Homework set 10 — APPM5440

From the textbook: 4.5a, 4.6, 5.1, 5.3.

Note: Problems 3, 4, and 5 are slightly outside the "core" curriculum. Make sure you understand the previous problems before spending time on them.

Problem 1: Set $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, and let $A \in \mathcal{B}(X, Y)$. Let

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

denote the representation of A in the standard basis. Equip X and Y with the supremum norms. Compute $||A||$.

Problem 2: Set $X = \mathbb{R}^2$ and $Y = \mathbb{R}$, and define $f : X \to Y$ by setting $f([x_1, x_2]) = x_1$. Prove that f is continuous. Prove that f is open. Prove that f does not necessarily map close sets to close sets.

Problem 3: Prove that the co-finite topology is first countable if and only if X is countable.

Problem 4: Prove that the co-finite topology on $\mathbb{R}$ weaker than the standard topology.

Problem 5: Consider the set $X = \mathbb{R}$. Let $\mathcal{S}$ denote the collection of sets of the form $(-\infty, a]$ or (a, ∞) for $a \in \mathbb{R}$.

(a) Let $\mathcal{B}$ denote the collection of sets obtained by taking finite intersections of sets in $\mathcal{S}$. Prove that if $G \in \mathcal{B}$, then either G is empty, or $G = (a, b]$ for some a and b such that $-\infty < a < b < \infty\}$.

(b) Let $\mathcal{T}$ denote the topology generated by the base $\mathcal{B}$. Prove that all sets in $\mathcal{B}$ are both open and closed in $\mathcal{T}$.

(c) Prove that $\mathcal{T}$ is first countable but not second countable. Hint: For any $x \in X$, any neighborhood base at x contains at least one set whose supremum is x.

(d) Prove that $\mathbb{Q}$ is dense in $\mathcal{T}$. (This proves that $(X, \mathcal{T})$ is separable but not second countable.)

(e) Prove that $(X, \mathcal{T})$ is not metrizable.