Homework set 12 — APPM5440

Problem 1: Let X denote the linear space of polynomials of degree 2 or less on $I = [0,\,1]$. For $f \in X$, set $||f|| = \sup_{x\in I} |f(x)|$. For $f \in X$, define

$$\varphi_1(f) = \int_0^1 f(x)\,dx, \quad \varphi_2(f) = f(0), \quad \varphi_3(f) = f'(1/2), \quad \varphi_4(f) = f'(1/3).$$

Prove that $\varphi_j \in X^*$ for $j = 1,2,3,4$. Prove that $\{\varphi_1,\,\varphi_2,\,\varphi_3\}$ forms a basis for X^*. Prove that $\{\varphi_1, \varphi_2, \varphi_4\}$ does not form a basis for X^*.

Problem 2: Consider the space $X = \mathbb{R}^2$ equipped with the Euclidean norm. Explicitly construct the set X^*. Set

$$\mathcal{S} = \{B_\varepsilon^\varphi(x_0) :\ \varphi \in X^*,\ \varepsilon > 0,\ x_0 \in X\},$$

where

$$B_\varepsilon^\varphi(x_0) = \varphi^{-1}(B_\varepsilon(\varphi(x_0))).$$

Give a geometric description of each set $B_\varepsilon^\varphi(x_0)$. Prove that the topology generated by the sub-basis $\mathcal{S}$ in X^* equals the norm topology on $\mathbb{R}^2$. This proves that on X, the weak topology equals the norm topology.

Problem 3: Let $X = l^3$. What is X^*? Set $D = \{x \in l^3 :\ ||x|| = 1\}$. Prove that the weak closure of D is the closed unit ball in l^3. (Hint: To prove that the closed unit ball is contained in the weak closure of D, you can for any element x such that $||x|| < 1$ explicitly construct a sequence $(x^{(n)})_{n=1}^\infty \subset D$ that weakly converges to x, such that $||x^{(n)}|| = 1$.)

Problem 4: Let X be a normed linear space, let M be a closed subspace, and let x_0 be an element not contained in M. Set

$$d = \text{dist}(M, x_0) = \inf_{y\in M} ||y - x_0||.$$

Prove that there exists an element $\varphi \in X^*$ such that $\varphi(x_0) = 1$, $\varphi(y) = 0$ for $y \in M$, and $||\varphi|| = 1/d$.