**Homework set 9 — APPM5440**

From the textbook: 4.5, 4.6.

**Problem:** Let $X$ be a topological space, let $K$ be a compact subset of $X$, and let $f : X \to \mathbb{R}$ be a continuous map. Prove that $\sup_{x \in K} f(x)$ is finite. Prove that $f$ attains its maximum value on $K$ (*i.e.* there exists an $x_0 \in K$ such that $f(x_0) = \sup_{x \in K} f(x)$).

**Problem:** Set $X = \mathbb{R}^2$ and $Y = \mathbb{R}$, and define $f : X \to Y$ by setting $f([x_1, x_2]) = x_1$. Prove that $f$ is continuous. Prove that $f$ is open. Prove that $f$ does not necessarily map close sets to close sets.

**Problem:** Prove that the co-finite topology (see problem set 8), is first countable if and only if $X$ is countable.

**Problem:** Prove that the co-finite topology on $\mathbb{R}$ weaker than the standard topology.

**Problem:** Consider the set $X = \mathbb{R}$. Let $\mathcal{S}$ denote the collection of sets of the form $(-\infty, a]$ or $(a, \infty)$ for $a \in \mathbb{R}$.

(a) Let $\mathcal{B}$ denote the collection of sets obtained by taking finite intersections of sets in $\mathcal{S}$. Prove that if $G \in \mathcal{B}$, then either $G$ is empty, or $G = (a, b]$ for some $a$ and $b$ such that $-\infty < a < b < \infty\}$.

(b) Let $\mathcal{T}$ denote the topology generated by the base $\mathcal{B}$. Prove that all sets in $\mathcal{B}$ are both open and closed in $\mathcal{T}$.

(c) Prove that $\mathcal{T}$ is first countable but not second countable. Hint: For any $x \in X$, any neighborhood base at $x$ contains at least one set whose supremum is $x$.

(d) Prove that $\mathbb{Q}$ is dense in $\mathcal{T}$. (This proves that $(X, \mathcal{T})$ is separable but not second countable.)

(e) Prove that $(X, \mathcal{T})$ is not metrizable.