^2$. This proves that on X, the weak topology equals the norm topology.

Problem 5: Let $X = l^3$. What is X^*? Set $D = \{x \in l^3 : ||x|| = 1\}$. Prove that the weak closure of D is the closed unit ball in l^3. (Hint: To prove that the closed unit ball is contained in the weak closure of D, you can for any element x such that $||x|| < 1$ explicitly construct a sequence $(x^{(n)})_{n=1}^{\infty} \subset D$ that weakly converges to x, such that $||x^{(n)}|| = 1$.)

Problem 6: Let X be a normed linear space, let M be a closed subspace, and let x_0 be an element not contained in M. Set

$$d = \text{dist}(M, x_0) = \inf_{y \in M} ||y - x_0||.$$

Prove that there exists an element $\varphi \in X^*$ such that $\varphi(x_0) = 1$, $\varphi(y) = 0$ for $y \in M$, and $||\varphi|| = 1/d$.