Applied Analysis (APPM 5440): Final Exam – Solutions
1.30pm – 5.00pm, Dec 11, 2005. Closed books.

In proofs, please state clearly what you assume, and what you will prove.

Problem 1: No motivation is required for the following questions: (2p each)

(a) Define what it means for a subset of a metric space to be *totally bounded*.

(b) Set $I = [0, 1)$. Specify which (if any) of the following inclusions are equalities: $C_{\rm c}(I) \subseteq C_0(I) \subseteq C_{\rm b}(I) \subseteq C(I)$.

(c) Let X be a Hilbert space, and define for $y \in X$ the functional φ_y by setting $\varphi_y(x) = (y, x)$. What do you know about the map $T : X \rightarrow X^* : y \mapsto \varphi_y$?

(d) Let $\mathcal{P}$ denote the set of all functions that can be written in the form $f(x) = \sum_{n=0}^{N} \big(a_n \cos(nx) + b_n \sin(nx)\big)$, for some finite integer N, and some complex numbers a_n and b_n. Is $\mathcal{P}$ dense in $C(\mathbb{T})$?

(e) Let $\mathcal{P}$ be as in (d). Is $\mathcal{P}$ dense in $L^2(\mathbb{T})$?

(f) Suppose that $f \in H^k(\mathbb{T})$. Specify for which k, if any, it is necessarily the case that f is continuous.

(g) Consider the metric space X consisting of all rational numbers, equipped with the metric $d(x, y) = |x - y|$. Which of the following sets are open: $A = \{q \in X : 0 < q^2 \leq 4\}$, $B = \{q \in X : 0 < q^2 \leq 2\}$, $C = \{q \in X : 0 < q < \infty\}$.

(h) Let X be a normed linear space, and let X^* define the (topological) dual of X. Define what it means for a sequence $(y_n)_{n=1}^{\infty} \subseteq X^*$ to converge in the weak-$*$ topology.

(a) For any $\varepsilon > 0$, there exists a finite set of point $(x_n)_{n=1}^{N}$ such that $\{B_\varepsilon(x_n)\}_{n=1}^{N}$ is a cover of the space.

(b) None.

(c) It is an isometric isomorphism. (*alt.:* It is a unitary map.)

(d) Yes.

(e) Yes.

(f) $k > 1/2$.

(g) B and C.

(h) There exists a $y \in X^*$ such that $|y_n(x) - y(x)| \rightarrow 0$ for any $x \in X$.

Problem 2: Let X be a finite-dimensional linear space, and let $||\cdot||_1$ and $||\cdot||_2$ be two norms on X.

(a) Prove that there exist numbers c and C such that $0 < c \leq C < \infty$, and

$$c||x||_2 \leq ||x||_1 \leq C||x||_2, \qquad \forall x \in X. \tag{1}$$

(3p)

(b) Let G be a subset of X. Define what it means for G to be open in the topology generated by the norm $||\cdot||_1$. (2p)

(c) Prove that if G is open in the topology generated by the norm $||\cdot||_1$, then G open in the topology generated by the norm $||\cdot||_2$. (You may use the inequality (1) regardless of whether you answered part (a).) (2p)

(a) Let $(e_j)_{j=1}^d$ be a basis for X.

Define a norm on X by setting $||x|| = ||\sum_{j=1}^d x_j e_j|| = \sum_{j=1}^d |x_j|$, and let $\mathcal{T}$ be the topology generated by $||\cdot||$.

Set $B = \{x \in X : ||x|| = 1\}$. This set is clearly compact in $\mathcal{T}$. (It is homeomorphic to the set $C = \{x \in \mathbb{R}^d : \sum_{j=1}^d |x_j| = 1\}$ which is bounded and closed in $\mathbb{R}^d$, and hence compact.)

For $p = 1, 2$, the map $x \mapsto ||x||_p$ is continuous in $\mathcal{T}$ (since $||x-y||_p \leq \sum_{j=1}^d |x_j - y_j| \, ||e_j||_p \leq ||x-y|| \big(\max_{1\leq j\leq d} ||e_j||_p\big)$. Thus, since B is compact, there exist constants $M_p < \infty$ such that $\sup_{x\in B} ||x||_p \leq M_p$. Similarly, there exist $m_p > 0$ such that $\inf_{x\in B} ||x||_p \geq m_p$ (m_p cannot equal zero because if it did, then $||\hat{x}||_p = 0$ for the non-zero minimizer $\hat{x}$). Now, for any $x \neq 0$,

$$\frac{||x||_1}{||x||_2} = \frac{||x/||x||\,||_1}{||x/||x||\,||_2} \leq M_1/m_2.$$

Thus, $C = M_1/m_2 < \infty$ works for the upper bound. Analogously, $c = m_1/M_2 > 0$ works for the lower bound.

(A briefer proof may still earn full points.)

(b) G is open in the topology generated by $||\cdot||_1$. $\Leftrightarrow$ For any $x \in G$, there exists an $\varepsilon > 0$ such that if $||x-y||_1 < \varepsilon_1$, then $y \in G$.

(c) Assume G is open in the 1-topology, and let $x \in G$. By assumption, there exists an $\varepsilon_1 > 0$ such that if $||x-y||_1 < \varepsilon_1$, then $y \in G$. Set $\varepsilon_2 = \varepsilon_1/C$. Then if $||x-y||_2 < \varepsilon_2$, we find that $||x-y||_1 \leq C||x-y||_2 < \varepsilon$, and then $y \in G$.

Problem 3: Set $I = [-1, 1]$, and consider the functions $f, g_1, g_2 \in C(I)$, given by $f(x) = x^2$, $g_1(x) = 1$, and $g_2(x) = x$. Set $A = \text{span}(g_1, g_2)$. Determine $\alpha = \text{dist}(A, f) = \inf_{g \in A} ||g - f||$. Is the minimizer unique? (4p)

Consider $\hat{g} = 1/2\, g_1 \in A$. Then $||f - \hat{g}|| = 1/2$, so $\alpha \leq 1/2$. Set $g = c_1 g_1 + c_2 g_2$, and assume that g is a minimizer (a minimizer must exist since A is finite dimensional). We know that $||f - g|| \leq 1/2$. Then

$$\begin{aligned} 1/2 \geq ||f - g|| \geq& |f(0) - g(0)| = |c_1|, \\ 1/2 \geq ||f - g|| \geq& |f(1) - g(1)| = |1 - c_1 - c_2|, \\ 1/2 \geq ||f - g|| \geq& |f(-1) - g(-1)| = |1 - c_1 + c_2|. \end{aligned}$$

Since $|c_1| \leq 1/2$, it follows that $|1 - c_1| \geq 1/2$. Then the second two inequalities imply that $c_2 = 0$, and thus $c_1 = 1/2$. It follows that $g = \hat{g}$ is the unique minimizer.

(This solution may seem slightly magical - how would you à priori know that $\hat{g}$ is the minimizer? Well, since f is even, it is reasonable to guess that the minimizer should be even, which means that $\hat{g} = c\, g_1$ for some c. Then you can very easily determine that c should be $1/2$.)

Problem 4: Set $I = [0, 1]$, let k be a continuous function on I^2, and consider the integral operator $T: C(I) \to C(I)$, given by

$$[Tf](x) = \int_0^1 k(x, y)\, f(y)\, dy.$$

Prove that T is compact. (4p)

Let B be a bounded subset of $C(I)$. In order to show that T is compact, we need to prove that $T(B)$ is pre-compact. The Arzelà-Ascoli theorem states that this is the case if and only if $T(B)$ is bounded and equicontinuous.

Set $M = \sup_{f \in B} ||f||$. Then $M < \infty$.

<u>Boundedness:</u> Since k is a continuous function on a compact set, k is bounded from above by some finite number C. If $f \in B$, then

$$||Tf|| = \sup_{x \in I} \left| \int_I k(x, y)\, f(y)\, dy \right| \leq \sup_{x \in I} \int_I |k(x, y)|\, |f(y)|\, dy \leq C\, M.$$

<u>Equicontinuity:</u> Fix $\varepsilon > 0$. We need to show that there exists a δ such that $|x - x'| < \delta \Rightarrow |g(x) - g(y)| < \varepsilon$, for any $g \in T(B)$. Since k is a continuous function on a compact set, it is uniformly continuous. Thus, there exists a δ such that $|(x, y) - (x', y')| < \delta \Rightarrow |k(x, y) - k(x', y')| < \varepsilon / M$. Then, if $g = Tf \in T(B)$, and $|x - x'| < \delta$, we find that

$$|g(x) - g(x')| = \left| \int_I \big(k(x, y) - k(x', y)\big)\, f(y)\, dy \right| < \int_I \frac{\varepsilon}{M}\, M\, dy = \varepsilon.$$

Problem 5: Let $X = l^1(\mathbb{N})$, and let $(\alpha_n)_{n=1}^{\infty}$ be numbers such that $|\alpha_n| \leq 2^{-n}$. Define the linear operator $T : X \to X$ by setting, for $x = (x_1, x_2, \dots)$, $(Tx)_j = \alpha_j x_1 + x_j$.

(a) Determine $\sup\{\frac{||Tx||}{||x||} : x \neq 0\}$. (3p)

(b) What is the range of T? (1p)

(c) Determine $\sup\{\frac{||x||}{||Tx||} : x \neq 0\}$. (2p)

(a) Note that

$$\sup\left\{\frac{||Tx||}{||x||} : x \neq 0\right\} = ||T|| = \sup\{||Tx|| : ||x|| = 1\}.$$

Suppose that $||x|| = 1$. Then

$$\begin{aligned}||Tx|| =& |(1+\alpha_1)x_1| + \sum_{n=2}^{\infty} |\alpha_n x_1 + x_n| \\ \leq& \Big(|1+\alpha_1| + \sum_{n=2}^{\infty} |\alpha_n|\Big)|x_1| + \sum_{n=2}^{\infty} |x_n| \\ \leq& \max\Big\{\Big(|1+\alpha_1| + \sum_{n=2}^{\infty} |\alpha_n|\Big),\, 1\Big\}.\end{aligned}$$

Set $\beta = |1+\alpha_1| + \sum_{n=2}^{\infty} |\alpha_n|$. Then $||T|| \leq \max(\beta, 1)$. To prove equality, set $x = e_1 = (1, 0, 0, \dots)$ if $\beta > 1$, and set $x = e_2 = (0, 1, 0, \dots)$ otherwise.

(b) T is in fact invertible: If $y \in X$, and if we set

$$x_n = \begin{cases} (1/(1+\alpha_1))\, y_1, & \text{for } n = 1, \\ (-\alpha_n/(1+\alpha_1))\, y_1 + y_n, & \text{for } n \geq 2, \end{cases}$$

then $x \in X$, and $Tx = y$. Thus, the range of T equals X.

(c) Via a computation analogous to the one in (a), we find that

$$\begin{aligned}\sup\left\{\frac{||x||}{||Tx||} : x \neq 0\right\} =& \sup\left\{\frac{||T^{-1}x||}{||x||} : x \neq 0\right\} = ||T^{-1}|| \\ =& \max\left\{\frac{1}{|1+\alpha_1|}\left(1 + \sum_{n=2}^{\infty} |\alpha_n|\right),\, 1\right\}.\end{aligned}$$

Problem 6: Let f be a bounded continuous function on $\mathbb{R}^2$ for which there exists a finite number C such that

$$|f(t,a) - f(t,b)| \leq C|a-b|, \qquad \forall\ t,a,b \in \mathbb{R}.$$

Consider the ODE

$$\text{(ODE)} \qquad \begin{cases} \dot{u}(t) = f(t,u(t)), \\ u(0) = 1. \end{cases}$$

State the contraction mapping theorem, and use it to prove that for some $\varepsilon > 0$, the equation (ODE) has a unique solution in $C^1([-\varepsilon,\varepsilon])$. (You do not need to give an optimal ε.) (5p)

Contraction mapping theorem: Let X be a complete metric space, and let $T : X \to X$ be a map for which there exists a $c < 1$ such that $d(Tx,Ty) \leq c\,d(x,y)$ for all $x,y \in X$. Then there exists a unique $x \in X$ such that $Tx = x$.

Set $X = C([0,\varepsilon])$, where $\varepsilon > 0$ will be specified later.

Rewrite (ODE) as an integral equation:

$$\text{(IE)} \qquad u(t) = 1 + \int_0^t f(s,u(s))\,ds = [Tu](t).$$

(The equation above defines T.) Clearly, T maps X to X. We will prove that T is a contraction if ε is small enough. For $u,v \in X$, we have

$$\begin{aligned} ||Tu - Tv|| &= \sup_{t\in[0,\varepsilon]} \Big| \int_0^t \big(f(s,u(s)) - f(s,v(s))\big)\,ds \Big| \\ &\leq \int_0^\varepsilon |f(s,u(s)) - f(s,v(s))|\,ds \\ &\leq \int_0^\varepsilon C|u(s) - v(s)|\,ds \leq C\varepsilon||u-v||. \end{aligned}$$

Pick an ε such that $0 < \varepsilon < 1/C$. Then T is a contraction on X, and (IE) has a unique solution u in X. From (IE), it follows directly that $u \in C^1([0,\varepsilon])$. Moreover, differentiating (IE), we see that u solves (ODE).

To prove the existence of a unique C^1 solution on $[-\varepsilon,0]$, simply repeat the proof with t replaced by $-t$.

(Note that since ε only depends on C, it is trivial to prove that there exists a unique solution in $C^1(\mathbb{R})$.)

Problem 7: Let X be a separable infinite-dimensional Hilbert space. Prove that there exists a family of closed linear subspaces $\{\Omega_t : t \in [0,1]\}$ such that Ω_s is a strict subset of Ω_t whenever $s < t$. (4p)

Let $(e_n)_{n=1}^{\infty}$ be an orthonormal basis for X.

Let $(q_n)_{n=1}^{\infty}$ denote an enumeration of the rational numbers in $[0,1]$.

Set $\Omega_s = \{x \in X : (e_n, x) = 0 \text{ if } q_n > s\}$.

Each Ω_s is obviously a closed linear subspace (since Ω_s is the orthogonal complement of the set $\{e_n : q_n > s\}$).

Moreover, if $s < t$, then obviously, $\Omega_s \subseteq \Omega_t$. To prove that the two spaces are not equal, pick a q_n such that $s < q_n < t$. Then $e_n \in \Omega_t$, but e_n does not belong to Ω_s.