2.3 GROWTH & DECAY

A very important eq^n is

$$\begin{cases} y' = ky \\ y(0) = y_0 \end{cases}$$

The sol^n is $y(t) = y_0 e^{kt}$.

$k > 0 \to$ <u>growth</u> e.g. bacteria in a petri dish, interest on ma

$k < 0 \to$ decay e.g. radioactive materials, cooling, ...

<u>Example</u> Money in a bank.

$A(t)$ = amount at time t.

r = interest rate

$$\underbrace{dA}_{\text{interest}} = \underset{\text{rate}}{r}\,\underset{\text{amount}}{A}\,\underset{\text{time}}{dt}$$

$$\frac{dA}{dt} = rA$$

If you have A_0 dollars at time $t=0$, then

$$A(t) = A_0 e^{rt}$$

Now assume we also deposit "s" dollars every year:

$$\begin{cases} \frac{dA}{dt} = rA + s \\ A(0) = A_0 \end{cases}$$

General sol^n is

$$A(t) = \underbrace{Ce^{rt}}_{\text{General hom sol}^n} \underbrace{- \frac{s}{r}}_{\text{particular sol}^n}$$

In order to determine C_1, we use the initial cond^n $A(0) = A_0$:

$$A_0 = C_1 e^{r\cdot 0} - \frac{s}{r} \quad \Rightarrow \quad C_1 = A_0 + \frac{s}{r}$$

The sol^n is therefore $A(t) = (A_0 + \frac{s}{r})e^{rt} - \frac{s}{r} = A_0 e^{rt} + \frac{s}{r}(e^{r\cdots}$

<u>Example</u> Set $A_0 = 0$ no initial deposit!

$s = 100$ we save \$100 per year

$r = 0.05$ 5% interest rate.

How much money is in the account after 100 years?

<u>Sol^n</u> $A(100) = \overset{0}{\cancel{100}}\, e^{0.05\cdot 100} + \frac{100}{0.05}\left(e^{0.05\cdot 100} - 1\right) \approx$

$\approx 295\,000$

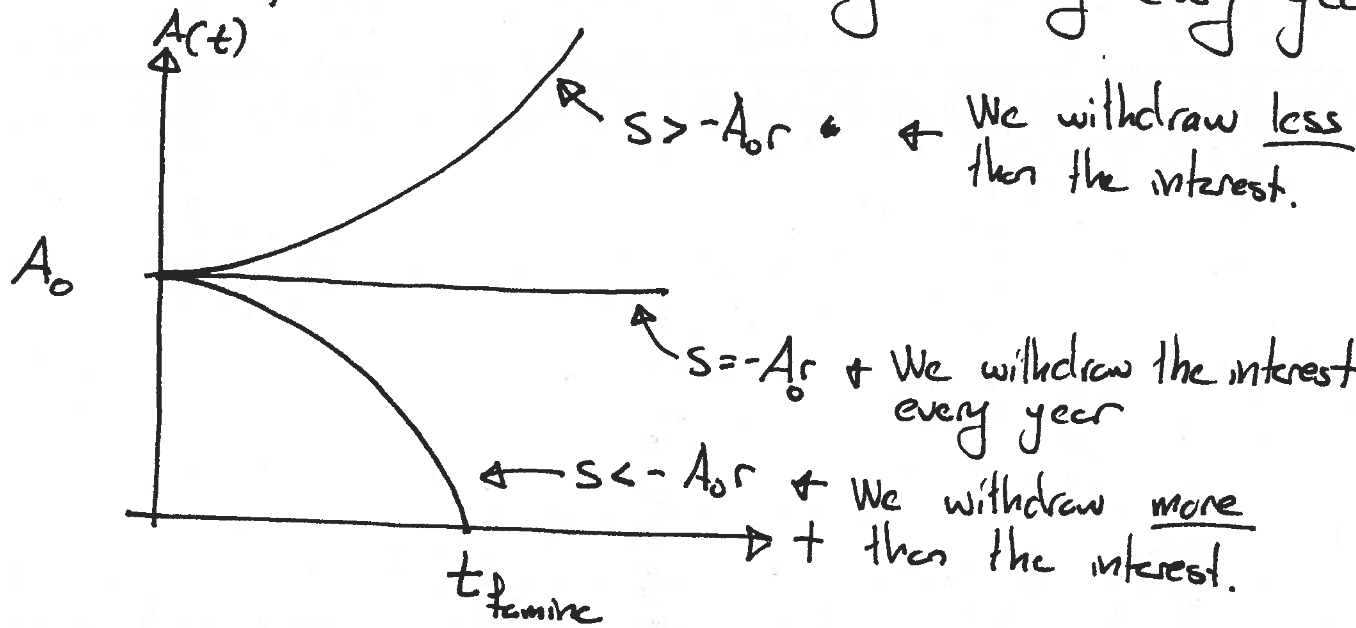
Note the linearity of the sol^n!

$$A(t) = \underbrace{A_0\, e^{rt}}_{\text{Contribution from initial deposit}} + \underbrace{\frac{s}{r}\left(e^{rt} - 1\right)}_{\text{Contribution from annual deposits}}$$

Notice that $\lim_{r\to 0} A(t) = \lim_{r\to 0} \left[A_0 e^{rt} + \frac{s}{r}(e^{rt}-1)\right] = A_0 + st$

This makes sense!
The term "st" is the sum of all deposits.

Note that s can be negative,
this corresponds to withdrawing money every year



Suppose that we withdraw more than the interest.
Then how long does the money last?
We seek the time t_{famine} such that $A(t_{famine}) = 0$

$$0 = \left(A_0 + \frac{s}{r}\right)e^{r t_{famine}} - \frac{s}{r} \quad \Rightarrow \quad e^{r t_{famine}} = \frac{s/r}{A_0 + s/r}$$

$$\Rightarrow t_{famine} = \frac{1}{r}\log\left(\frac{s/r}{A_0 + s/r}\right)$$

Example Consider a mortgage over $T=30$ years.
The interest rate is $r=0.06$ (6%)
The monthly payment is \$800.

(a) How much money can you borrow?
(b) How much do you pay in total interest?
(c) Suppose you do a 15 year mortgage for the same amount. Then what will the monthly payment be? How much is the total interest?

Sol^n (a) Let p denote the annual payment $p = 12 \times 800 = 9600$
Let $D(t)$ denote the debt in year t. Then

$$(*) \quad \begin{cases} \dfrac{dD}{dt} = rD - p \\ D(0) = D_0 \end{cases}$$

Initial debt!

We seek to determine D_0 given that $D(T)=0$!

The sol^n of $(*)$ is $D(t) = \left(D_0 - \frac{p}{r}\right)e^{rt} + \frac{p}{r}$ CHE

We know that $D(30)=0$:

$$0 = \left(D_0 - \frac{p}{r}\right)e^{r\cdot 30} + \frac{p}{r} \Rightarrow$$

$$\Rightarrow D_0 = \frac{p}{r}\left(1-e^{-rT}\right) = \frac{9600}{0.06}\left(1-e^{-0.06\cdot 30}\right) \approx \$134000$$

(b) The total interest payment is

$$\underbrace{pT}_{\text{Total payments}} - \underbrace{D_0}_{\substack{\text{How much money}\\ \text{you got in year 0}}} = 9600 \cdot 30 - 134000 \approx \$288k - \$134k = \$154k$$

(c) If $D_0 = \cancel{\ }\ 134000$ and $\cancel{\ }$ $T=15$, then p must satisfy

$$0 = \left(D_0 - \frac{p}{r}\right)e^{rT} + \frac{p}{r} \Rightarrow p = \frac{rD_0}{1-e^{-rT}} =$$

$$= \frac{0.06 \cdot 134000}{1 - \exp(-0.06 \cdot 15)} \approx 13548$$

Note that p is the annual payment, so the monthly payment is $\frac{p}{12} \approx 1129$

The total interest is now

$$pT - D_0 \approx 13548 \cdot 15 - 134000 \approx 69\,000$$

Note how much less you pay in total interest!

NEWTON COOLING

The goal here is to model the temperature of an object that is put in an environment of ambient temperature T_{amb}.
Let $T(t)$ denote the temperature of the object, and let T_0 denote its initial temperature.
The key assumption is that the rate of change in T is proportionate to the difference between T and T_{amb}. In a formula:

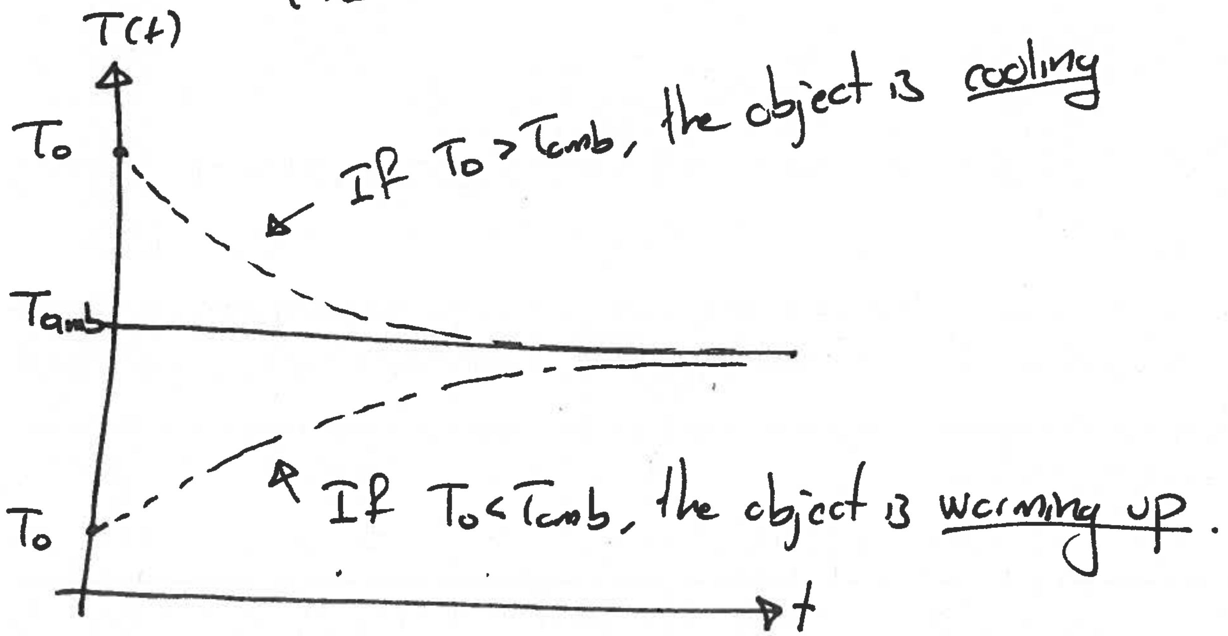
$$\begin{cases} \dfrac{dT}{dt} = k(T_{amb} - T) \\ T(0) = T_0 \end{cases}$$

(This assumption is not always realistic!)

The exact sol^n is

$$T(t) = T_{amb} + (T_0 - T_{amb})e^{-kt}$$

Note that $\lim_{t \to \infty} T(t) = T_{amb}$:



Example Consider a roast in an oven. Suppose the roast starts at 50F and that the oven is 400F. After 75 minutes, we measure the temperature and find that it is 125F.

At what time is the temperature 150F inside?

Sol^n The idea is to use the formula

$$T(t) = (T_0 - T_{amb})e^{-kt} + T_{amb}.$$

We see immediately that $T_0 = 50$ $T_{amb} = 400$.

We need to use the data point $T(75) = 125$ to determine k!

2.4-c

$$\overset{T(75)}{125} = (\overset{T_0}{50} - \overset{T_{amb}}{400})\, e^{-75k} + \overset{T_{amb}}{400}$$

$$\Rightarrow \quad -275 = -350\, e^{-75k}$$

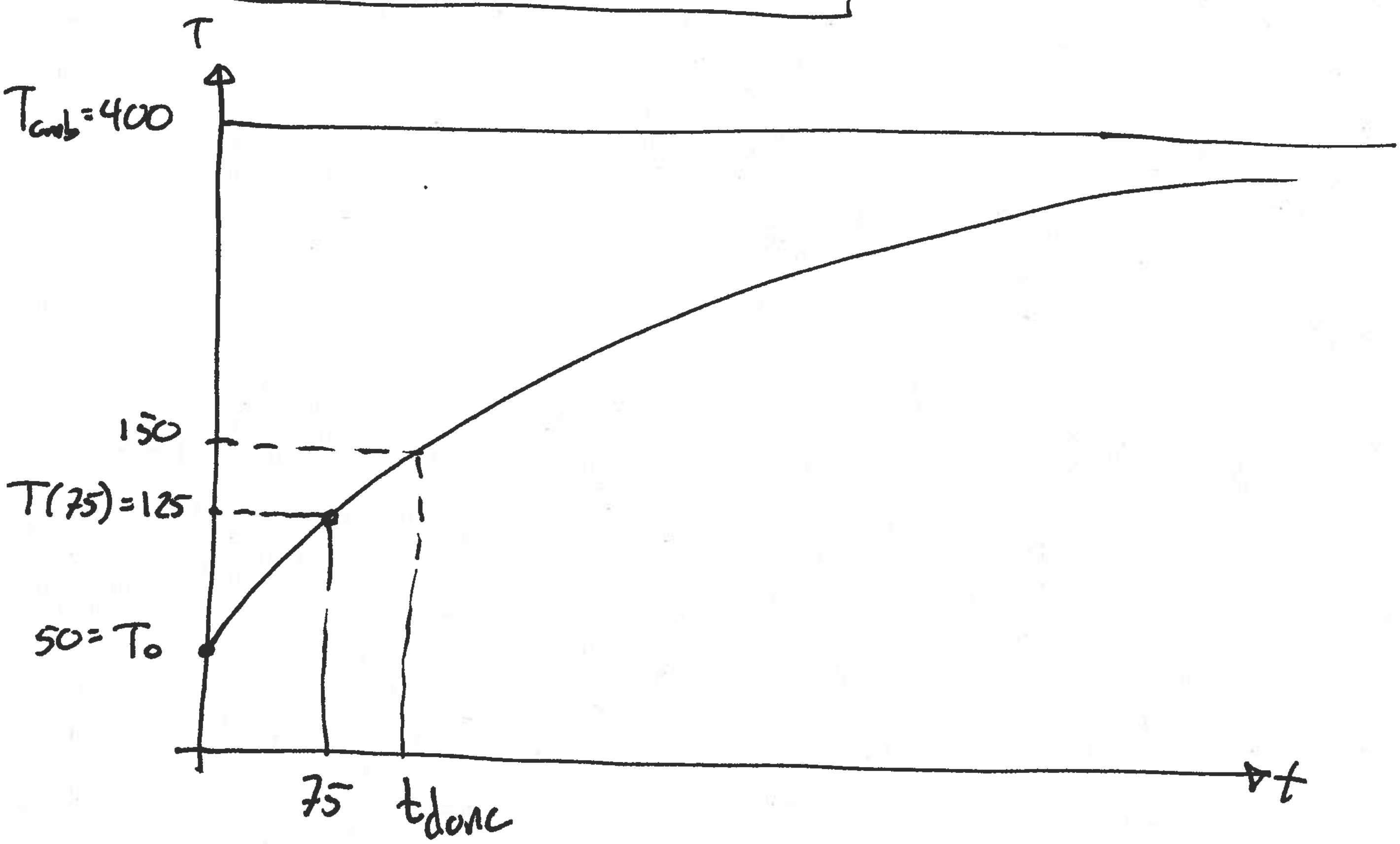
$$\Rightarrow \quad k = -\frac{1}{75} \log \frac{275}{350} \approx 0.0032155.$$

We can now determine t_{done} such that $T(t_{done}) = 150$

$$150 = (50-400)e^{-k t_{done}} + 400$$

$$\Rightarrow \quad t_{done} = -\frac{1}{k} \log \frac{150-400}{50-400} = -\frac{1}{0.0032155\ldots} \log \frac{250}{350} \approx 105 \text{ minutes}$$

$\boxed{\text{Answer: } 105 \text{ minutes}}$



Example A corpse is found and is determined to have temperature 85F. After 2h, the temperature is 74F. The room temperature is 68F. Estimate the time of death!

Sol^n $T(t) = (T_0 - T_{amb})\, e^{-kt} + T_{amb}$

Given data: $T_0 = 85$

$T_{amb} = 68$

$T(2) = 74$

We seek t_{death} such that $T(t_{death}) = 98.6$.

First we determine k using that $T(2) = 74$:

$$74 = (85-68)\, e^{-2k} + 68 \quad \Rightarrow$$

$$\Rightarrow \quad k = -\frac{1}{2} \log \frac{74-68}{85-68} = 0.52073\ldots$$

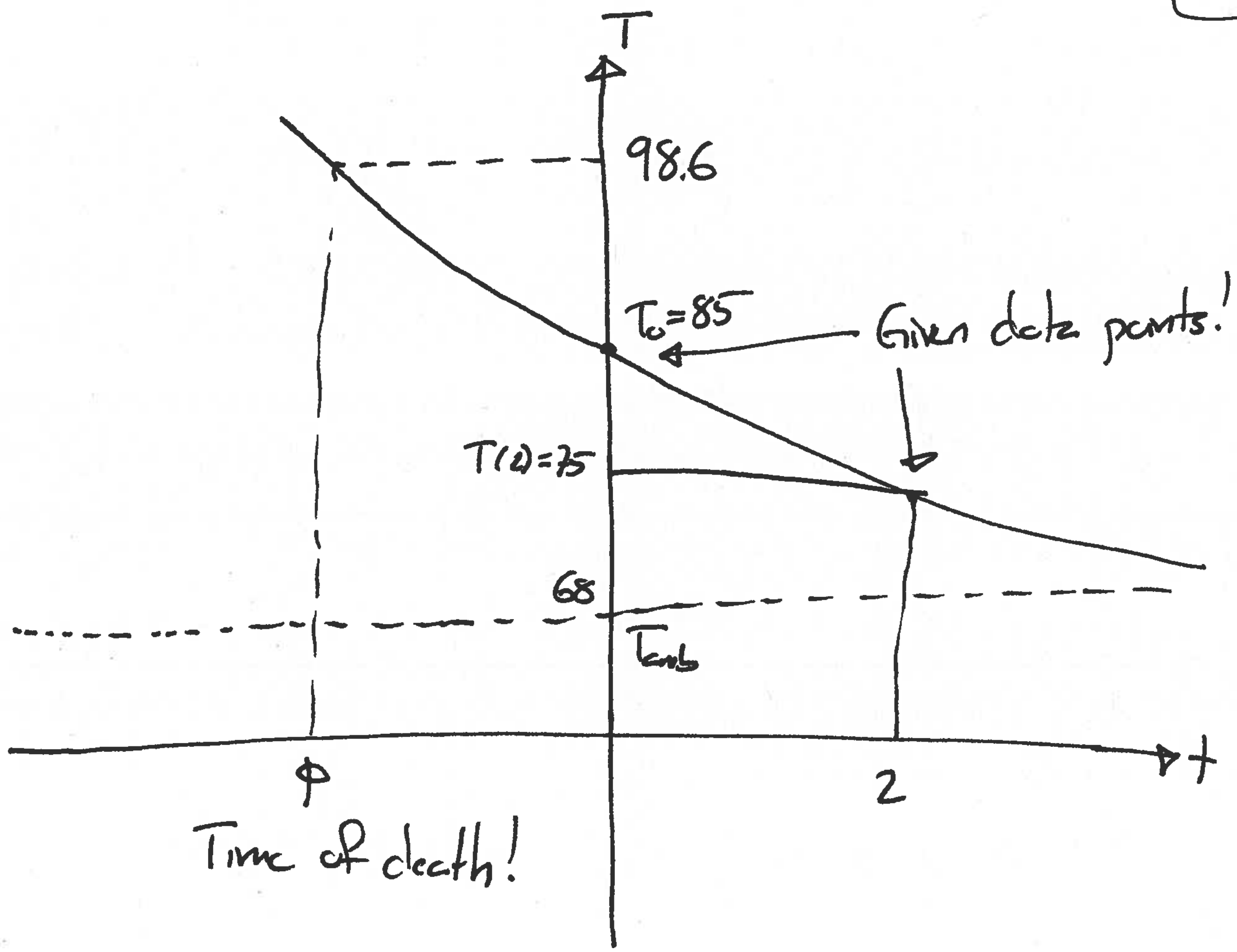
Now determine t so that $T(t) = 98.6$:

$$\cancel{T(t) = 98.6}$$

$$98.6 = (85-68)\, e^{-kt} + 68 \quad \Rightarrow$$

$$\Rightarrow \quad t = -\frac{1}{k} \log \frac{98.6-68}{85-68} \approx -1.13h = -1h\ 8min$$

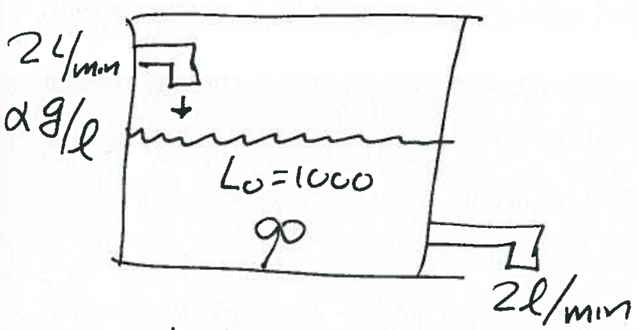
Answer About 1h 8min before the discovery.


T
98.6
$T_0 = 85$
Given data points!
$T(2) = 75$
68
T_{amb}
ϕ
2
t
Time of death!

MIXING

$\boxed{2.4\text{-}P}$

Example Tank with L_0 liter capacity.



$\cancel{y(t)}$

Inflow: 2 l/min of saline sol'n with conc. α g/l

Outflow: 2 l/min of saline sol'n ~~with conc~~

Let y_0 = amount of salt in tank at $t=0$

$y(t)$ = — " — time t.

$$\frac{dy}{dt} = \begin{bmatrix}\text{conc.}\\ \text{in}\end{bmatrix}\begin{bmatrix}\text{flowrate}\\ \text{in}\end{bmatrix} - \begin{bmatrix}\text{conc}\\ \text{out}\end{bmatrix}\begin{bmatrix}\text{flowrate}\\ \text{out}\end{bmatrix}$$

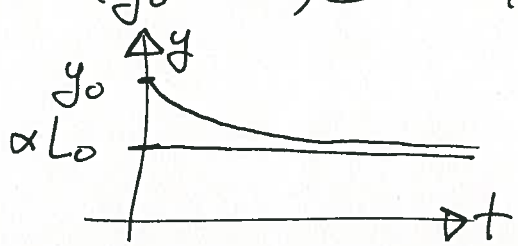
$$= \underset{g/l}{\alpha}\ \underset{l/\text{min}}{2} - \underset{g/l}{\frac{y}{L_0}}\ \underset{l/\text{min}}{2} = 2\alpha - \frac{2}{L_0}y$$

$$y' + \frac{2}{L_0}y = 2\alpha$$

$$y = \underbrace{Ce^{-\frac{2}{L_0}t}}_{y_h} + \underbrace{\alpha L_0}_{y_p}$$

I.C. $\Rightarrow$ $y_0 = C + \alpha L_0 \Rightarrow C = y_0 - \alpha L_0$

$$y(t) = (y_0 - \alpha L_0)e^{-\frac{2}{L_0}t} + \alpha L_0 = y_0 e^{-\frac{2}{L_0}t} + \alpha L_0\left(1 - e^{-\frac{2}{L_0}t}\right)$$

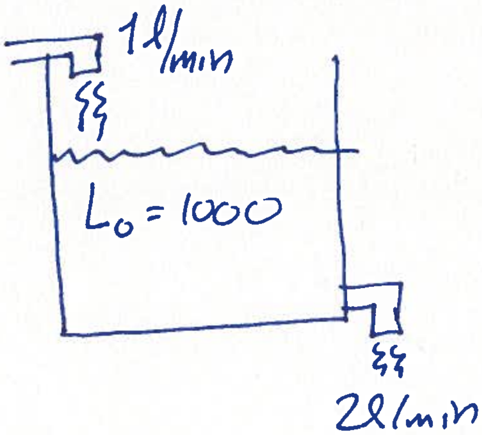


Example Consider a mixing problem that starts with $L_0 = 1000$ l of pure water (no salt). $\boxed{2.4\text{-}g}$

Inflow: 1 l/min at concentration c

Outflow: 2 l/min at concentration [?].

Determine $y(t)$ = amount of salt in tank at time



SOLUTION

First observe that there is a net outflow of 1 l/min (= outflow 2 − inflow 1).

So at time t, there is $L_0 - t$ liters in the tank. Then

$$\frac{dy}{dt} = \begin{bmatrix}\text{flow rate}\\\text{in}\end{bmatrix}\begin{bmatrix}\text{concentration}\\\text{in}\end{bmatrix} - \begin{bmatrix}\text{flow rate}\\\text{out}\end{bmatrix}\begin{bmatrix}\text{conc.}\\\text{out}\end{bmatrix} =$$

$$= 1 \cdot c - 2\frac{y}{L_0 - t}$$

We need to solve $\quad y' + \frac{2}{L_0 - t}\, y = c. \quad (*)$

Use an integrating factor.

$$p(t) = \frac{2}{L_0 - t} \Rightarrow P(t) = -2\log(L_0 - t) = \log\frac{1}{(L_0 - t)^2}$$

We find $e^{P(t)} = \frac{1}{(L_0 - t)^2}$

$\boxed{2.4\text{-}h}$

Multiply $(*)$ by $\frac{1}{(L_0-t)^2}$ to get

$$\frac{1}{(L_0-t)^2}y' + \frac{2}{(L_0-t)^3}y = \frac{c}{(L_0-t)^2}$$

$$\frac{d}{dt}\left(\frac{1}{(L_0-t)^2}y\right) = \frac{c}{(L_0-t)^2}$$

General sol^n is, for any constant B:

$$\frac{1}{(L_0-t)^2}y(t) = \frac{c}{(L_0-t)} + B \Rightarrow y = C(L_0-t) + B(L_0-t)$$

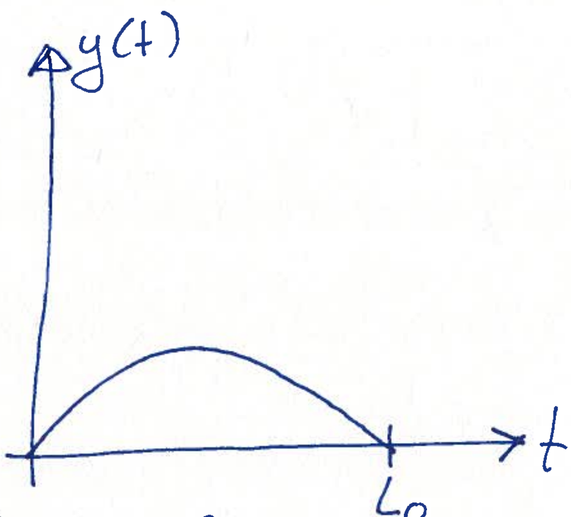
Use initial cond^n to ~~f~~ determine B:

$$0 = c(L_0-0) + B(L_0-0)^2 \Rightarrow B = -\frac{c}{L_0}$$

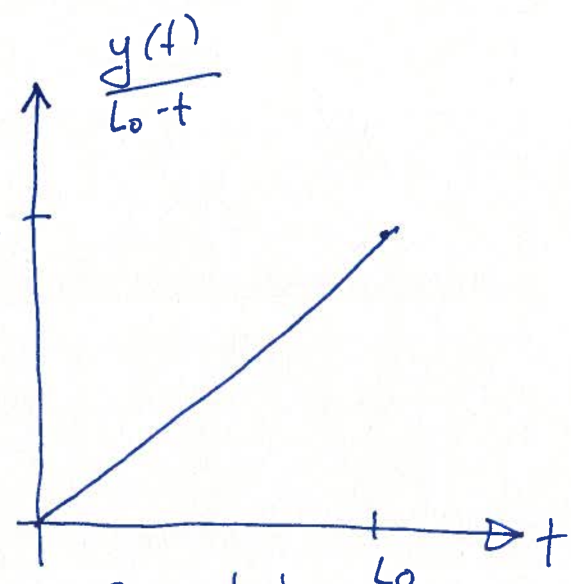
The final solution is then

$$y(t) = c\,(L_0-t) - \frac{c}{L_0}(L_0-t)^2 =$$

$$= c\left(t - \frac{1}{L_0}t^2\right)$$



Amount of salt



Concentration