Finding particular solutions to linear differential equations with constants coefficients

Let a_{n-1}, a_{n-2}, $\ldots$, a_1, a_0 be real numbers and consider the equation

$$y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_1\, y' + a_0\, y = f. \qquad (1)$$

Let r_1, r_2, $\ldots$, r_n be the roots of the characteristic equation

$$r^n + a_{n-1}\, r^{n-1} + \cdots + a_1\, r + a_0 = 0. \qquad (2)$$

If all the r_j's are distinct, then the general solution to (1) is

$$y(t) = \underbrace{c_1\, e^{r_1\, t} + c_2\, e^{r_2\, t} + \cdots + c_n\, e^{r_n\, t}}_{=y_{\rm h}(r)} + y_{\rm p}(t).$$

Note that if any r_j is a complex number, then you can rewrite the formula for $y_{\rm h}(t)$ using sines and cosines; if any r_j is a double root, then use the basis functions $e^{r_j\, t}$ and $t\, e^{r_j\, t}$, *etc.*

Now, how do you find a particular solution $y_{\rm p}$? The easiest way is to simply look it up in a table:

If:		then try:
$f(t) = b_0 + b_1\, t + \cdots b_p\, t^p$		$y_{\rm p}(t) = d_0 + d_1\, t + \cdots + d_p\, t^p$
$f(t) = e^{r\, t}$	(r is not a root of (2))	$y_{\rm p}(t) = d\, e^{r\, t}$
$f(t) = e^{r\, t}$	(r is a single root of (2))	$y_{\rm p}(t) = d\, t\, e^{r\, t}$
$f(t) = e^{r\, t}$	(r is a double root of (2))	$y_{\rm p}(t) = d\, t^2\, e^{r\, t}$
$f(t) = (b_0 + b_1\, t + \cdots + b_p\, t^p)\, e^{r\, t}$	(r is not a root of (2))	$y_{\rm p}(t) = (d_0 + d_1\, t + \cdots + d_p\, r^p)\, e^{r\, t}$
$f(t) = (b_0 + b_1\, t + \cdots + b_p\, t^p)\, e^{r\, t}$	(r is a single root of (2))	$y_{\rm p}(t) = (d_0 + d_1\, t + \cdots + d_p\, r^p)\, t\, e^{r\, t}$
$f(t) = (b_0 + b_1\, t + \cdots + b_p\, t^p)\, e^{r\, t}$	(r is a double root of (2))	$y_{\rm p}(t) = (d_0 + d_1\, t + \cdots + d_p\, r^p)\, t^2\, e^{r\, t}$
$f(t) = \cos(r\, t)$	($i\, r$ is not a root of (2))	$y_{\rm p}(t) = d_1\, \cos(r\, t) + d_2\, \sin(r\, t)$
$f(t) = \sin(r\, t)$	($i\, r$ is not a root of (2))	$y_{\rm p}(t) = d_1\, \cos(r\, t) + d_2\, \sin(r\, t)$
$f(t) = b_1\, \sin(r\, t) + b_2\, \cos(r\, t)$	($i\, r$ is not a root of (2))	$y_{\rm p}(t) = d_1\, \cos(r\, t) + d_2\, \sin(r\, t)$
$f(t) = b_1\, \sin(r\, t) + b_2\, \cos(r\, t)$	($i\, r$ is a single root of (2))	$y_{\rm p}(t) = d_1\, t\, \cos(r\, t) + d_2\, t\, \sin(r\, t)$

In each case, you have to determine the coefficients in $y_{\rm p}$ by plugging the "guess" into equation (1) and matching the terms.