APPM 2360: Section exam 1

7.00pm – 8.30pm, February 11, 2009.

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) recitation section (4) your instructor's name, and (5) a grading table. Text books, class notes, and calculators are NOT permitted. A one-page crib sheet is allowed.

Problem 1: (28 points) For this problem, give the answer only. No motivation is required. Each correct answer earns 4 points (no partial credits will be given).

(a) Give the *general* solution to the equation $y' = -t\,y^2$.

(b) Give a solution to the equation $y' = -t\,y^2$ that satisfies $y(0) = 7$.

(c) Identify the equilibrium points to the equation $y' = y^3 - y$.
(You do not need to determine whether they are stable.)

(d) Consider the initial value problem

$$\begin{cases} y' &= (1 + t\,y)\,y \\ y(0) &= 7. \end{cases}$$

Compute an approximation to $y(1/2)$ by performing *one step* of the forwards Euler method (in other words, use the step size $h = 1/2$).

(e) Which of the following equations are both linear and homogeneous:
(1) $y'' + t^2\,y = 0$
(2) $y'' + t\,y^2 = 0$
(3) $y' + 2\,y + 1 = 0$
(4) $y' + 2\,y + t^2 = 0$.

(f) Determine all values of a for which Picard's theorem guarantees that the following initial value problem has at most one solution:

$$\begin{cases} y' &= t\,y^a \\ y(1) &= 0. \end{cases}$$

(g) The equation

$$y' + 2\,y = 8t^3 + 2 \tag{1}$$

has the particular solution

$$y_{\text{p}} = 4t^3 - 6t^2 + 6t - 2$$

Find the solution y of equation (1) that satisfies $y(0) = 2$.

Solutions to problem 1:

(a) $y(t) = \dfrac{1}{C + \frac{1}{2}\,t^2}$

(b) $y(t) = \dfrac{1}{\frac{1}{7} + \frac{1}{2}\,t^2}$

(c) $y = -1,\ 0,\ 1$

(d) $y(1/2) \approx 10.5$

(e) only (1)

(f) $a \geq 1$ or $a = 0$

(g) $y(t) = 4\,e^{-2\,t} + 4\,t^3 - 6\,t^2 + 6\,t - 2$

Comments:

(a) Use separation of variables: $-dy/y^2 = t\,dt$ and so $1/y = \frac{1}{2}\,t^2 + C$.

(b) $y(0) = 1/C$ so $C = 1/7$.

(c) We have $f(y) = y^3 - y = (y^2 - 1)\,y = (y-1)\,(y+1)\,y$. Now just read off the zeros!

(d) We have $y(1/2) \approx y(0) + h\,y'(0) = 7 + 0.5 \cdot (1 + 0 \cdot 7) \cdot 7 = 7 + 0.5 \cdot 7 = 10.5$.

(e) (2) is nonlinear. (3) and (4) are linear but non-homogeneous.

(f) Set $f(t, y) = t\,y^a$. Then $\frac{\partial f}{\partial y} = t\,a\,y^{a-1}$ which is continuous around the initial value if and only if $a \geq 1$ or $a = 0$.

The answer answer $a \geq 1$ gives full credit.

(g) The homogeneous solution is $y_{\text{h}} = C\,e^{-2\,t}$. It follows that the general solution is $y = C\,e^{-2\,t} + 4t^3 - 6t^2 + 6t - 2$. The initial condition implies that $2 = C - 2$ and so $C = 4$.

For question 2 — 4, motivate your answers. A correct answer with no work may receive no credit, while an incorrect answer with some correct work may result in partial credit.

Problem 2: (18 points) Consider the equation $y' + t\,y = t$.

(a) (9 points) Construct the general solution.

(b) (9 points) Construct the solution that satisfies $y(1) = 2$.

Solution:

(a) The integrating factor is $\mu(t) = e^{\int t\,dt} = e^{t^2/2}$. Multiplying both sides of the equation by $\mu(t)$ we obtain

$$\frac{d}{dt}\left[e^{t^2/2}\,y\right] = t\,e^{t^2/2}.$$

Integrate:

$$e^{t^2/2}\,y(t) = C + e^{t^2/2}.$$

Solve for y:

$$y(t) = C\,e^{-t^2/2} + 1.$$

(b) Inserting $y(1) = 2$ we find that

$$2 = C\,e^{-1/2} + 1$$

which implies that

$$C = e^{1/2}.$$

Inserting this C in the general solution yields

$$y(t) = e^{(1-t^2)/2} + 1.$$

Problem 3: (18 points) Consider the equation

$$y' = y - \frac{y}{t} + \frac{1}{t}. \tag{2}$$

(a) (9 points) Set $z(t) = t\,y(t)$. Construct a differential equation for z that is equivalent to equation (2).

(b) (9 points) Construct the general solution to (2).

Solution:

(a) We have $y = z/t$, and consequently

$$y' = \frac{z'}{t} - \frac{z}{t^2}.$$

Inserting into (2), we obtain

$$\frac{z'}{t} - \frac{z}{t^2} = \frac{z}{t} - \frac{z}{t^2} + \frac{1}{t}$$

which simplifies to

$$z' - z = 1. \tag{3}$$

(b) The general solution of (3) is

$$z = C\,e^t - 1.$$

Using that $y = z/t$ we then immediately get

$$y = \frac{C\,e^t}{t} - \frac{1}{t}.$$

Alternative solution for (b): Equation (2) can be written

$$y' + \left(\frac{1}{t} - 1\right)\,y = \frac{1}{t}.$$

This can be solved via the integrating factor

$$\mu(t) = \exp\int\left(\frac{1}{t} - 1\right)\,dt = \exp\left(\log(t) - t\right) = t\,e^{-t}.$$

We get

$$\frac{d}{dt}\left[t\,e^{-t}\,y\right] = e^{-t}.$$

Integrate:

$$t\,e^{-t}\,y = -e^{-t} + C.$$

Solving for y we obtain the answer given above.

Problem 4: (18 points)

(a) (6 points) A radioactive material, Dirtyum, is known to decay at a rate proportional to the amount present. It takes 2 years for 10 kg to decay down to 1 kg. Find $A(t)$, the amount of Dirtyum as a function of time, if there are 100 tons of Dirtyum at time $t = 0$.

(b) (6 points) Initially ($t = 0$), 100 tons of Dirtyum are deposited in an underground storage facility. In addition to this initial amount, more Dirtyum is dumped continuously in the facility at a rate totaling 5 tons per year. Write down a differential equation for $A(t)$ and an initial equation describing this situation.

(c) (6 points) Solve the differential equation in part (b) to find $A(t)$ and find what is the amount of Dirtyum in the facility as $t \to \infty$.

Note: Your answers may contain unevaluated formulas (for instance, "as $t \to \infty$, the amount of dirtyum will approach $3\cos(\log(4))$ tons").

Solution:

(a) Consider first a sample of dirtyum that is decaying without being replenished. Let $y(t)$ denote the amount of material at time t (in kg). We are given that $y(0) = 10$, so $y(t) = 10\,e^{-r\,t}$ for some constant r. To determine r, we use that $y(2) = 1$ which implies that $1 = 10\,e^{-2\,r}$. Solving for r, we find that

$$r = \frac{\log(10)}{2}.$$

If we initially start with $A_0 = 100$ tons, then the amount of material at time t is

$$z(t) = 100\,e^{-t\,\log(10)/2}\,\text{tons} = 100 \cdot 10^{-t/2}\,\text{tons}.$$

(b) Now let $A(t)$ denote the amount of dirtyum in a pile that contained $A_0 = 100$ tons at time $t = 0$ (in years), and to which $s = 5$ tons is added per year. The equation for $A(t)$ is then

$$A'(t) = -r\,A(t) + s. \tag{4}$$

The initial condition is

$$A(0) = A_0 = 100.$$

(c) The homogeneous solution of (4) is $A_{\text{h}} = C\,e^{-r\,t}$ and a particular solution is $A_{\text{p}} = s/r$. Consequently,

$$A(t) = A_{\text{h}}(t) + A_{\text{p}}(t) = C\,e^{-r\,t} + \frac{s}{r}.$$

We use the initial condition to determine C:

$$A_0 = C\,e^{-r\,0} + \frac{s}{r} = C + \frac{s}{r}.$$

It follows that $C = A_0 - s/r$, whence

$$A(t) = \left(A_0 - \frac{s}{r}\right)\,e^{-r\,t} + \frac{s}{r}.$$

As $t \to \infty$, the first term tends to zero and so

$$\lim_{t\to\infty} A(t) = \frac{s}{r} = \frac{5}{\log(10)/2} = \frac{10}{\log(10)}.$$

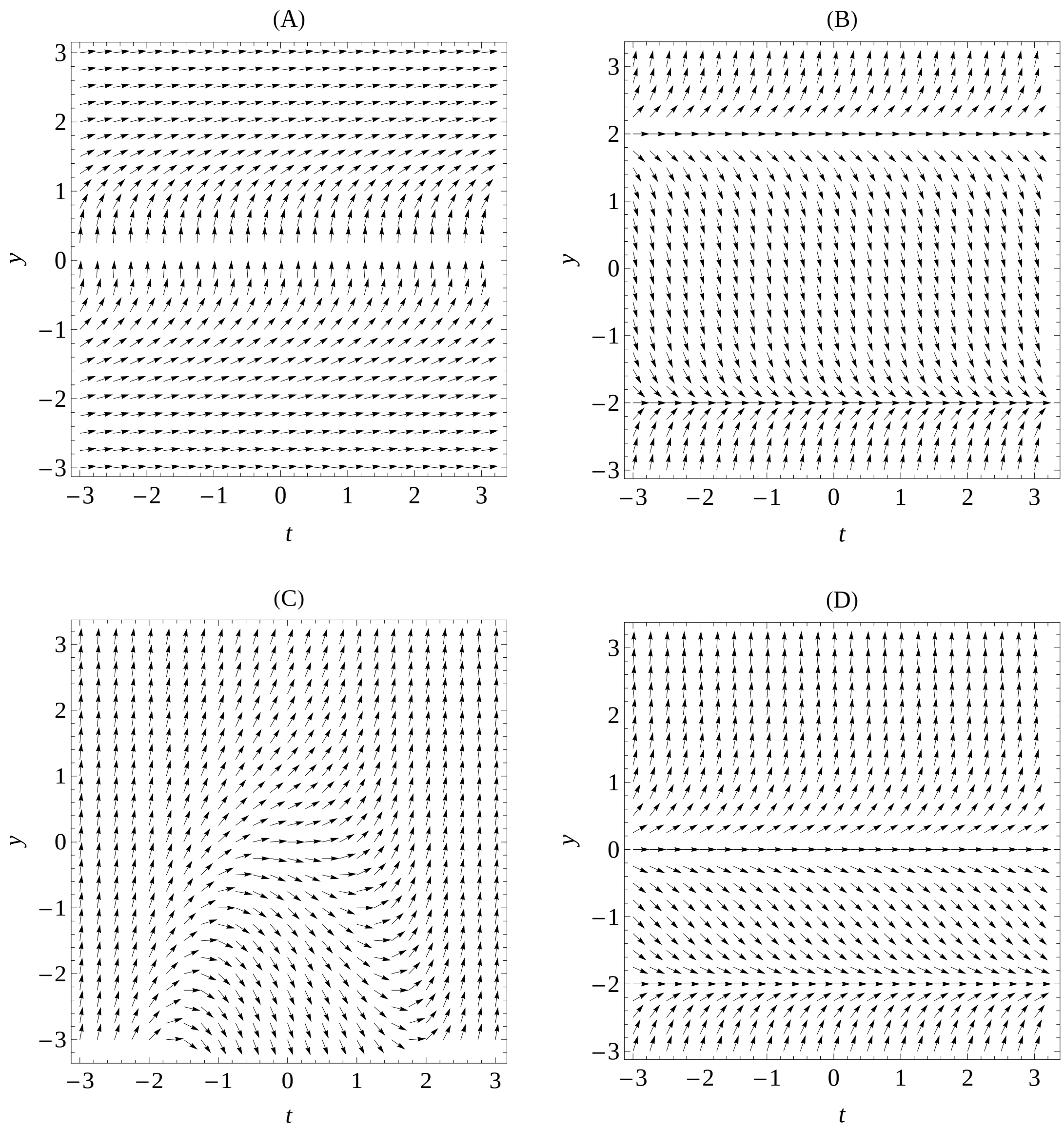
Problem 5: (18 points) For this problem, simply state the answer, no motivation is necessary. Consider the following first order ordinary differential equations:

(1) $y' = y + t^2,$ (2) $y' = y^2,$ (3) $y' = \sin(y),$

(4) $y' = y^2 - 4,$ (5) $y' = y^2 + 2\,y,$ (6) $y' = 1/y^2.$

(a) (9 points) Match direction fields (A)—(D) to the above differential equations. (Note that there are more equations than direction fields, so two equations have no corresponding direction fields.)

(b) (9 points) Identify the equilibria shown in **EACH GRAPH** and give their stability.



Solution:

Field (A): Equation (6). No equilibrium points.

Field (B): Equation (4). $y = 2$ is an unstable eq. point. $y = -2$ is a stable eq. point.

Field (C): Equation (1). No equilibrium points.

Field (D): Equation (5). $y = 0$ is an unstable eq. point. $y = -2$ is a stable eq. point.