#1

(a) False

$$(AB)^T = B^T A^T \neq A^T B^T$$

(b) ~~Tr~~ True

~~the det triangular matrix~~

the determinant of triangular matrix is the product of diagonal entries.

(c) True

Since $|A| = |A^T|$

And $|A| \neq 0$

$\Rightarrow$ A^T is invertible

(d) True

Because for any $x \in W$

$0 \cdot x \in W \Rightarrow 0 \in W$.

(e) True

If $|A| \neq 0$, the rank of matrix $= n$ $\Rightarrow$ columns are linearly independent. And R^n of dimension n $\Rightarrow$ columns are basis.

#2. $F(x) = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$

(a). $F(x) \cdot F(y) = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x+y & 1 \end{bmatrix}$

$F(x+y) = \begin{bmatrix} 1 & 0 \\ x+y & 1 \end{bmatrix}$

$\Rightarrow$ $F(x) \cdot F(y) = F(x+y)$

#2.

(b) $\frac{dF}{dx} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$,

Find F^{-1}

$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ x & 1 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -x & 1 \end{array}\right]$

$\Rightarrow F^{-1} = \begin{bmatrix} 1 & 0 \\ -x & 1 \end{bmatrix}$

$\frac{dF^{-1}}{dx} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$

$\Rightarrow \frac{dF}{dx} + \frac{d(F^{-1})}{dx} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(c). $B = \frac{dF}{dx} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

since $B^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ for all $n > 2$,

$B^n = B^2 \cdot B^{n-2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot B^{n-2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

then $e^B = I + \sum_{n=1}^{\infty} \frac{B^n}{n!}$

$= I + \left(B + \frac{B^2}{2} + \frac{B^3}{6} + \cdots + \frac{B^n}{n!} + \cdots\right)$

$= I + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

#3.

(a) Since A and U are row-equivalent.

rank(A) = rank(U).

RREF.

$$U = \begin{bmatrix} 2 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ circle the pivot.

$\Rightarrow$ rank(U) = 3 $\Rightarrow$ rank(A) = 3.

(b). By RREF.

$$\begin{bmatrix} 2 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 2x_1 + x_2 \qquad - x_5 &= 0 \\ \cancel{2}\, x_3 \quad - 3x_5 &= 0 \\ x_4 + x_5 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_2 &= -2x_1 + x_5 \\ x_3 &= 3x_5 \\ x_4 &= -x_5 \end{aligned}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} \cancel{-2x_1+x_5}\ x_1 \\ -2x_1 + x_5 \\ 3x_5 \\ -x_5 \\ x_5 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 1 \\ 3 \\ -1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ basis of solution space $= \left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ -1 \\ 1 \end{pmatrix} \right\}$.

(c). Yes.

Because there are more unknowns than the equation and rank(A) = 3 < 4.

The system is consistent.

$\Rightarrow$ infinitely many solution.

#4.

(a) $\begin{pmatrix} \lambda & 4 \\ 1 & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

has unique solution

$\Rightarrow \begin{vmatrix} \lambda & 4 \\ 1 & \lambda \end{vmatrix} \neq 0 \Rightarrow \lambda^2 - 4 \neq 0$

$\Rightarrow \lambda \neq \pm 2$

And the unique solution must be $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(b) $z^2 - \lambda z + 1 = 0$ has a double root

$\Rightarrow (-\lambda)^2 - 4 = 0$

$\Rightarrow \lambda = \pm 2$

then $\begin{vmatrix} \lambda & 4 \\ 1 & \lambda \end{vmatrix} = 0$

the matrix is singular and the homogenous system is always consistent

$\Rightarrow$ system has infinitely many solution

(c) ~~Let $\lambda = 2$~~

~~z_0 be the double root of $z^2 - 2z + 1 = 0$~~

$\Rightarrow$ ~~$(z-1)^2 = 0$~~

~~$z = 1$ (double root) $\Rightarrow z_0 =$~~

z_0 be the double root of $z^2 - \lambda z + 1 = 0$.

(x_0, y_0) be the solution of system.

$$z_0 = \frac{\lambda}{2},$$

$$\Rightarrow \quad z_0^2 - \lambda z_0 + 1 = 0$$

$$\Rightarrow \quad \lambda^2 = 4.$$

And. $\begin{cases} \lambda x_0 + 4y = 0 \\ x_0 + \lambda y_0 = 0. \end{cases}$

$$\Rightarrow \quad \lambda x_0 = -4y_0$$

$$x_0 = -\lambda y_0.$$

$$x_0^2 = x_0 \cdot x_0 = -\lambda y_0 \cdot x_0 = 4y_0^2$$

$$\Rightarrow \quad x_0^2 - 4y_0^2 = 0.$$

And. $1 - 5z_0^2 = 1 - 5 \cdot \left(\frac{\lambda}{2}\right)^2$

$= 1 - \frac{5}{4}\lambda^2$ And $\lambda^2 = 4$

$= 1 - 5 = -4.$

$$\Rightarrow \quad x_0^2 + (1 - 5z_0^2) y_0^2 = 0$$

#5.

(a). for. $M_1, M_2 \in W$. $M_1 = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & c_1 \end{pmatrix}$ $M_2 = \begin{pmatrix} a_2 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$M_1 + M_2 = \begin{pmatrix} a_1+a_2 & 0 & 0 \\ 0 & b_1+b_2 & 0 \\ 0 & 0 & c_1+c_2 \end{pmatrix} \in W$$

$cM_1 \in W$

$\Rightarrow$. W is a subspace of V.

(b). ①. $A_1\ A_2\ A_3$ are lin. Independent

$c_1A_1 + c_2A_2 + c_3A_3 = 0$.

$$\Rightarrow \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow c_1 = c_2 = c_3 = 0$

$\Rightarrow A_1\ A_2\ A_3$ are Linearly independent.

②. $\text{span}\{A_1\ A_2\ A_3\} = W$.

~~since~~ let $M \in W$, $M = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

then $M = aA_1 + bA_2 + cA_3$

$\Rightarrow$ $\text{span}\{A_1\ A_2\ A_3\} = M$

$\Rightarrow \{A_1, A_2\ A_3\}$ are basis for W.

(c). No. since $A_4 = A_1 + 2A_2$

$\Rightarrow \{A_1\ A_2\ A_4\}$ are Linear dependent.

Extra credic.

$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ not invertible.

$\Rightarrow |A| = \begin{vmatrix} a & b \\ c & -a \end{vmatrix} = 0 \Rightarrow -a^2 - bc = 0.$

$\Rightarrow a^2 + bc = 0$

$A^2 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$

$= \begin{bmatrix} a^2+bc & 0 \\ 0 & a^2+bc \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\vdots$

$\therefore A^n = A^2 \cdot A^{n-2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for $n \geq 2$.