APPM 2360: Section exam 2

7.00pm – 8.30pm, February 11, 2009.

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) recitation section (4) your instructor's name, and (5) a grading table. Text books, class notes, and calculators are NOT permitted. A one-page crib sheet is allowed.

Problem 1: (36 points) Give a brief answer to each question. Box your answer. Each correct answer earns 4 points. No work given for this question will be graded.

(a) Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be non-zero vectors in $\mathbb{R}^7$. Let r denote the dimension of Span$\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}$. What are the possible values of r?

(b) Set $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, and $X = A\,B\,C$.

Compute $\det(X)$. *Hint:* $\det(A) = -3$ and $\det(C) = -2$.

(c) Which of the following sets constitute basis sets for $\mathbb{P}_2$ (the set of all polynomials of degree at most 2):

$$S_1 = \{1,\, t,\, t^2\}, \quad S_2 = \{1,\, t^2\}, \quad S_3 = \{1 + t,\, 1 - t,\, t^2\}, \quad S_4 = \{0,\, 1,\, t,\, t^2\}.$$

(d) Assuming the standard addition and multiplication rules, determine which of the following sets are vectors spaces.

(i) All 4×5 matrices A such that $a_{2,3} = 0$;

(ii) All vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ that satisfy $x_1 + x_2 = 3$.

(e) Set $A = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$. Find $A^T B$.

(f) Let A be an 8×5 matrix. If the rank of A is 2, what is the dimension of the vector space $V = \{\mathbf{x} \in \mathbb{R}^5 :\ A\,\mathbf{x} = 0\}$.

(g) For which values of t is the matrix $A = \begin{bmatrix} 1 & t \\ 1 & (1+t) \end{bmatrix}$ invertible?

(h) "If a 4×4 matrix A is invertible, then A^5 is also invertible." Answer TRUE or FALSE.

(i) "The dimension of the vector space of all $m \times n$ matrices is $m+n$." Answer TRUE or FALSE.

Solution:

(a)	$r = 1, 2, 3$	(d)	(i)	(g)	For all t
(b)	18	(e)	$\begin{bmatrix} -4 & -2 \\ 9 & 3 \end{bmatrix}$	(h)	TRUE
(c)	S_1 and S_3	(f)	3	(i)	FALSE

Notes:

(a) r is the rank of the non-zero 7×3 matrix $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$.

(b) Use that $\det(X) = \det(A\,B\,C) = \det(A)\,\det(B)\,\det(C) = (-3)\,\det(B)\,(-2)$, and then note that since B is diagonal, we have $\det(B) = 2\,(1/2)\,3 = 3$.

(c) S_1 is the canonical basis. Since $\dim(\mathbb{P}_2) = 3$, S_2 and S_4 cannot be bases since any basis must have precisely 3 elements. To see that S_3 is a basis, note that it has three elements, and is a spanning set since you can recover the standard basis from it: $1 = (1/2)\,(1+t) + (1/2)\,(1-t)$ and $t = (1/2)\,(1+t) - (1/2)\,(1-t)$.

(d) The set in (ii) does not contain the zero vector.

(e) $A^{\mathrm{T}}\,B = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + 2\cdot(-3) & 1\cdot 0 + (-2)\cdot 1 \\ 0\cdot 2 + 3\cdot 3 & 0\cdot 0 + 3\cdot 1 \end{bmatrix}$.

(f) The rank of the null-space equals the number of free variables in the RREF of A, which in this case is $5 - 2$ since there are 5 columns and 2 pivot elements.

(g) $\det(A) = 1\cdot(1+t) - t\cdot 1 = 1$ so $\det(A) \neq 0$ for all t.

(h) Note that $\left(A^5\right)^{-1} = \left(A^{-1}\right)^5$.

(i) The actual dimension is $m\,n$.

Problem 2: (16 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}.$$

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ denote the columns of A, so that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$

(a) (8 points) Is A invertible? If yes, find the inverse; if not, substantiate your claim.

(b) (4 points) Is the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

(c) (4 points) Determine for which real numbers t (if any), the vector $\mathbf{b} = \begin{bmatrix} 1 \\ t \\ 1 \end{bmatrix}$ belongs to $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Hint: Verify your calculations in (a) carefully since a wrong answer may influence your answers to (b) and (c).

Solution:

(a) We attempt to solve the matrix equation $AX = I$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & -1 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -2 & -1 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right].$$

It worked, so A is invertible, and

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -1 \\ -2 & 2 & 1 \end{bmatrix}.$$

(b) $\boxed{S \text{ is linearly independent.}}$

Since A is invertible, the solution $A\mathbf{x} = \mathbf{0}$ has only the zero solution.

(c) $\boxed{\text{For every } t.}$

Since A is invertible, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.

Problem 3: (16 points) Set

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 0 \\ -1 & 0 & -2 & 1 & 1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}.$$

(a) (8 points) Give the general solution of the equation $A\,\mathbf{x} = \mathbf{b}$.

(b) (8 points) Construct a basis for the vector space $V = \text{Null}(A) = \{\mathbf{x} : \ A\,\mathbf{x} = \mathbf{0}\}$.

Hint: A is row equivalent to the matrix $\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

Solution:

(a) We attempt to solve via row operations:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 1 & 3 \\ 0 & 2 & 2 & 0 & 0 & -2 \\ -1 & 0 & -2 & 1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 1 & 3 \\ 0 & 2 & 2 & 0 & 0 & -2 \\ 0 & -2 & -2 & 1 & 2 & 6 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 1 & 3 \\ 0 & 2 & 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{array}\right].$$

The system is consistent, and has the free variables $x_3 = s$ and $x_5 = t$. Then

$$\begin{aligned} x_1 &= 1 - 2\,x_3 - x_5 = 1 - 2\,s - t \\ x_2 &= -1 - x_3 = -1 - s \\ x_3 &= s \\ x_4 &= 4 - 2\,x_5 = 4 - 2\,t \\ x_5 &= t. \end{aligned}$$

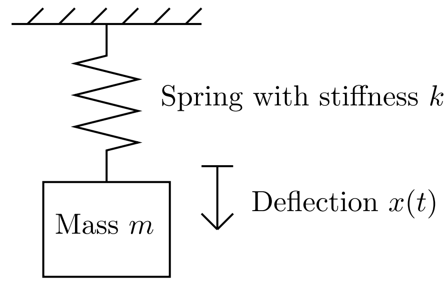
(b) We read off the general solution of $A\,\mathbf{x}_{\text{h}} = 0$ directly from the given RREF:

$$\mathbf{x}_{\text{h}} = \begin{bmatrix} -2\,s - t \\ -s \\ s \\ -2\,t \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} s + \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\mathbf{v}_2} t.$$

It is obvious that $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for V. That S is linearly independent is also clear since if $c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 = 0$, then it must be that $c_1 = 0$ (look at the third entry) and $c_2 = 0$ (look at the fifth entry).

Problem 4: (16 points) Consider the mechanical spring-mass system shown in the picture. Let $x = x(t)$ denote the deflection of the spring from its equilibrium position (counted positive downwards) at time t. The governing equation is

$$\ddot{x} + 9\,x = 0. \tag{1}$$



(a) (4 points) Suppose that the body has mass m and the spring has stiffness k. What equation must m and k satisfy for the system to be modeled by (1)?

(b) (4 points) Give the general solution of (1).

(c) (8 points) Suppose that at time $t = 1$, the mass is at position $x = 0$ and has (downwards) velocity v_0. Determine the solution $x(t)$ of (1) that matches these conditions. (Your answer should depend on the parameter v_0.)

Solution:

(a) $\boxed{k/m = 9}$

The governing equation is $m\,\ddot{x} = -k\,x$ which can be written $\ddot{x} + (k/m)\,x = 0$.

(b) $x(t) = A\,\cos(3\,t) + B\,\sin(3\,t)$ where A and B are arbitrary constants.

(c) The initial conditions are $x(1) = 0$ and $\dot{x}(1) = v_0$, so we get the equations

$$\begin{aligned} 0 &= x(1) = A\,\cos(3) + B\,\sin(3), \\ v_0 &= \dot{x}(1) = -3\,A\,\sin(3) + 3\,B\,\cos(3). \end{aligned}$$

Write the system as a matrix equation:

$$\begin{bmatrix} \cos(3) & \sin(3) \\ -3\,\sin(3) & 3\,\cos(3) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ v_0 \end{bmatrix}.$$

The determinant of the coefficient matrix is 3. Then from the formula for the inverse of a 2×2 matrix we get

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3\,\cos(3) & -\sin(3) \\ 3\,\sin(3) & \cos(3) \end{bmatrix} \begin{bmatrix} 0 \\ v_0 \end{bmatrix} = \frac{v_0}{3} \begin{bmatrix} -\sin(3) \\ \cos(3) \end{bmatrix}.$$

Consequently,

$$x(t) = \frac{v_0}{3} \big(-\sin(3)\,\cos(3\,t) + \cos(3)\,\sin(3\,t)\big).$$

Problem 5: (16 points) Let k be a real number and consider the system

$$\begin{aligned}(1-k)\,x+(1-k)\,y&=1,\\ k\,y&=k.\end{aligned}$$

(a) (4 points) Find all the values of k, if any, for which this system has infinitely many solutions.

(b) (4 points) Find all the values of k, if any, for which this system has no solutions.

(c) (8 points) Find all the values of k, if any, for which this system has a unique solution. Determine x and y in this case.

Solution: We first calculate the determinant of the system matrix: $(1-k)\,k-(1-k)\,0=k\,(1-k)$. If $k=0$ or $k=1$, then the determinant is zero, so we need to investigate these cases separately. If $k\neq 0,1$, then the system has a unique solution.

<u>Case 1 — $k=0$:</u> The second equation vanishes, and the first takes the form $x+y=1$. In this case, we have infinitely many solutions.

<u>Case 2 — $k=1$:</u> The first equation takes the form "$0=1$" which clearly does not have a solution.

<u>Case 3 — $k\neq 0,1$:</u> In this case, we have a unique solution. The second equation takes the form $y=1$. Since $k\neq 1$, we can divide the first equation by $(1-k)$ which gives $x+y=1/(1-k)$. Using that $y=1$, we then find that $x=1/(1-k)-1$.

It only remains to summarize the different cases:

(a) The system has infinitely many solutions when $k=0$.

(b) The system has no solutions when $k=1$.

(c) When $k\neq 0,1$, the system has the unique solution

$$\begin{aligned}x&=\frac{1}{1-k}-1,\\ y&=1.\end{aligned}$$