

## Homework set 8 — APPM5440

From the textbook: 3.7, 4.1, 4.2, 4.3.

**Problem:** Let  $X$  be an infinite set. Define the set  $\mathcal{T}$  of subsets of  $X$  as follows:  $G \in \mathcal{T} \Leftrightarrow G = \emptyset$  or  $G^c$  is finite. Prove that  $\mathcal{T}$  is a topology. This topology is called the *co-finite* topology.

**Problem:** The Hausdorff property is only one of many so called “separability” conditions on topological spaces. As an example, we say that a topological space  $X$  is  $T_j$ , for  $j = 0, 1, 2, 3, 4$  if:

$T_0$ : Given  $x, y \in X$ , there either exists an open set containing  $x$  but not  $y$ , or vice versa.

$T_1$ : Given  $x, y \in X$ , there exists an open set that contains  $x$  but not  $y$ .

$T_2$ : Given  $x, y \in X$ , there exist disjoint open sets  $G, H$  such that  $x \in G, y \in H$ . (Note that  $T_2$  is the same as Hausdorff.)

$T_3$ :  $X$  is  $T_1$ , and: Given any closed set  $A$ , and any point  $x \in A^c$ , there exist disjoint open sets  $G, H$  such that  $x \in G, A \subseteq H$ .

$T_4$ :  $X$  is  $T_1$ , and: Given any two closed disjoint sets  $A$  and  $B$ , there exists disjoint open set  $G$ , and  $H$  such that  $A \subseteq G, B \subseteq H$ .

Prove that if  $i < j$ , then any  $T_j$  space is  $T_i$ . Prove that the co-finite topology is  $T_1$  but not  $T_2$ . Prove that a topological space is  $T_1$  if and only if the set  $\{x\}$  is closed for every  $x \in X$ .