APPM 2360: Section exam 2

7.00pm – 8.30pm, February 11, 2009.

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) recitation section (4) your instructor's name, and (5) a grading table. Text books, class notes, and calculators are NOT permitted. A one-page crib sheet is allowed.

Problem 1: (36 points) Give a brief answer to each question. Box your answer. Each correct answer earns 4 points. No work given for this question will be graded.

(a) Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be non-zero vectors in $\mathbb{R}^7$. Let r denote the dimension of Span$\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}$. What are the possible values of r?

(b) Set $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, and $X = A\,B\,C$.

Compute $\det(X)$. *Hint:* $\det(A) = -3$ and $\det(C) = -2$.

(c) Which of the following sets constitute basis sets for $\mathbb{P}_2$ (the set of all polynomials of degree at most 2):

$$S_1 = \{1,\, t,\, t^2\}, \quad S_2 = \{1,\, t^2\}, \quad S_3 = \{1+t,\, 1-t,\, t^2\}, \quad S_4 = \{0,\, 1,\, t,\, t^2\}.$$

(d) Assuming the standard addition and multiplication rules, determine which of the following sets are vectors spaces.

(i) All 4×5 matrices A such that $a_{2,3} = 0$;

(ii) All vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ that satisfy $x_1 + x_2 = 3$.

(e) Set $A = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$. Find $A^T B$.

(f) Let A be an 8×5 matrix. If the rank of A is 2, what is the dimension of the vector space $V = \{\mathbf{x} \in \mathbb{R}^5 :\ A\,\mathbf{x} = 0\}$.

(g) For which values of t is the matrix $A = \begin{bmatrix} 1 & t \\ 1 & (1+t) \end{bmatrix}$ invertible?

(h) "If a 4×4 matrix A is invertible, then A^5 is also invertible." Answer TRUE or FALSE.

(i) "The dimension of the vector space of all $m \times n$ matrices is $m + n$." Answer TRUE or FALSE.

For question 2 — 5, motivate your answers. A correct answer with no work may receive no credit, while an incorrect answer with some correct work may result in partial credit.

Problem 2: (16 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}.$$

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ denote the columns of A, so that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) (8 points) Is A invertible? If yes, find the inverse; if not, substantiate your claim.

(b) (4 points) Is the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

(c) (4 points) Determine for which real numbers t (if any), the vector $\mathbf{b} = \begin{bmatrix} 1 \\ t \\ 1 \end{bmatrix}$ belongs to Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Hint: Verify your calculations in (a) carefully since a wrong answer may influence your answers to (b) and (c).

Problem 3: (16 points) Set

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 0 \\ -1 & 0 & -2 & 1 & 1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}.$$

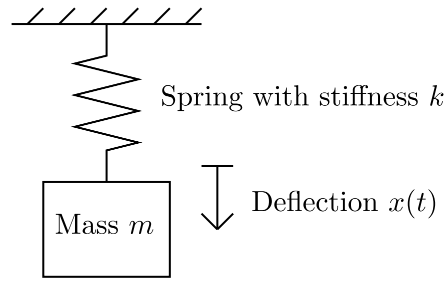
(a) (8 points) Give the general solution of the equation $A\mathbf{x} = \mathbf{b}$.

(b) (8 points) Construct a basis for the vector space $V = \text{Null}(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$.

Hint: A is row equivalent to the matrix $\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

Problem 4: (16 points) Consider the mechanical spring-mass system shown in the picture. Let $x = x(t)$ denote the deflection of the spring from its equilibrium position (counted positive downwards) at time t. The governing equation is

$$\ddot{x} + 9\,x = 0. \qquad (1)$$



(a) (4 points) Suppose that the body has mass m and the spring has stiffness k. What equation must m and k satisfy for the system to be modeled by (1)?

(b) (4 points) Give the general solution of (1).

(c) (8 points) Suppose that at time $t = 1$, the mass is at position $x = 0$ and has (downwards) velocity v_0. Determine the solution $x(t)$ of (1) that matches these conditions. (Your answer should depend on the parameter v_0.)

Problem 5: (16 points) Let k be a real number and consider the system

$$\begin{aligned} (1-k)\,x + (1-k)\,y &= 1, \\ k\,y &= k. \end{aligned}$$

(a) (4 points) Find all the values of k, if any, for which this system has infinitely many solutions.

(b) (4 points) Find all the values of k, if any, for which this system has no solutions.

(c) (8 points) Find all the values of k, if any, for which this system has a unique solution. Determine x and y in this case.