Separable differential equations:

Consider the DE: $$\frac{dy}{dt} = h(t)\,g(y)$$

Solution recipe:

1. Collect terms: $$\frac{1}{g(y)}\,dy = h(t)\,dt$$
2. Integrate: $$\int \frac{1}{g(y)}\,dy = \int h(t)\,dt + C$$

   In other words, find primitive functions H and G such that $H'(t) = h(t)$ and $G'(y) = \frac{1}{g(y)}$.

   The solution is then

   $$G(y) = H(t) + C. \tag{1}$$
3. If you have an initial condition, you can determine C.
4. If possible, you can solve (1) for y, if you so desire.

Note: If you are given an initial condition, you can use *definite integrals* in step 3:

$$\int_{y_0}^{y} \frac{1}{g(z)}\,dz = \int_{t_0}^{t} h(s)\,ds.$$

Rigorous verification that the solution method works:

Consider a separable DE

$$\frac{dy}{dt} = h(t)\,g(y). \tag{2}$$

On the previous slide, we in a questionable way derived the solution:

$$G(y(t)) = H(t) + C. \tag{3}$$

where

$$G'(y) = \frac{1}{g(y)} \qquad \text{and} \qquad H'(t) = h(t).$$

Let us differentiate the solution (3), using the chain rule,

$$\frac{dy}{dt}\,G'(y(t)) = H'(t).$$

This simplifies to

$$\frac{dy}{dt}\,\frac{1}{g(y)} = h(t).$$

Multiply by $g(y)$ to see that we do indeed satisfy the DE (2).

Example: Consider the equation

$$\frac{dy}{dt} = -2t\,y.$$

First note that $y = 0$ is an equilibrium point.

Then collect terms (assuming $y \neq 0$):

$$\frac{dy}{y} = -2t\,dt.$$

Then integrate both sides:

$$\log|y| = -t^2 + C.$$

Solve for y:

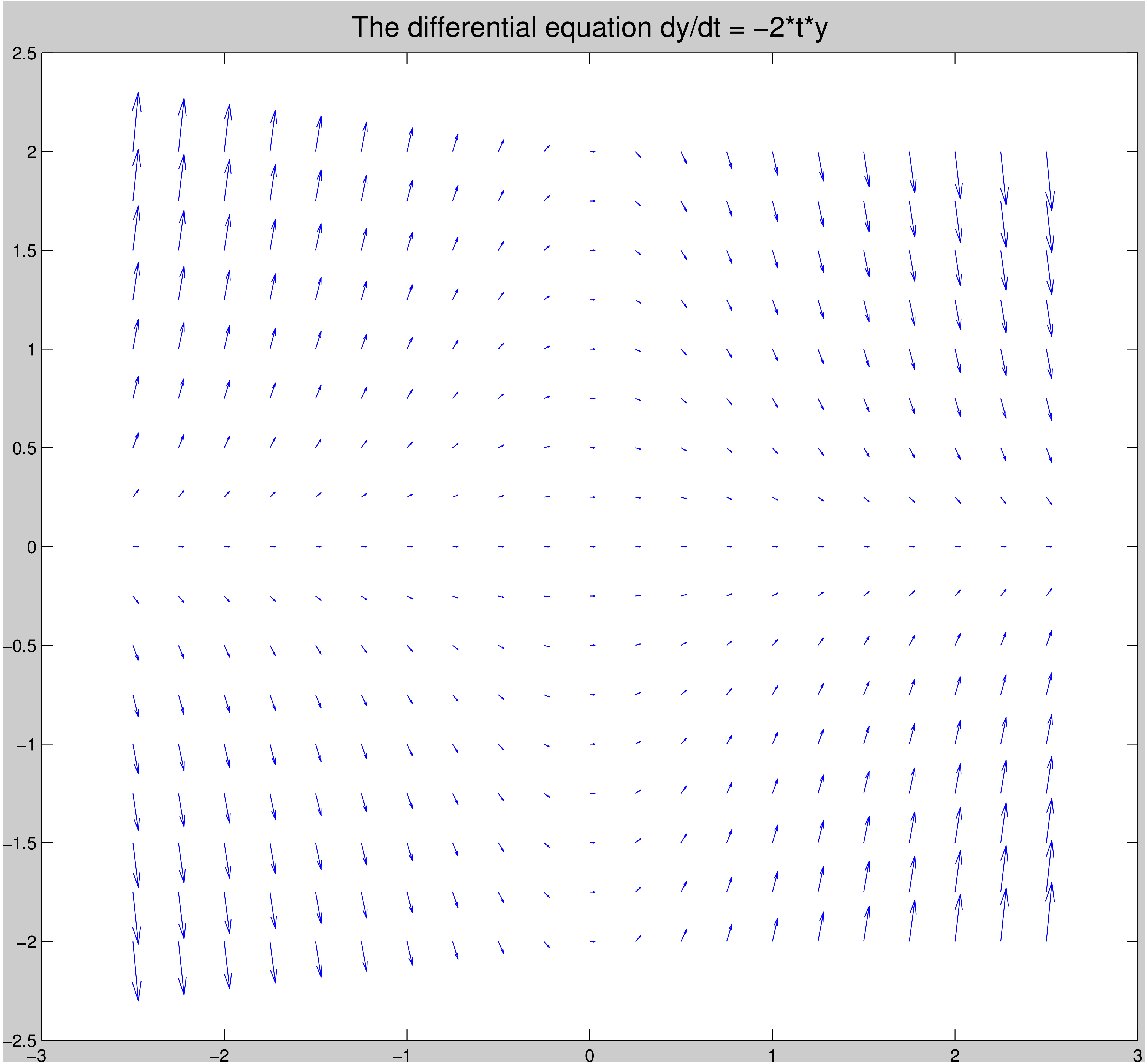
$$|y| = e^{-t^2+C} = e^C\,e^{-t^2} = \{\text{Set } D = e^C\} = D\,e^{-t^2}.$$

Observe that $D > 0$. Removing the absolute value, we find

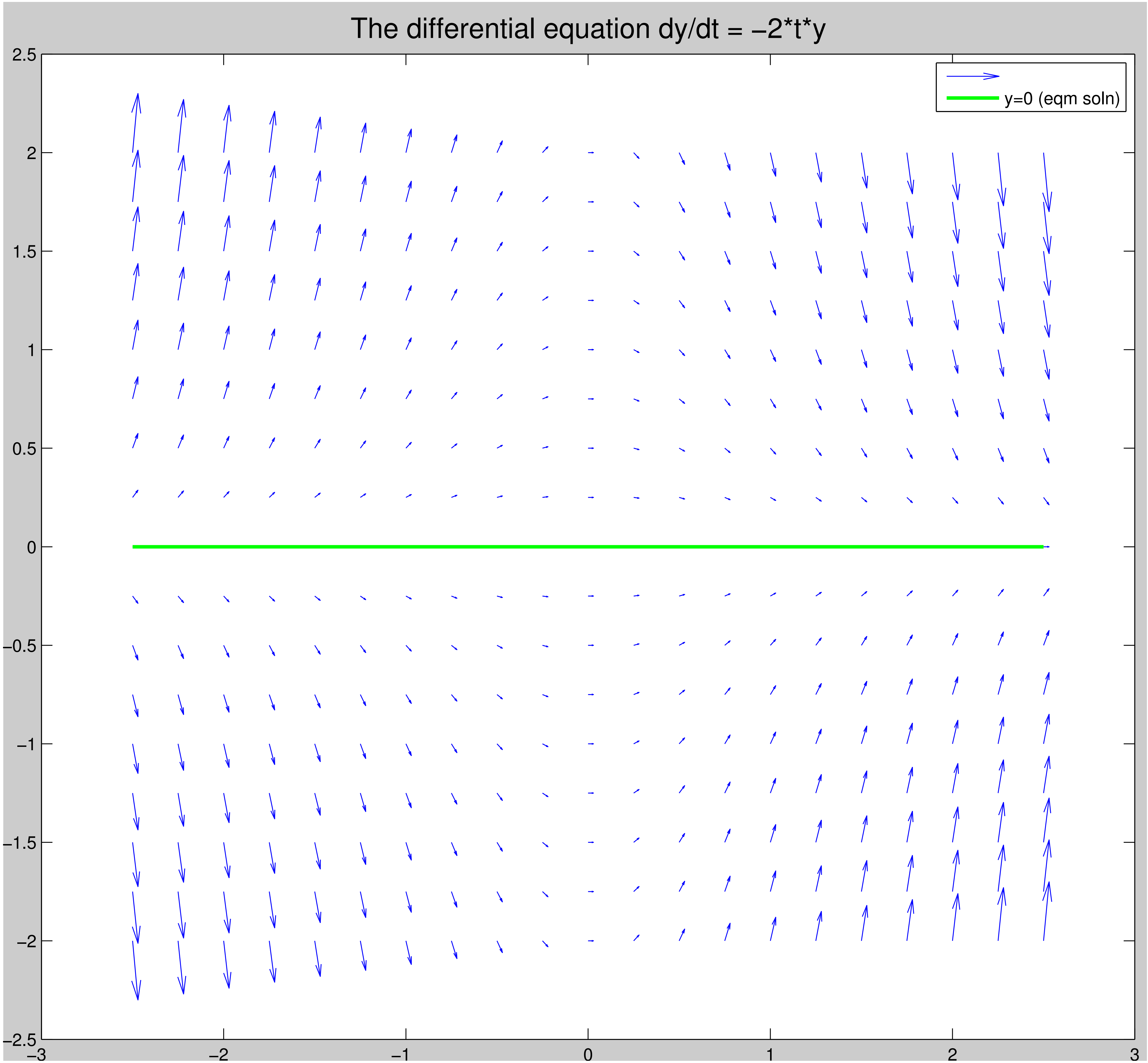
$$y = \pm D\,e^{-t^2}.$$

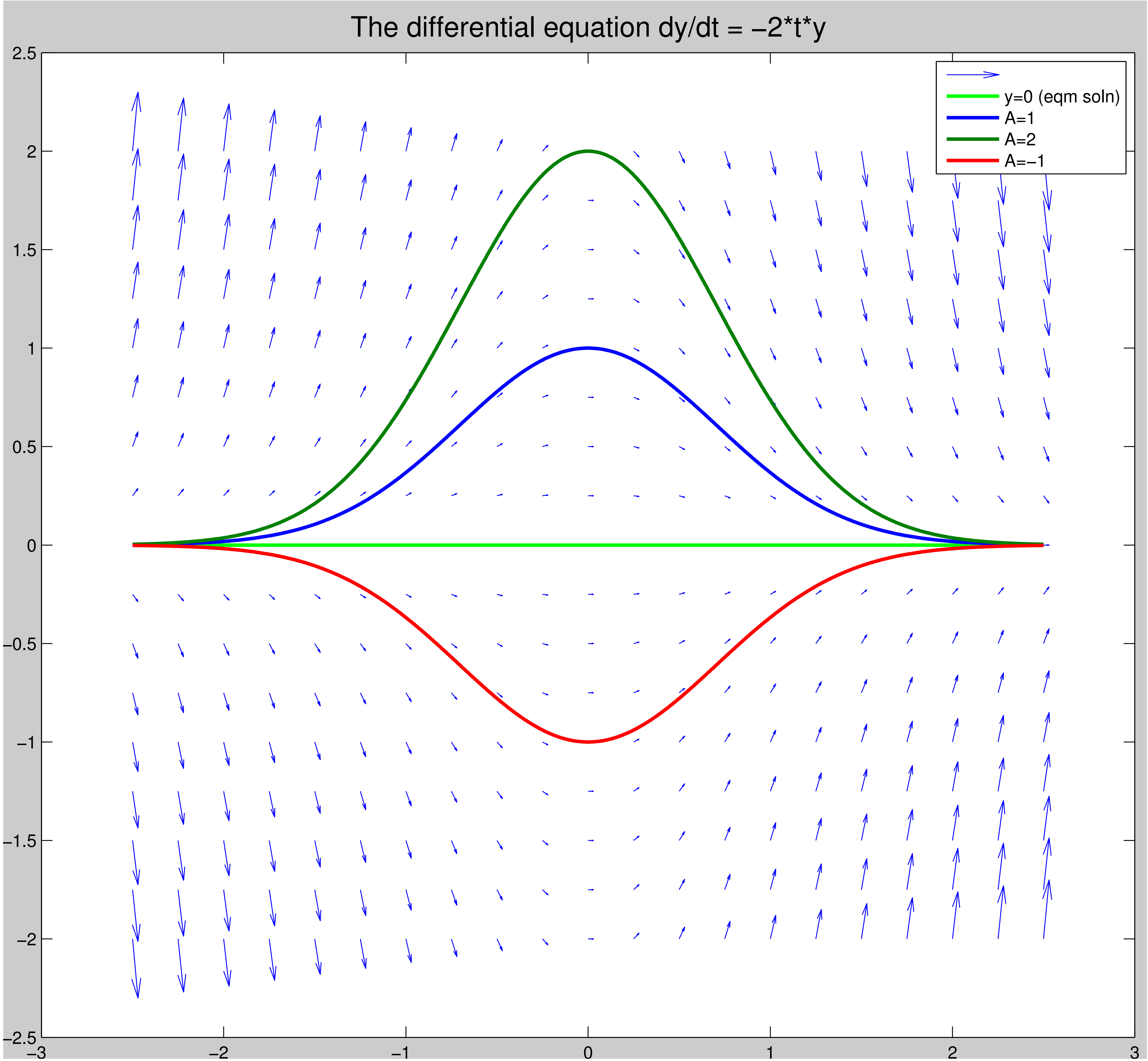
We can summarize all solutions we found as:

$$\boxed{y(t) = A\,e^{-t^2} \text{ where } A \text{ is any real number.}}$$



The differential equation dy/dt = −2*t*y
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−3
−2
−1
0
1
2
3


The differential equation dy/dt = −2*t*y
y=0 (eqm soln)
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−3
−2
−1
0
1
2
3



The differential equation dy/dt = −2*t*y
y=0 (eqm soln)
A=1
A=2
A=−1
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−3
−2
−1
0
1
2
3

Example: Consider the equation

$$\frac{dy}{dt} = -\frac{t}{y}.$$

First observe that there are no equilibrium solutions. ($y = 0$ is a *singular point.*)

Collect terms:

$$y\,dy = -t\,dt.$$

Integrate:

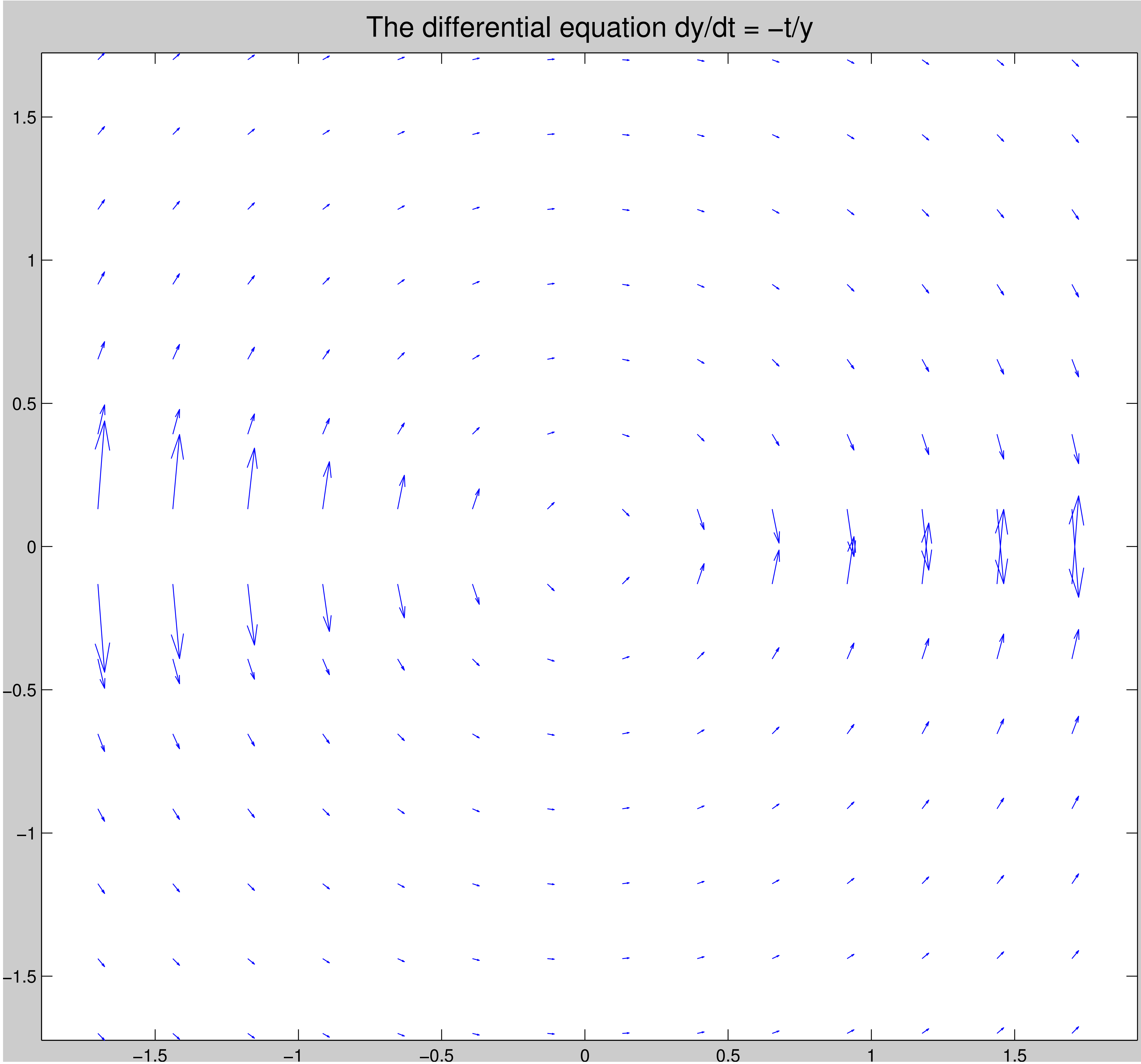
$$\frac{1}{2}y^2 = -\frac{1}{2}t^2 + C.$$

Reformulate slightly:

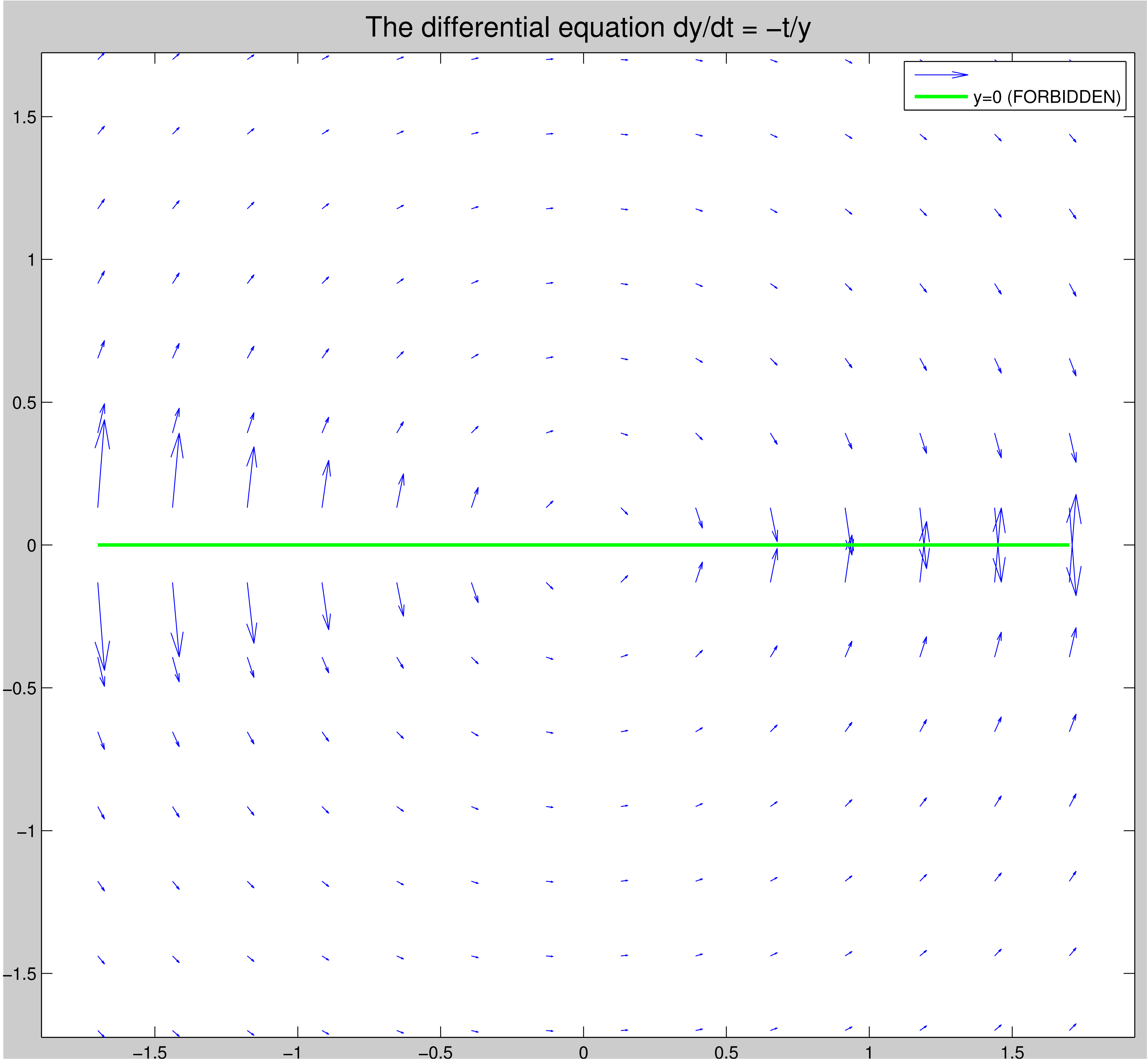
$$y^2 + t^2 = 2C.$$

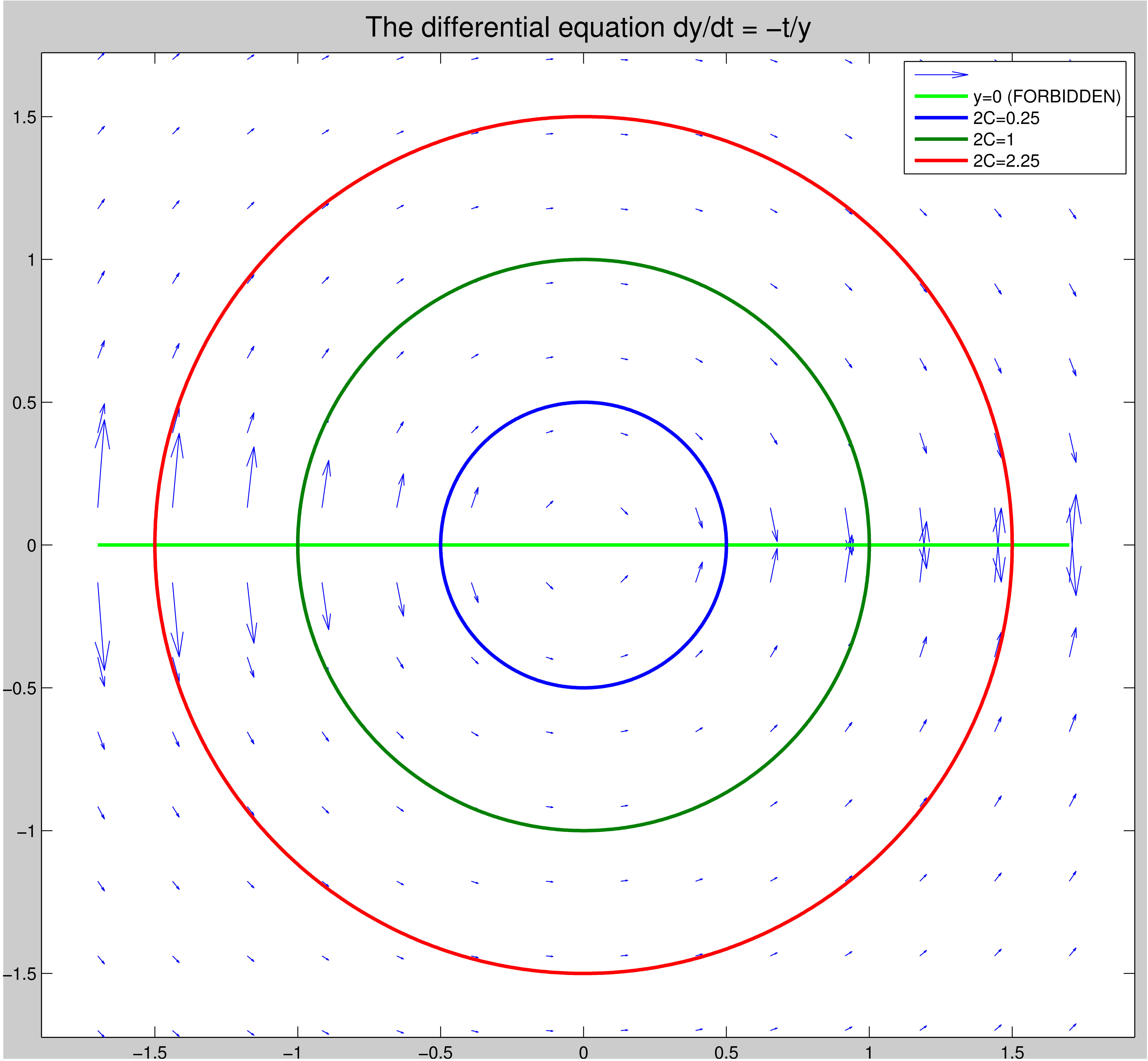
You *could* solve for y to get:

$$y(t) = \pm\sqrt{2C - t^2}.$$



The differential equation dy/dt = −t/y
1.5
1
0.5
0
−0.5
−1
−1.5
−1.5
−1
−0.5
0
0.5
1
1.5


The differential equation dy/dt = −t/y
y=0 (FORBIDDEN)
1.5
1
0.5
0
−0.5
−1
−1.5
−1.5
−1
−0.5
0
0.5
1
1.5



The differential equation dy/dt = −t/y
y=0 (FORBIDDEN)
2C=0.25
2C=1
2C=2.25
1.5
1
0.5
0
−0.5
−1
−1.5
−1.5
−1
−0.5
0
0.5
1
1.5

Section 1.4: Euler's method for solving DEs

Consider a DE:

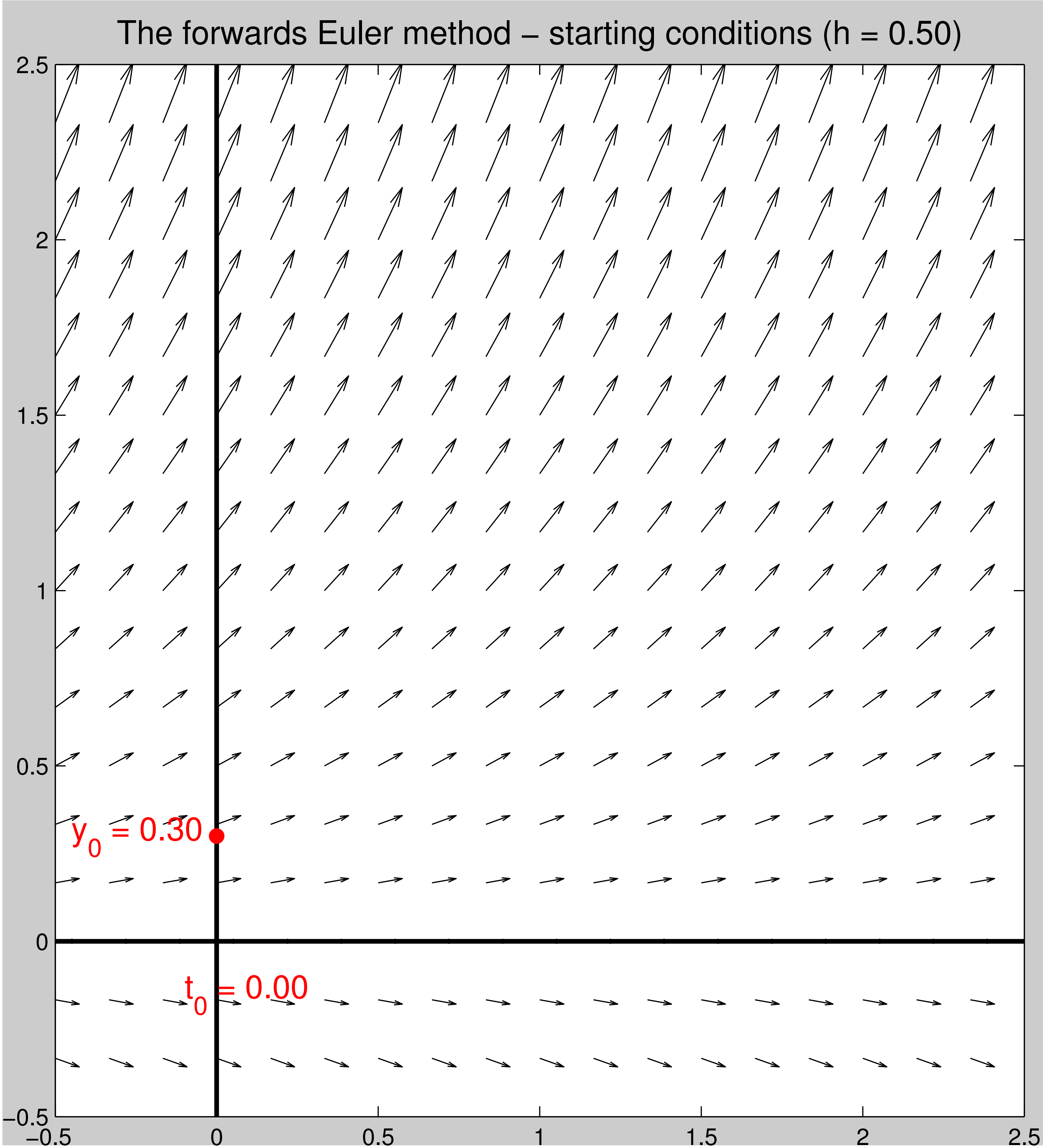
$$\begin{cases} y'(t) = f(t, y), \\ y(a) = y_0. \end{cases}$$

We seek a solution on the interval $I = [a, b]$.

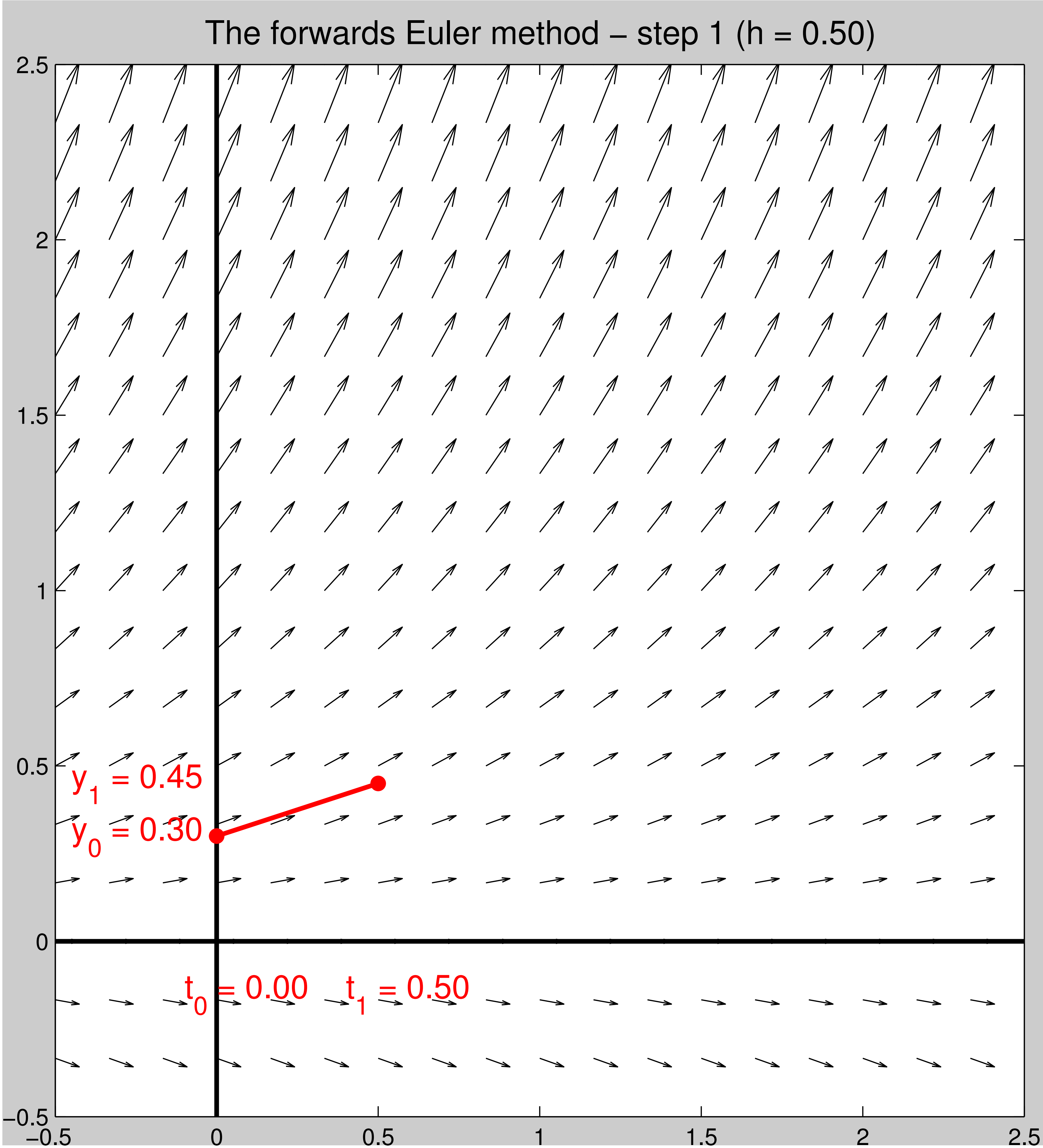
How would you find an approximate solution using a computer?

In the example that follows we solve the very simple equation

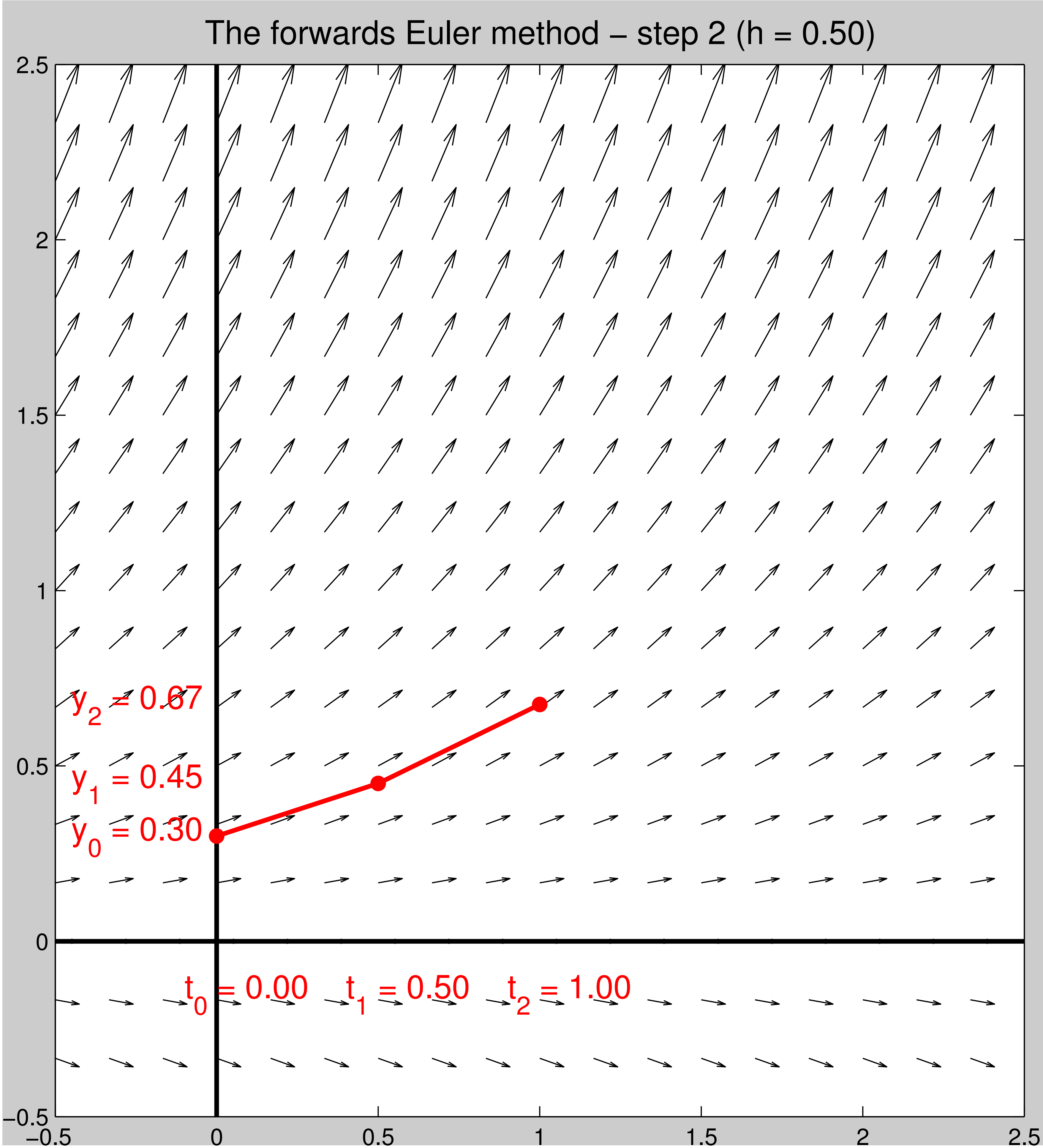
$$\begin{cases} y'(t) = y, \\ y(0) = y_0 = 0.3 \end{cases}$$



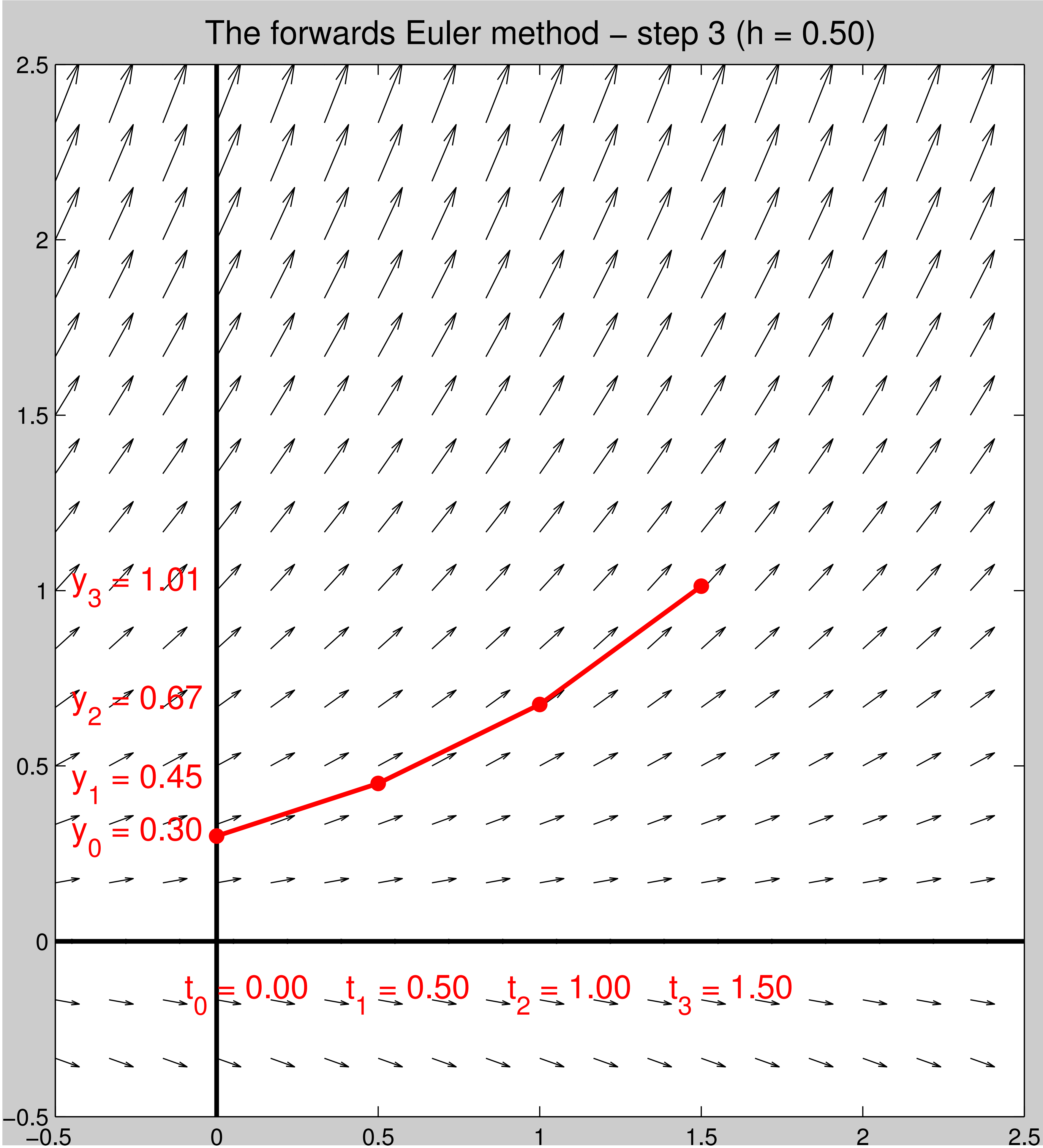
$$\begin{cases} y'(t) = y, \\ y(0) = y_0 = 0.3 \end{cases}$$



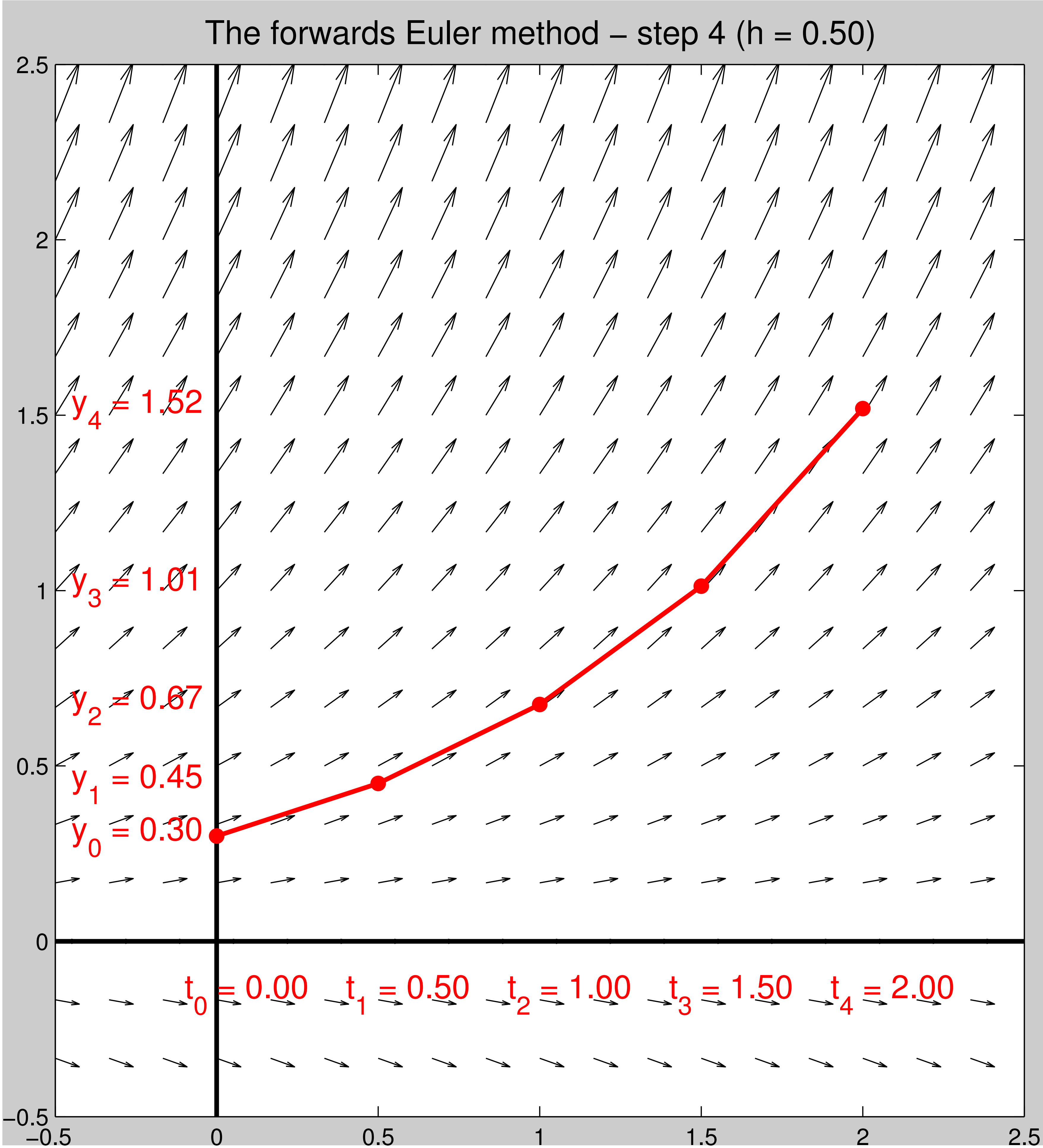
$$\begin{cases} y'(t) = y, \\ y(0) = y_0 = 0.3 \end{cases}$$



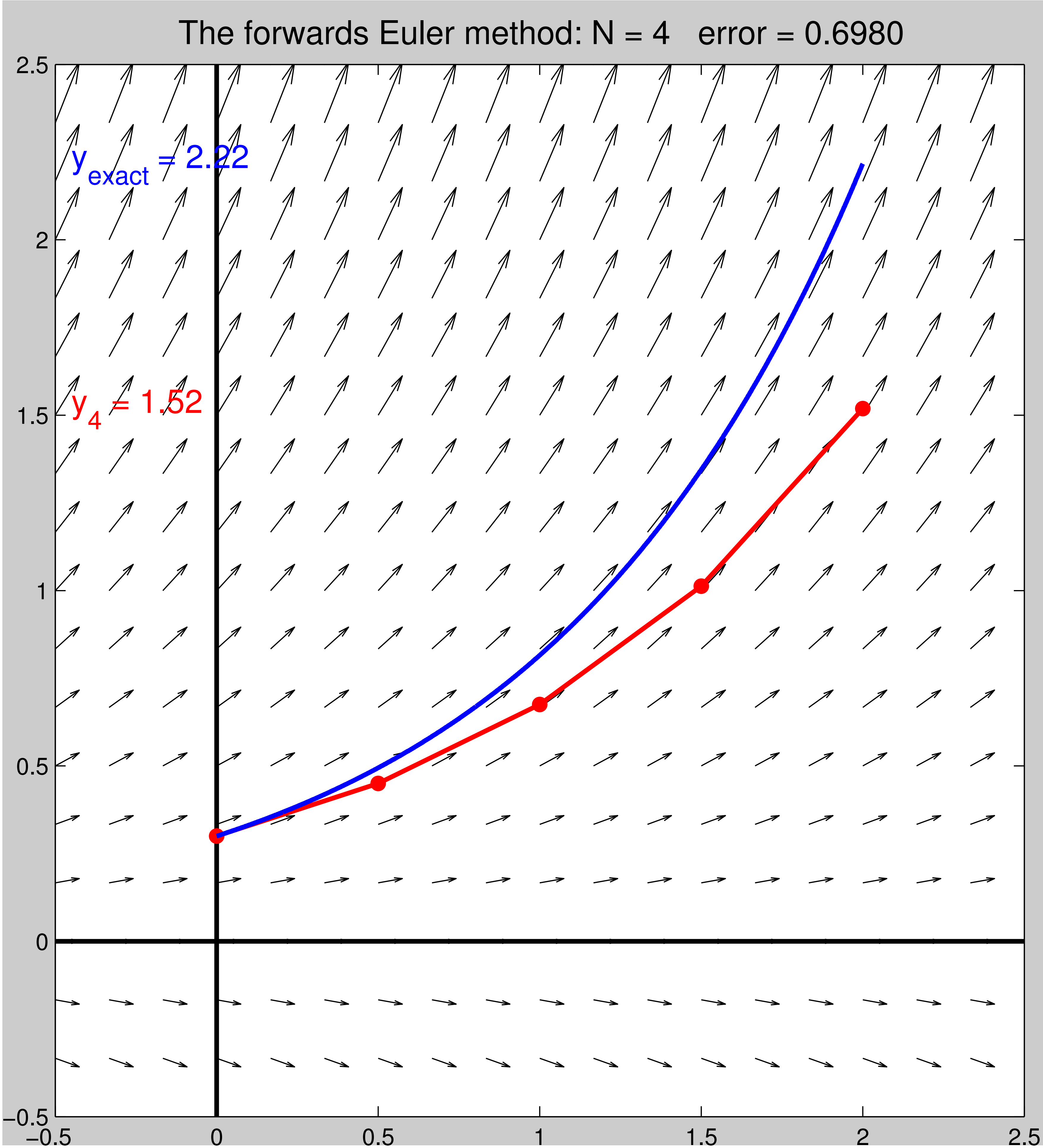
$$\begin{cases} y'(t) = y, \\ y(0) = y_0 = 0.3 \end{cases}$$



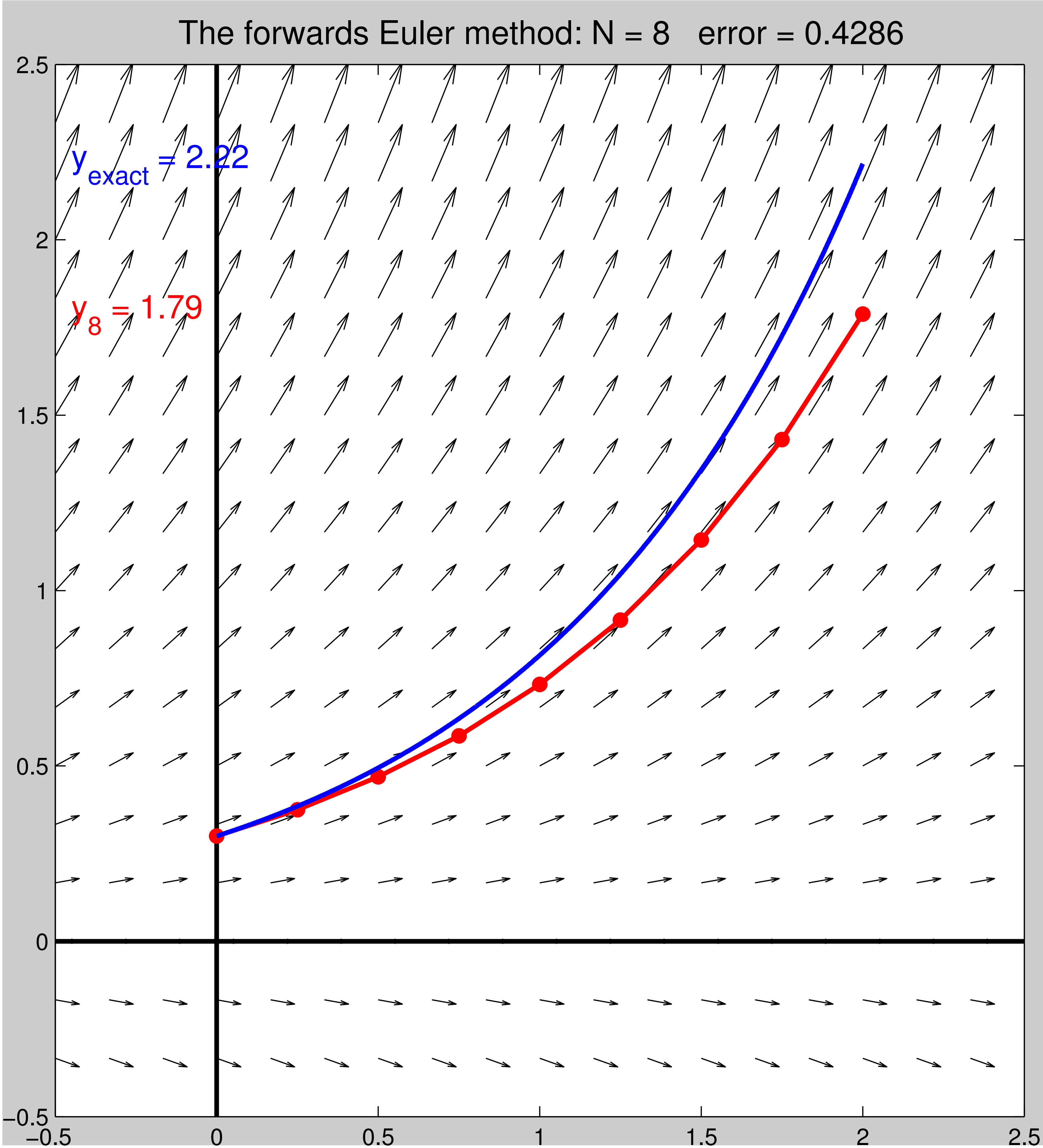
$$\begin{cases} y'(t) = y, \\ y(0) = y_0 = 0.3 \end{cases}$$



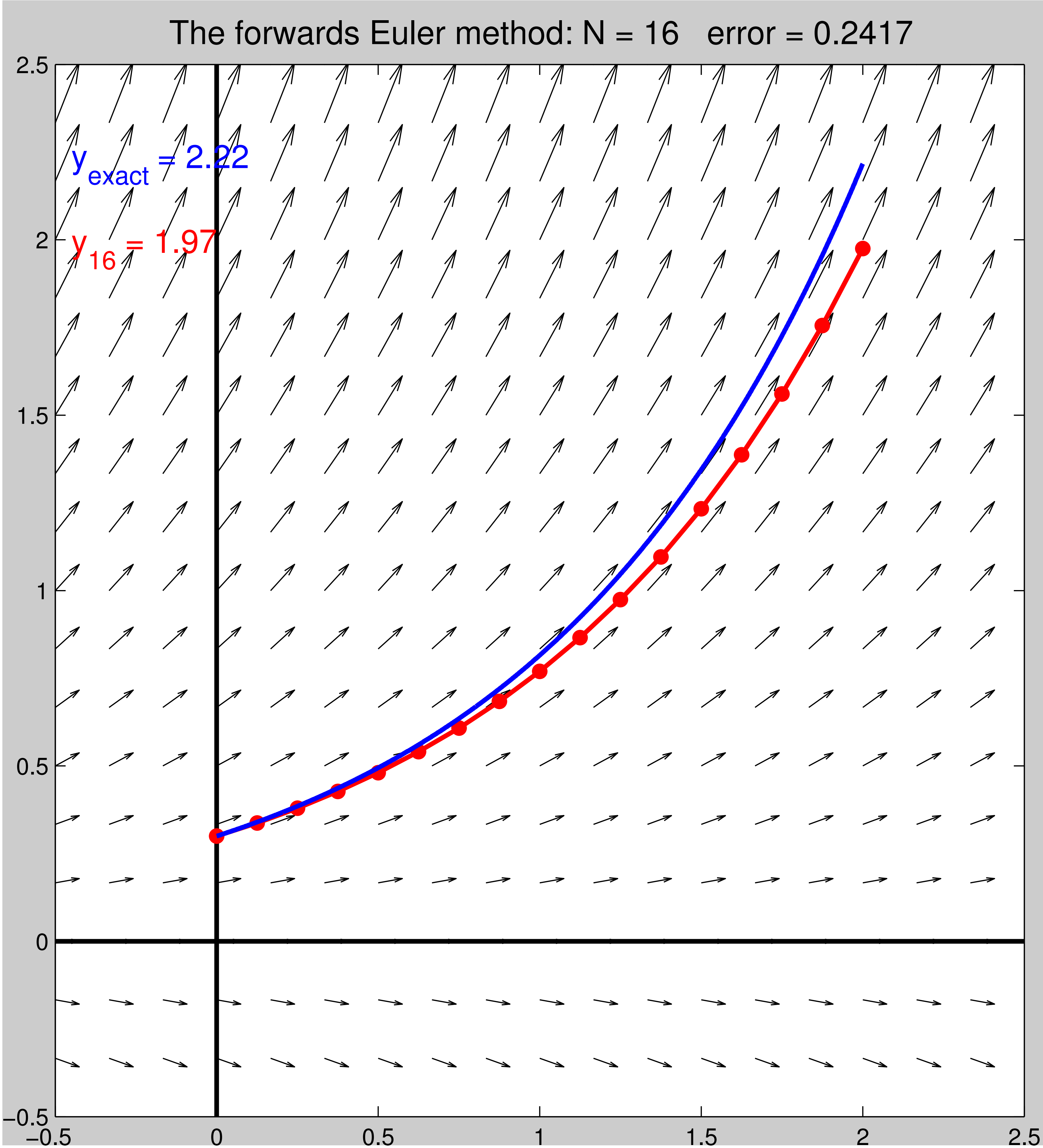
$$\begin{cases} y'(t) = y, \\ y(0) = y_0 = 0.3 \end{cases}$$



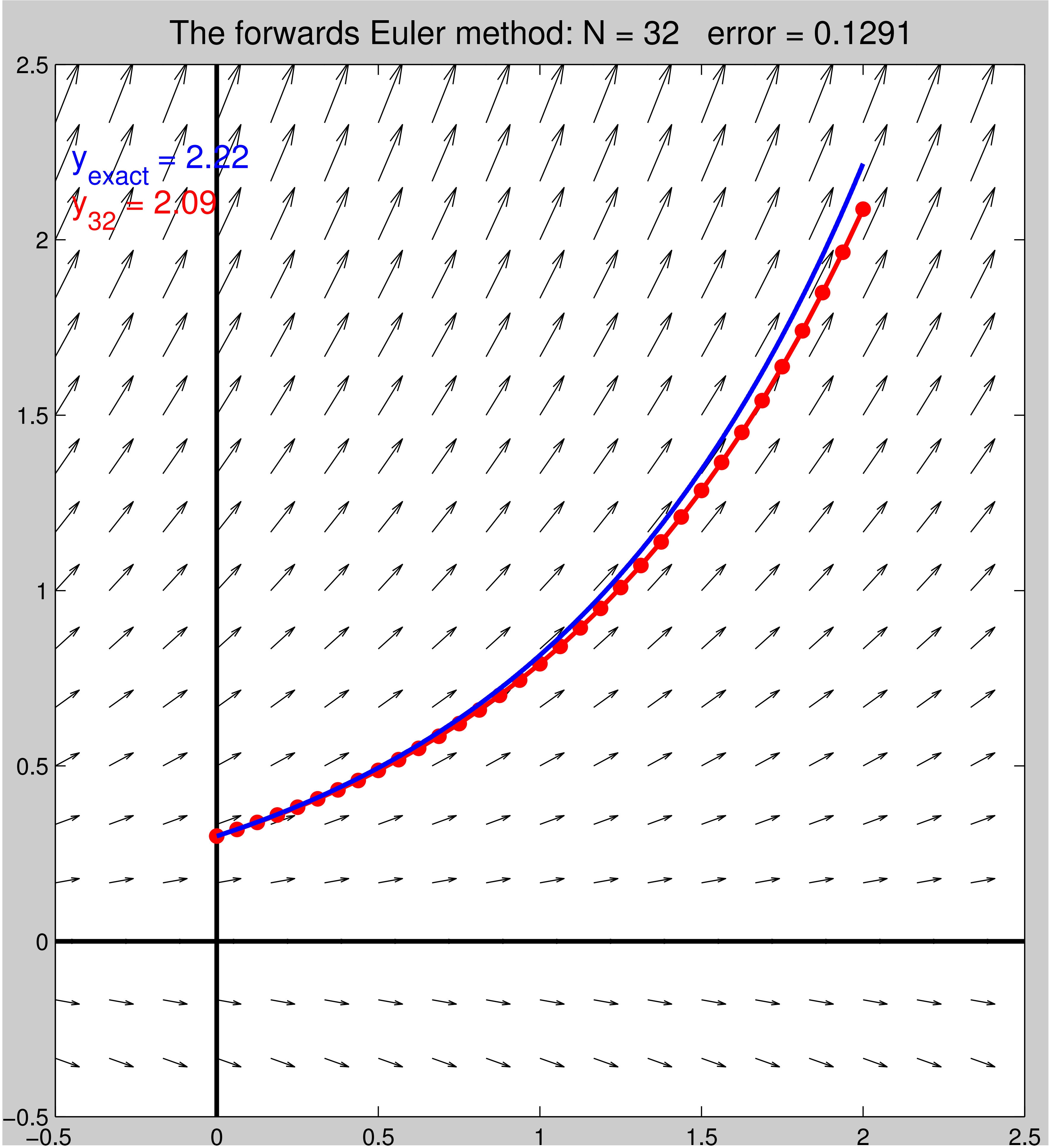
$$\begin{cases} y'(t) = y, \\ y(0) = y_0 = 0.3 \end{cases}$$



$$\begin{cases} y'(t) = y, \\ y(0) = y_0 = 0.3 \end{cases}$$



$$\begin{cases} y'(t) = y, \\ y(0) = y_0 = 0.3 \end{cases}$$



$$\begin{cases} y'(t) = y, \\ y(0) = y_0 = 0.3 \end{cases}$$

Section 1.4: Euler's method for solving DEs

Consider a DE:

$$\begin{cases} y'(t) = f(t, y), \\ y(a) = y_0. \end{cases}$$

We seek a solution on the interval $I = [a, b]$.

Split the interval into N points, separated a distance $h = \dfrac{b-a}{N}$:

$$t_0 = a, \qquad t_1 = a + h, \qquad t_2 = a + 2h \qquad t_3 = a + 3h, \qquad \cdots \qquad t_N = b.$$

Now approximate $y(t)$ by a sequence of straight lines:

$$\begin{aligned} y_0 &= y_0 \\ y_1 &= y_0 + h\,f(t_0,\, y_0), \\ y_2 &= y_1 + h\,f(t_1,\, y_1), \\ y_3 &= y_2 + h\,f(t_2,\, y_2), \end{aligned}$$

Section 1.4: Euler's method for solving DEs

Key points about Euler's method (a.k.a. "the Forwards Euler method"):

- You should know the formula.

- You should know that the error depends on the number of intervals used.
  Roughly, if you *double* the number of intervals, you *half* the error.
  Technically, we say the error E satisfies $E = O(h)$ or, equivalently, $E = O(1/N)$.

- This method is extremely simple to use, which is why we teach it in APPM2360.
  However, *it is a very bad method*.

- There are other easy-to-use methods that are much better!
  If you need to code up a method, then read up a little.
  (Or take more APPM courses!)

- Even better, there are black-box numerical integrators that are extremely good and also very easy to use.
  - You specify a desired error, the black-box figures out what h should be.
  - The step-size changes from step-to-step!

Consider a DE: $\begin{cases} y'(t) = f(t, y), \\ y(a) = y_0. \end{cases}$

An "ad hoc" scheme whose error decays as $1/N^2$ as $N \to \infty$

First compute y_1 using forwards Euler: $y_1 = y_0 + h\, f(t_0, y_0)$.

Then proceed via the formula: $y_{n+1} = y_{n-1} + 2\, h\, f(t_n, y_n)$

The second order "Runge-Kutta" method:

Given y_n, compute two intermediate values:

$$\begin{aligned} k_1 =& f(t_n, y_n), \\ k_2 =& f(t_n + (1/2)h, y_n + (1/2)\, h\, k_1). \end{aligned}$$

Then $y_{n+1} = y_n + h\, k_2$.

The fourth order "Runge-Kutta" method:

Given y_n, compute four intermediate values:

$$\begin{aligned} k_1 =& f(t_n, y_n), \\ k_2 =& f(t_n + (1/2)h, y_n + (1/2)\, h\, k_1), \\ k_3 =& f(t_n + (1/2)h, y_n + (1/2)\, h\, k_2), \\ k_4 =& f(t_n + h, y_n + h\, k_3). \end{aligned}$$

Then $y_{n+1} = y_n + h\, \frac{1}{6}\, (k_1 + 2k_2 + 2k_3 + k_4)$.