Linear stability analysis: Consider an autonomous system of *non-linear* equations

$$\dot{x} = f(x,y),$$
$$\dot{y} = g(x,y).$$

Step 1: Find every equilibrium point $(\hat{x},\hat{y})$ for which $f(\hat{x},\hat{y}) = g(\hat{x},\hat{y}) = 0$.

Step 2: For each equilibrium point $(\hat{x},\hat{y})$, compute the Jacobian

$$J = \begin{bmatrix} f_x(\hat{x},\hat{y}) & f_y(\hat{x},\hat{y}) \\ g_x(\hat{x},\hat{y}) & g_y(\hat{x},\hat{y}) \end{bmatrix}.$$

Step 3: Compute the eigenvalues and eigenvectors of J.

Step 4: Inspect the eigenvalues to determine the nature of the equilibrium point. E.g.:

$\lambda_1 < 0$	and	$\lambda_2 > 0$	$\Rightarrow$	Unstable (saddle point)
$\lambda_{1,2} = \alpha \pm i\beta$	with	$\alpha < 0$	$\Rightarrow$	Asympt.-stable (spiral)

Warning 1: Linear analysis may fail if J is singular.

Warning 2: Linear analysis is inconclusive if $\lambda_{1,2} = \pm i\beta$. The solution can be a center, or a repelling spiral (unstable), or an attracting spiral (asy-stable).

Example: Find equilibria and determine their type for $\begin{cases} \dot{x} = y \\ \dot{y} = x(x-1) \end{cases}$

Determine Jacobian: $J = \begin{bmatrix} 0 & 1 \\ 2x-1 & 0 \end{bmatrix}$.

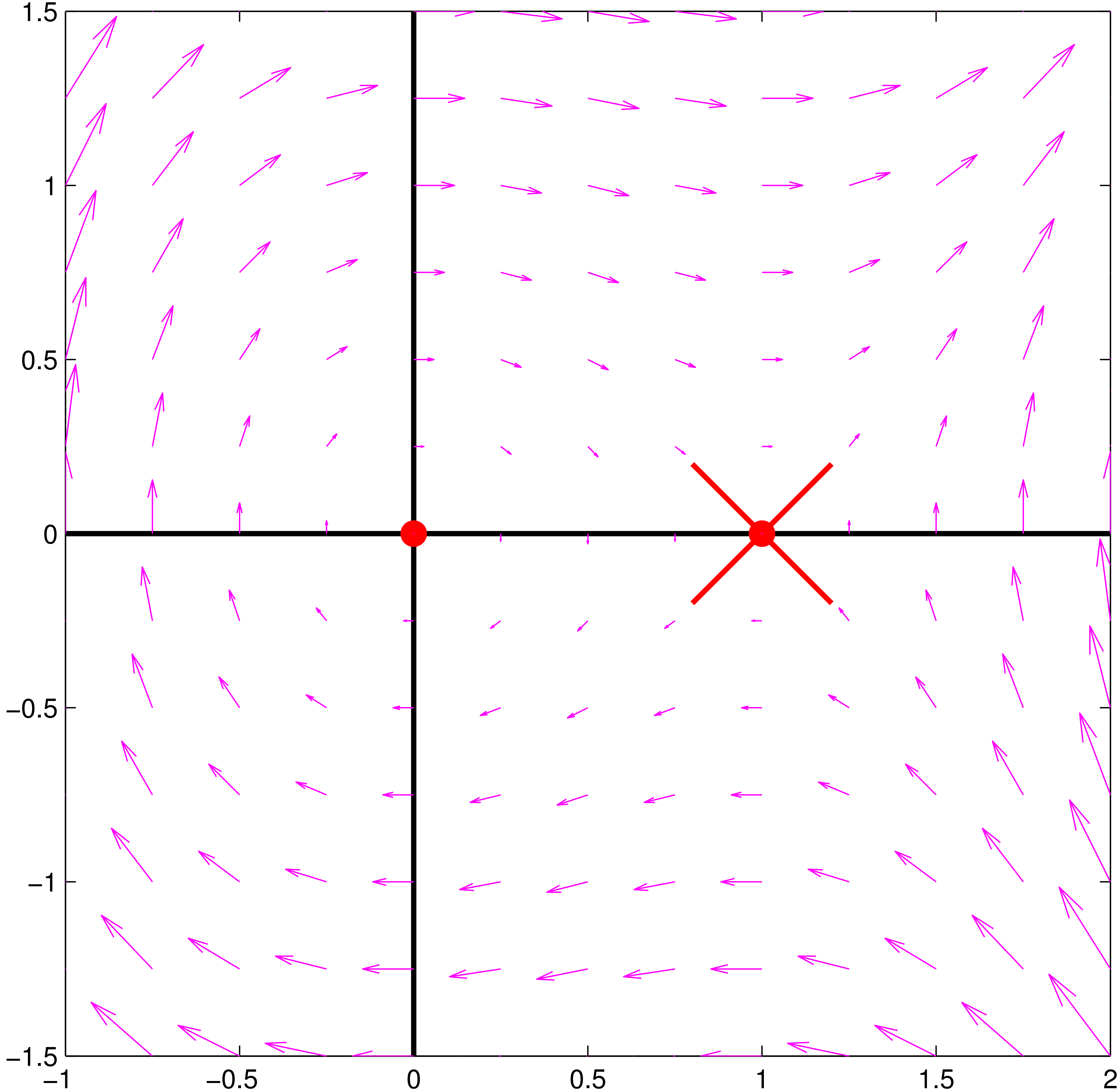
Find equilibria: (A) $(x_0, y_0) = (1, 0)$ and (B) $(x_0, y_0) = (0, 0)$.

Analyze point (A): $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $\lambda_1 = 1$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda_2 = -1$ $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

unstable (saddle point)

Analyze point (B): $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. $\lambda_1 = i$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ $\lambda_2 = -i$ $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

test is inconclusive (spiral of some sort)



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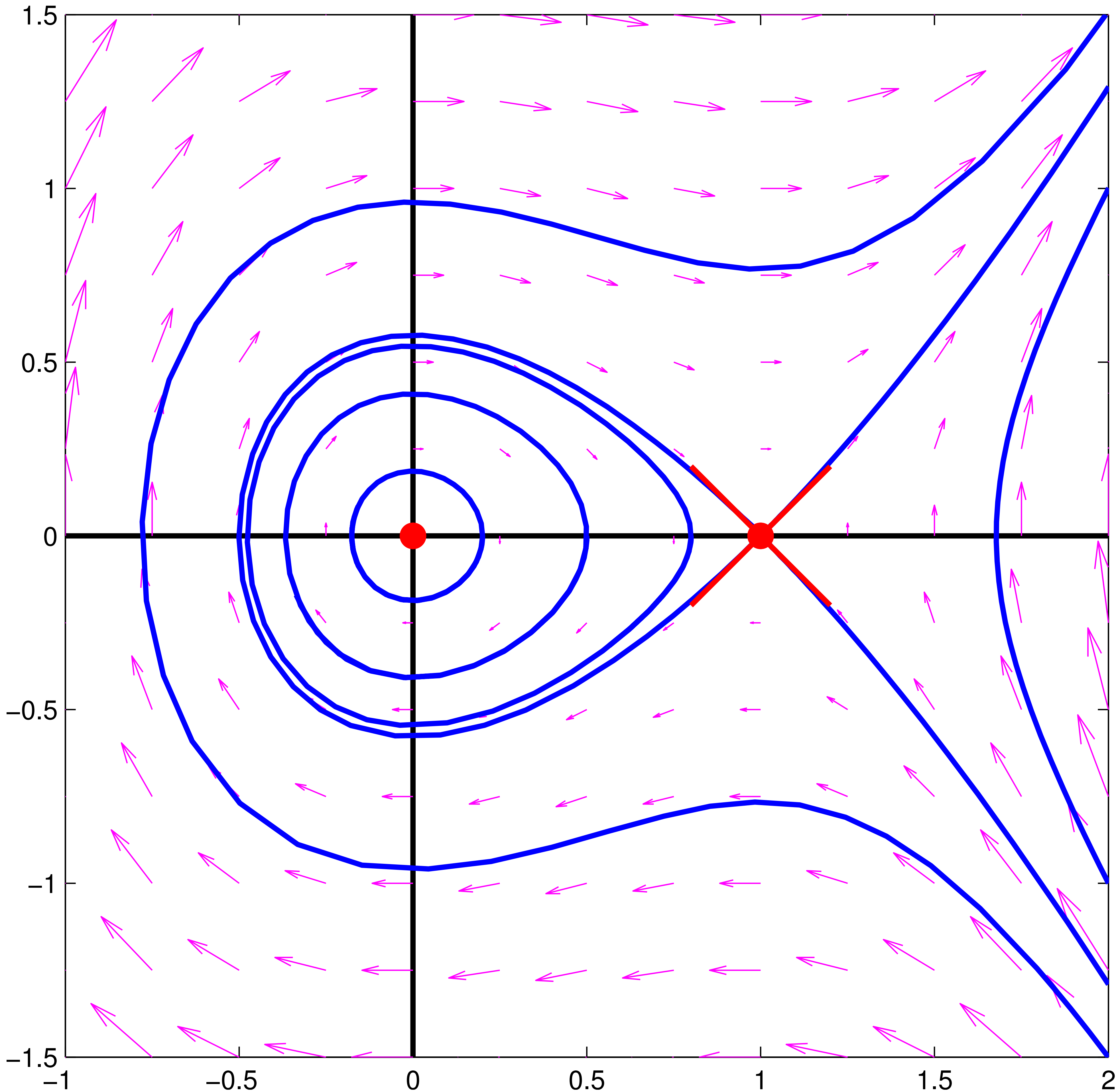
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2

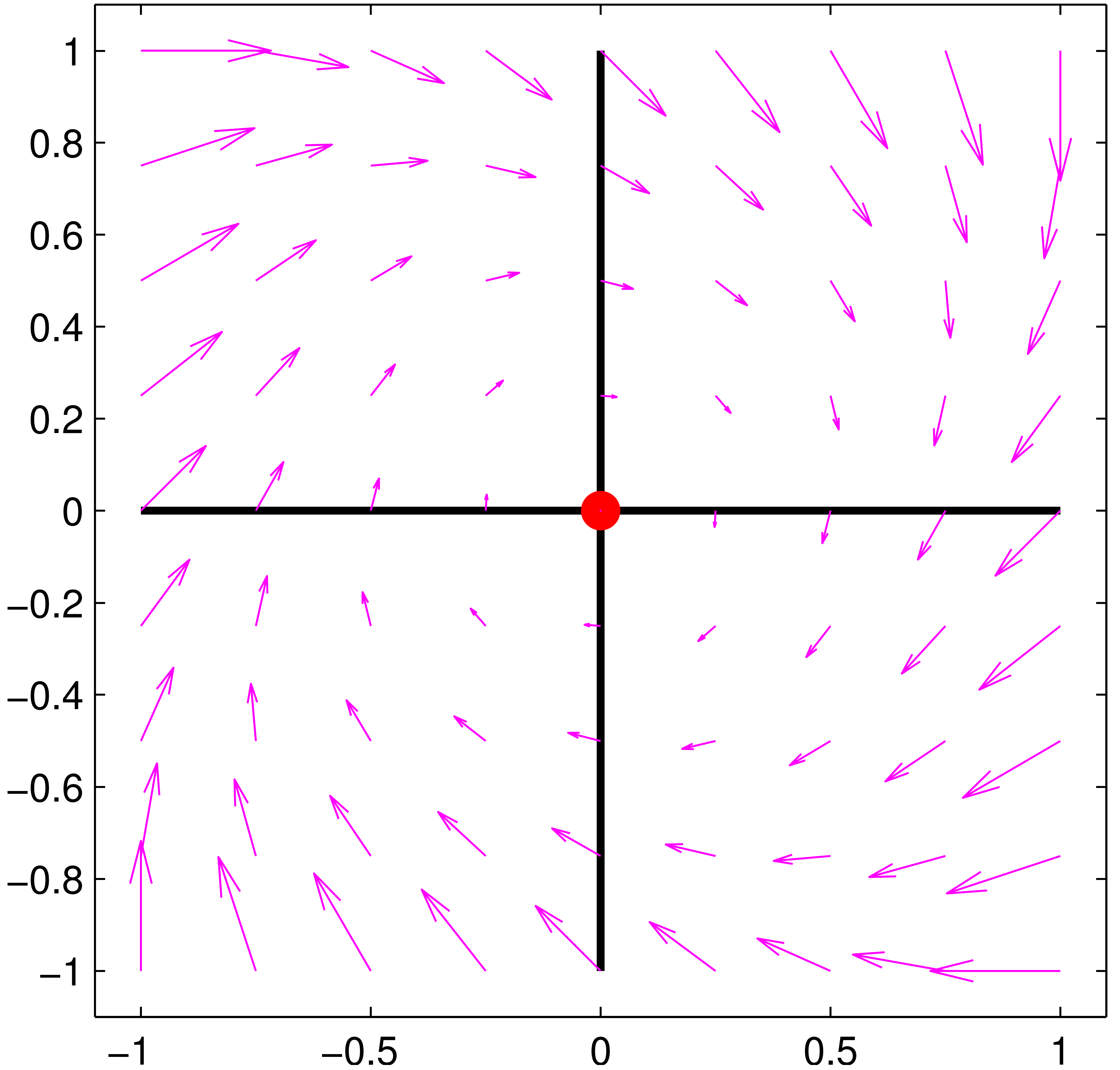
Example: Find equilibria and determine their type for $\begin{cases} \dot{x} = y - x^3 \\ \dot{y} = -x - y^3 \end{cases}$

Determine Jacobian: $J = \begin{bmatrix} -3x^2 & 1 \\ -1 & -3y^2 \end{bmatrix}$.

Find equilibria: (A) $(x_0, y_0) = (0, 0)$.

Analyze point (A): $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. $\lambda_1 = i$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ $\lambda_2 = -i$ $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

test is inconclusive (spiral of some sort)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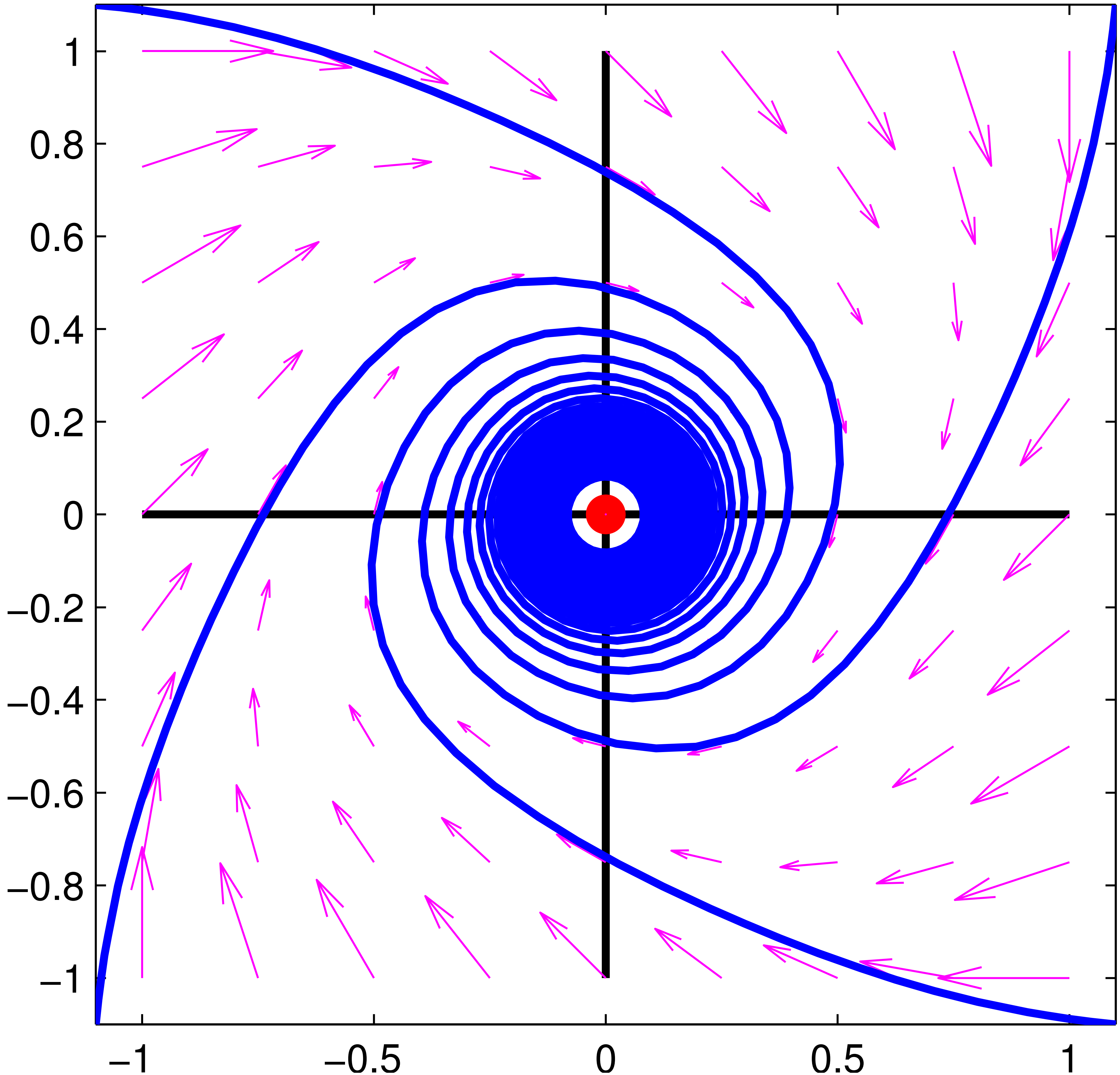
0
0.5
1



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0.4
0.2
0
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−0.4
−0.6
−0.8
−1
−1
−0.5
0
0.5
1

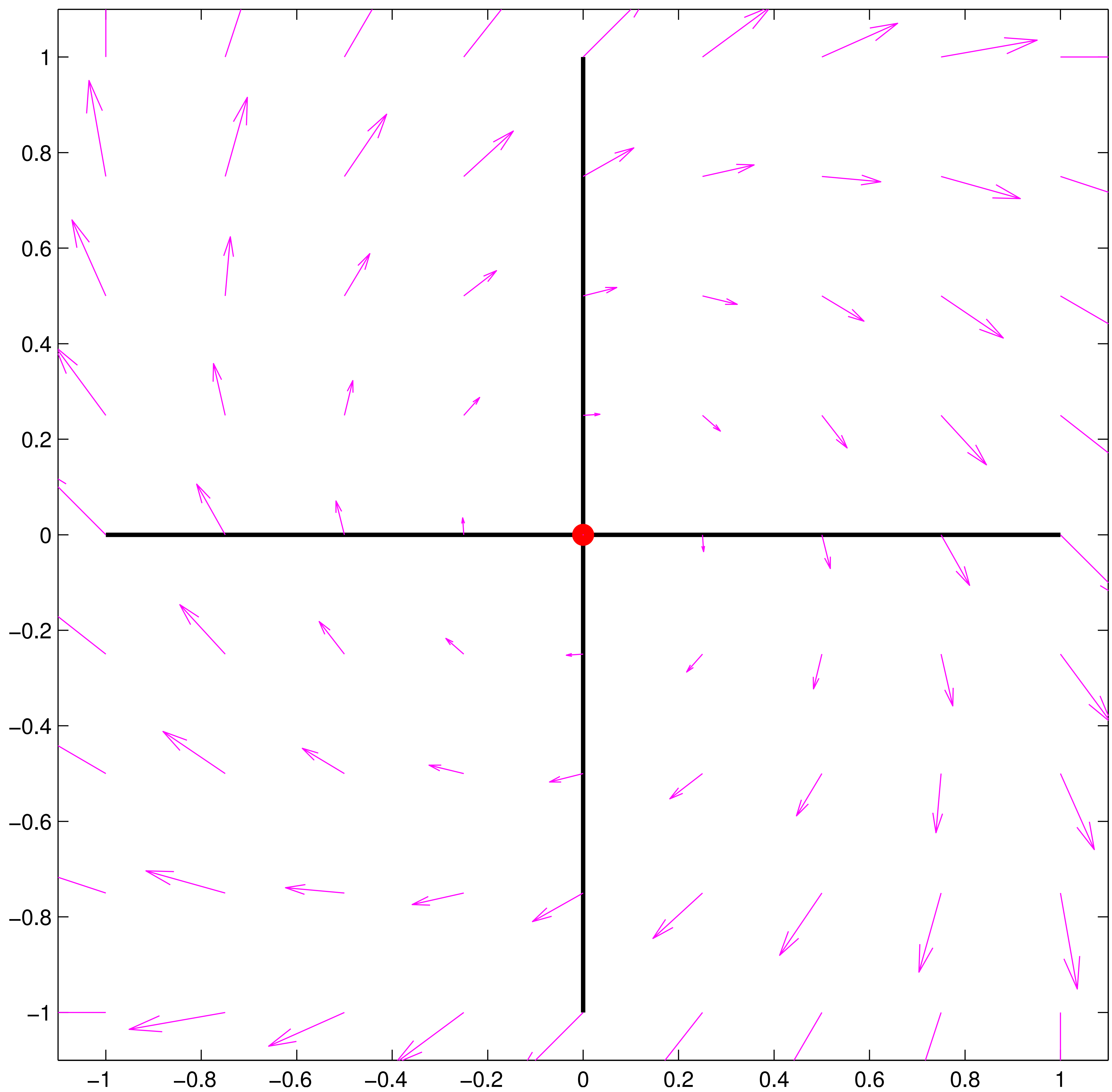
Example: Find equilibria and determine their type for $\begin{cases} \dot{x} = y + x^3 \\ \dot{y} = -x + y^3 \end{cases}$

Determine Jacobian: $J = \begin{bmatrix} 3x^2 & 1 \\ -1 & 3y^2 \end{bmatrix}$.

Find equilibria: (A) $(x_0, y_0) = (0, 0)$.

Analyze point (A): $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. $\lambda_1 = i$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ $\lambda_2 = -i$ $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

test is inconclusive (spiral of some sort)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0.4
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0
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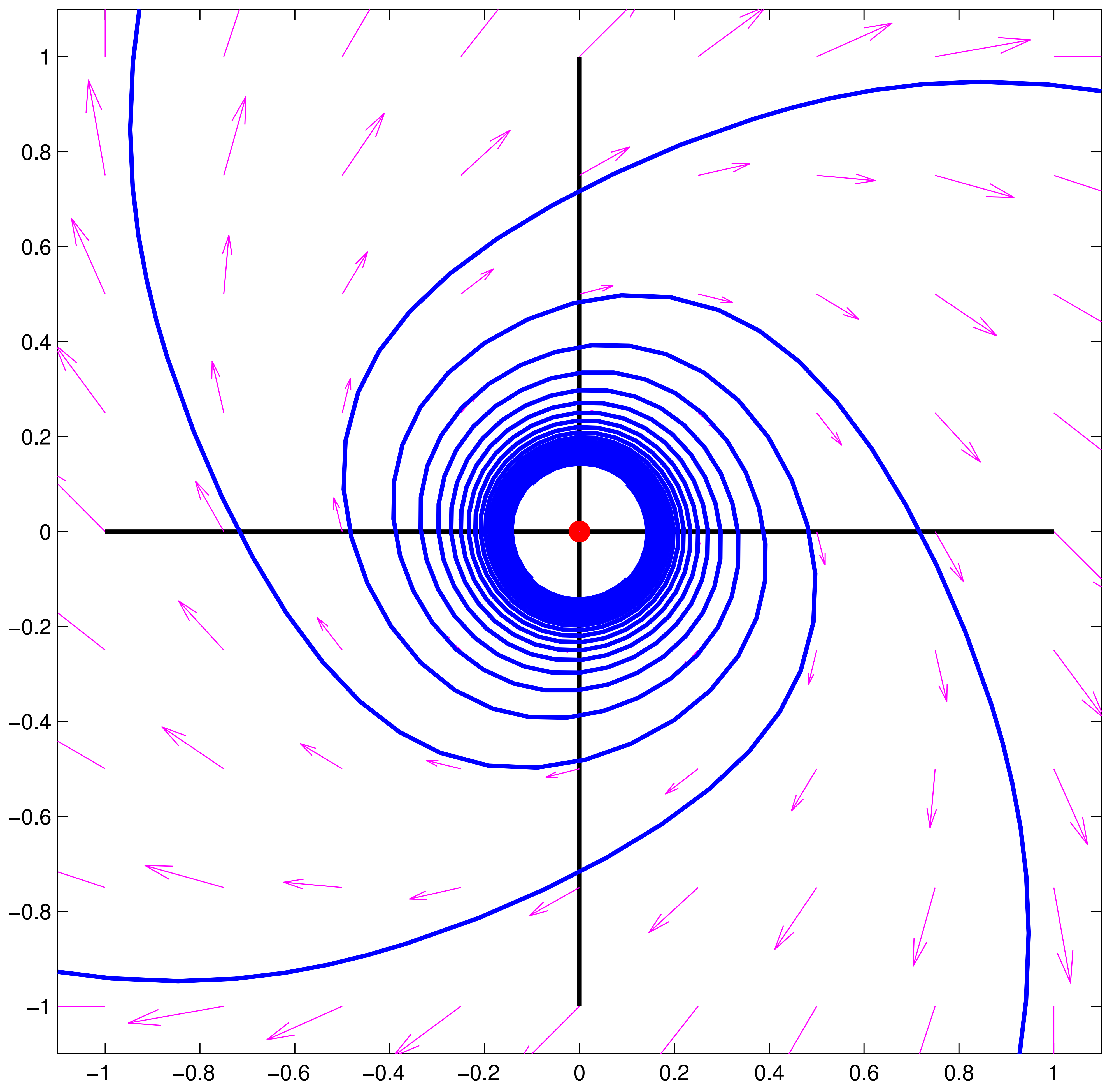
0.8
1



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−1
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−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

Example: Find equilibria and determine their type for $\begin{cases} \dot{x} = y \\ \dot{y} = -\sin(x) \end{cases}$

(Note: This is the system form of the mathematical pendulum $\ddot{x} + \sin(x) = 0$.)

Determine Jacobian: $J = \begin{bmatrix} 0 & 1 \\ -\cos(x) & 0 \end{bmatrix}$.

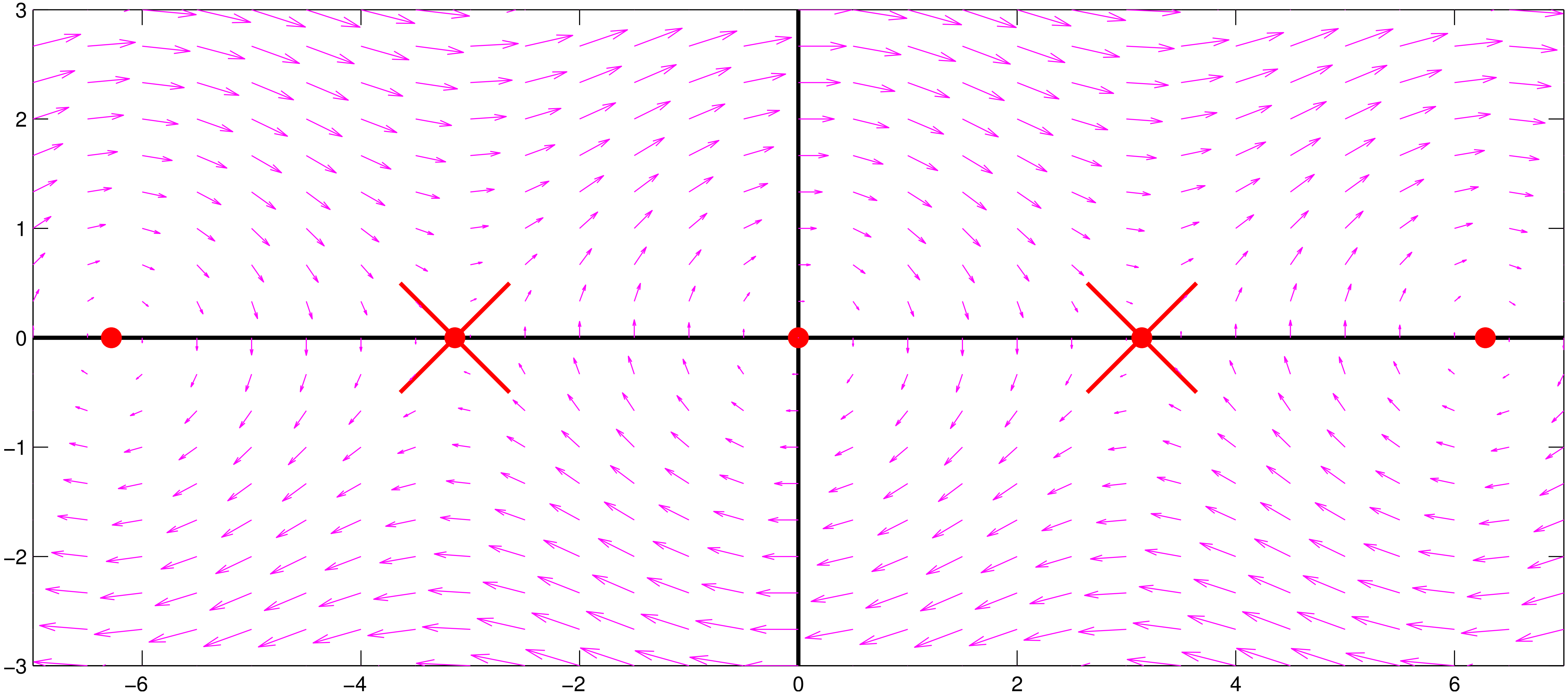
Find equilibria: (A) $(x, y) = (\pi\, 2n, 0)$ and (B) $(x, y) = (\pi\,(2n+1), 0)$.

Analyze point (A): $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. $\lambda_1 = i$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ $\lambda_2 = -i$ $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

test is inconclusive (spiral of some sort)

Analyze point (B): $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $\lambda_1 = 1$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda_2 = -1$ $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

saddle point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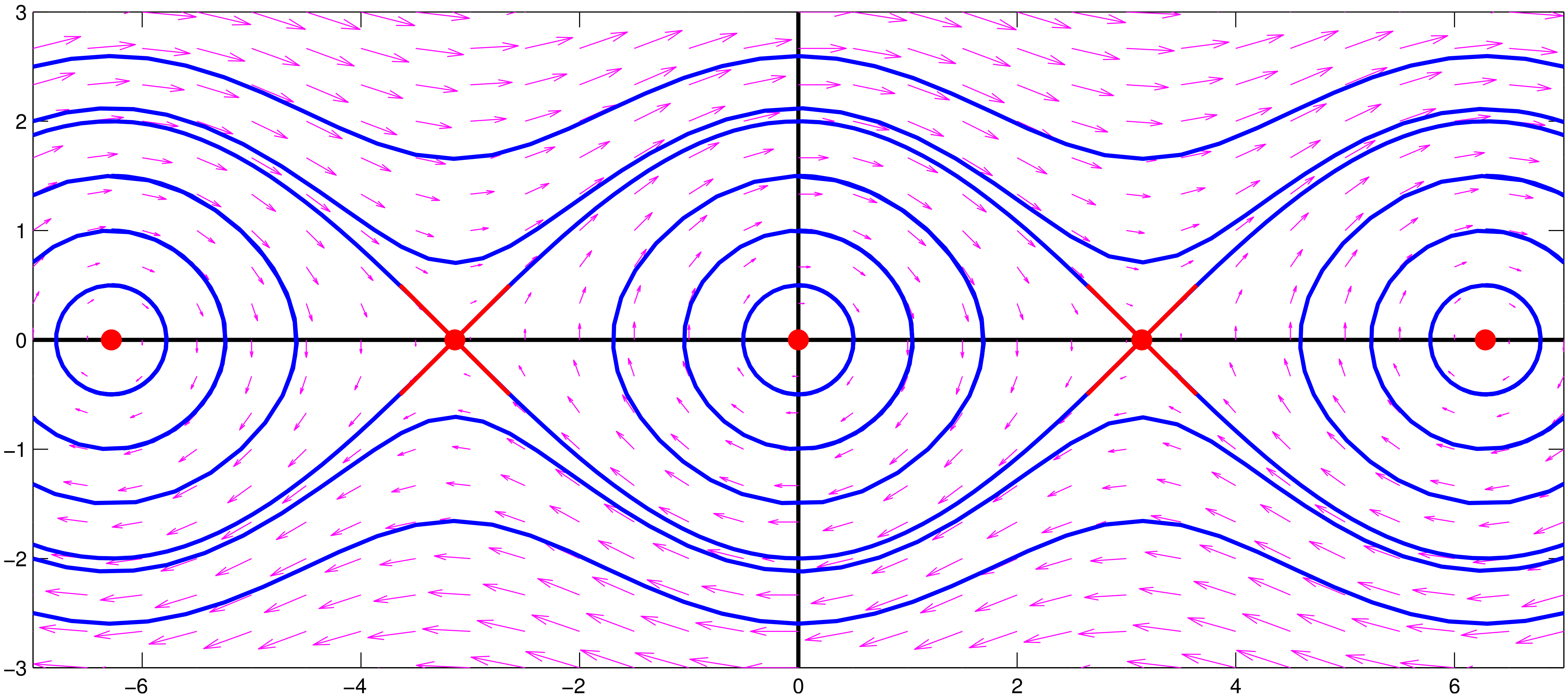
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−6
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0
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6

Example: Find equilibria and determine their type for $\begin{cases} \dot{x} = y \\ \dot{y} = -\sin(x) - y \end{cases}$

Note: This is the system form of the mathematical pendulum $\ddot{x} + \dot{x} + \sin(x) = 0$.

Observe that the term $\dot{x}$ represents *friction.* The system is now losing energy.

Determine Jacobian: $J = \begin{bmatrix} 0 & 1 \\ -\cos(x) & -1 \end{bmatrix}$.

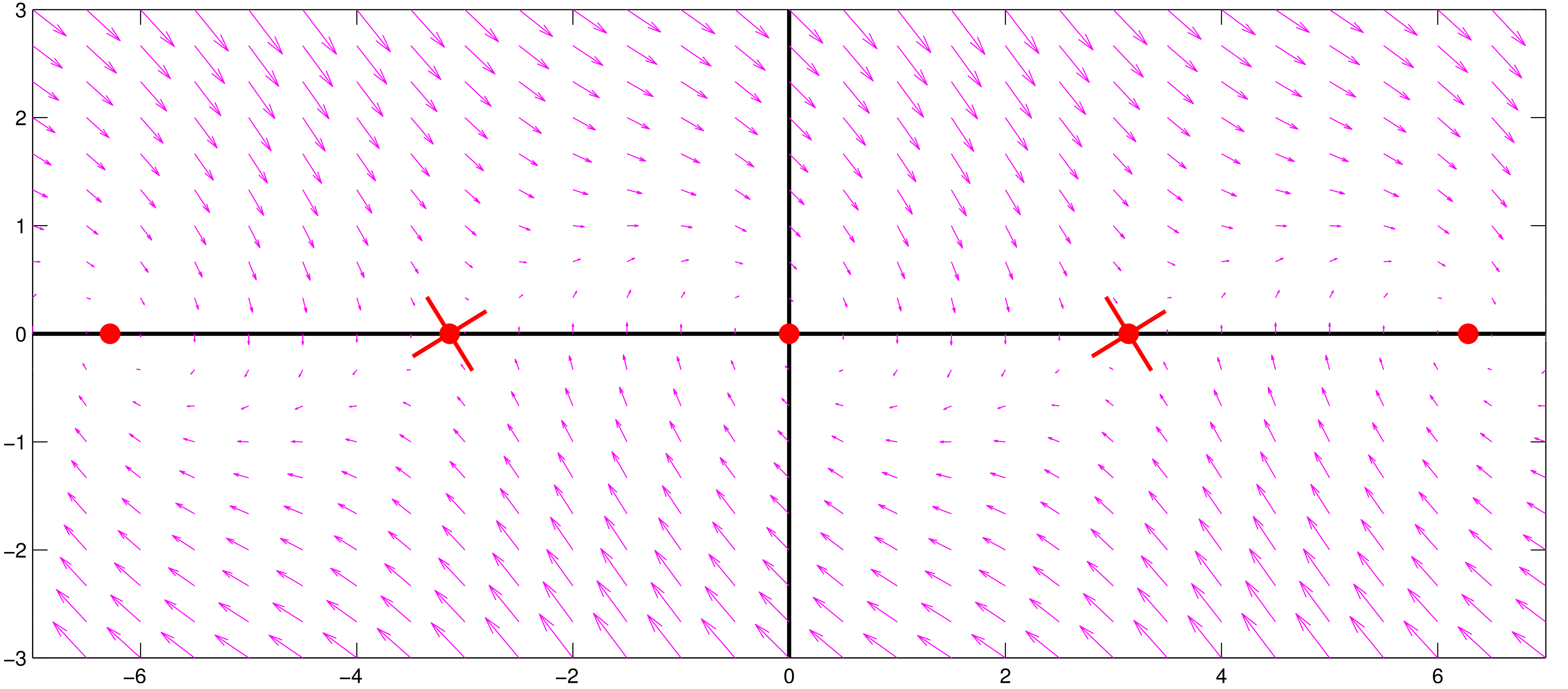
Find equilibria: (A) $(x, y) = (\pi\, 2n, 0)$ and (B) $(x, y) = (\pi\,(2n+1), 0)$.

Analyze point (A): $J = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$. $\lambda_{1,2} = -\frac{1}{2} \pm i\sqrt{3}/2$.

asymptotically stable spiral

Analyze point (B): $J = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$. $\lambda_{1,2} = -\frac{1}{2} \pm \sqrt{5}/2$

saddle point



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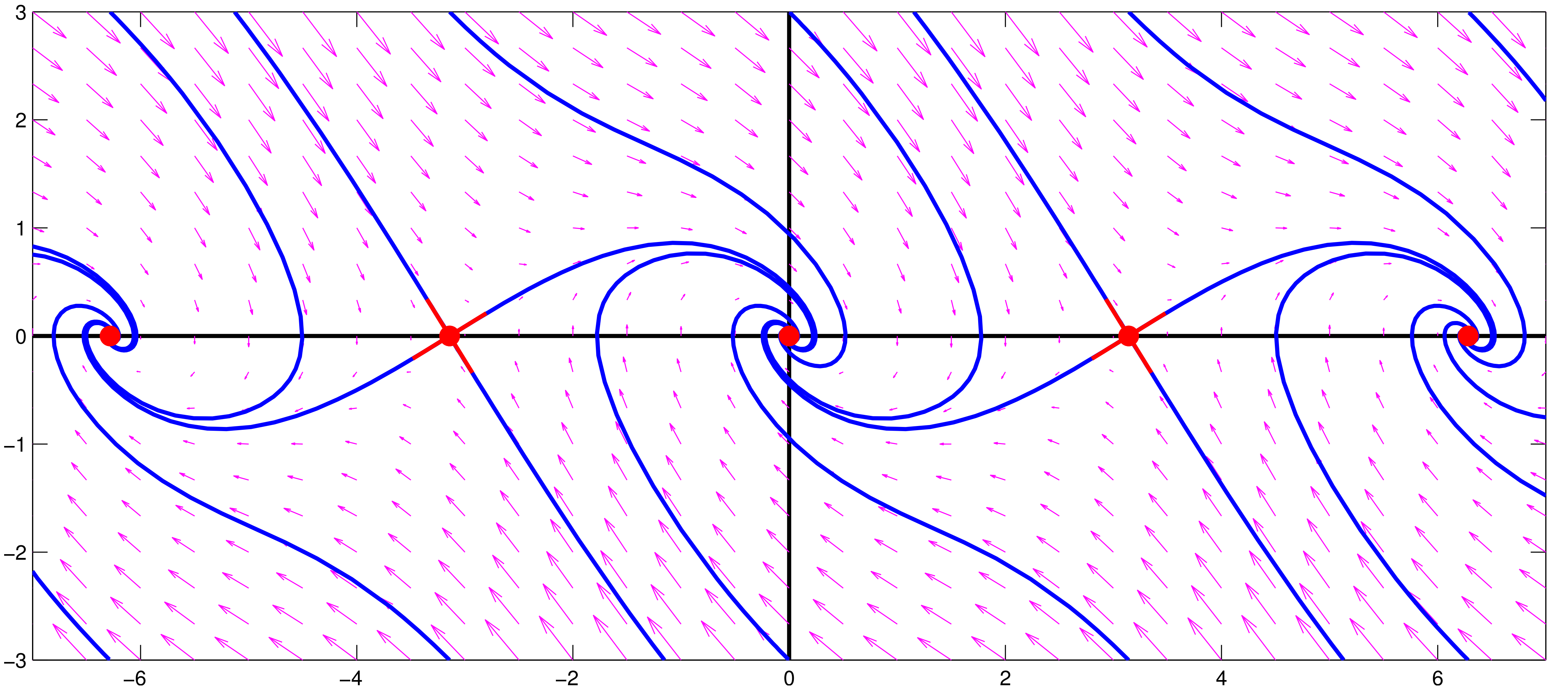

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