Let us consider a number of differential equations:

Equation:	Solution
$y'(t) = t^3$	$y(t) = \frac{1}{4}\,t^4 + C$
$y'(t) = 2\,y(t)$	$y(t) = C\,e^{2\,t}$
$y'(t) = y(t)^2 - 4$	$y(t) = 2\,\frac{1 + C\,e^{4\,t}}{1 - C\,e^{4\,t}}$

The functions are solutions to the corresponding equation *for any real number C.*

How do you know which one is the "correct" one?

Typically, the value of y at some initial point t_0 is given.

Example:

$$\begin{cases} y' = 2\,y \\ y(1) = 3 \end{cases}$$

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Solution:

The "general solution" is $y(t) = C\,e^{2\,t}$.

Now insert the "initial value":

$$3 = C\,e^{2}.$$

We see that

$$C = 3\,e^{-2},$$

so the solution is

$$y(t) = 3\,e^{-2}\,e^{2\,t} = 3\,e^{2\,t-2}.$$

In this class, we will describe mathematical techniques for determining solutions of an initial value problem such as

$$\begin{cases} y' = & f(t,y) \\ y(t_0) = & y_0 \end{cases}$$

Before we do that, we will describe a technique that can tell much about the solution — without doing any "real" work!

Example: $\begin{cases} y' = t^3 \\ y(t_0) = y_0 \end{cases}$

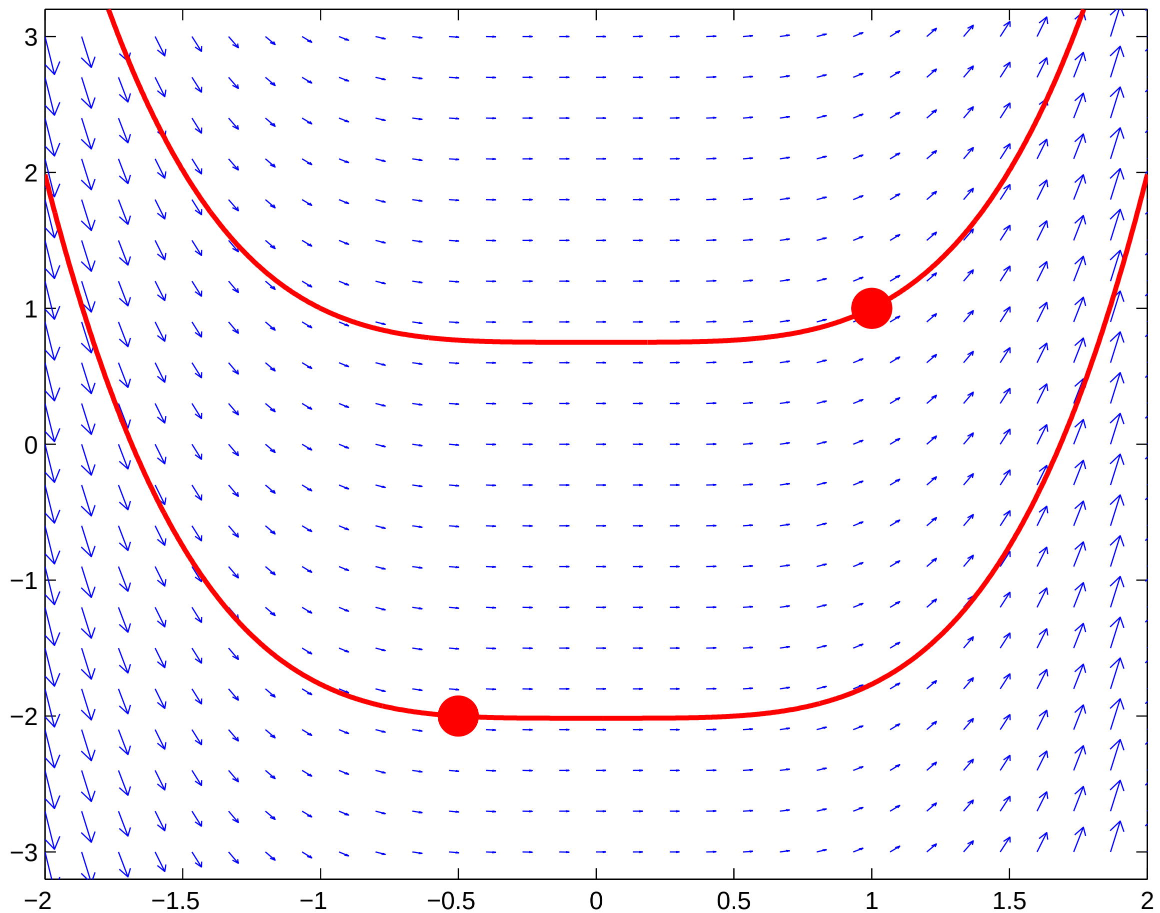
Consider the t-y plane. A solution $y = y(t)$ that passes through the point (t, y) has derivative $y'(t) = 2\,y$. Therefore, we can at any point (t, y) draw a short arrow that points in the direction of the curve $y(t)$. These arrows let us get an inkling of what the solutions may look like.

Problem: Try to sketch out the solutions that pass through the points:

(a) $(t_0,\, y_0) = (1,\, 1)$

(b) $(t_0,\, y_0) = (-1/2,\, -2)$

Example: $\begin{cases} y' = t^3 \\ y(t_0) = y_0 \end{cases}$



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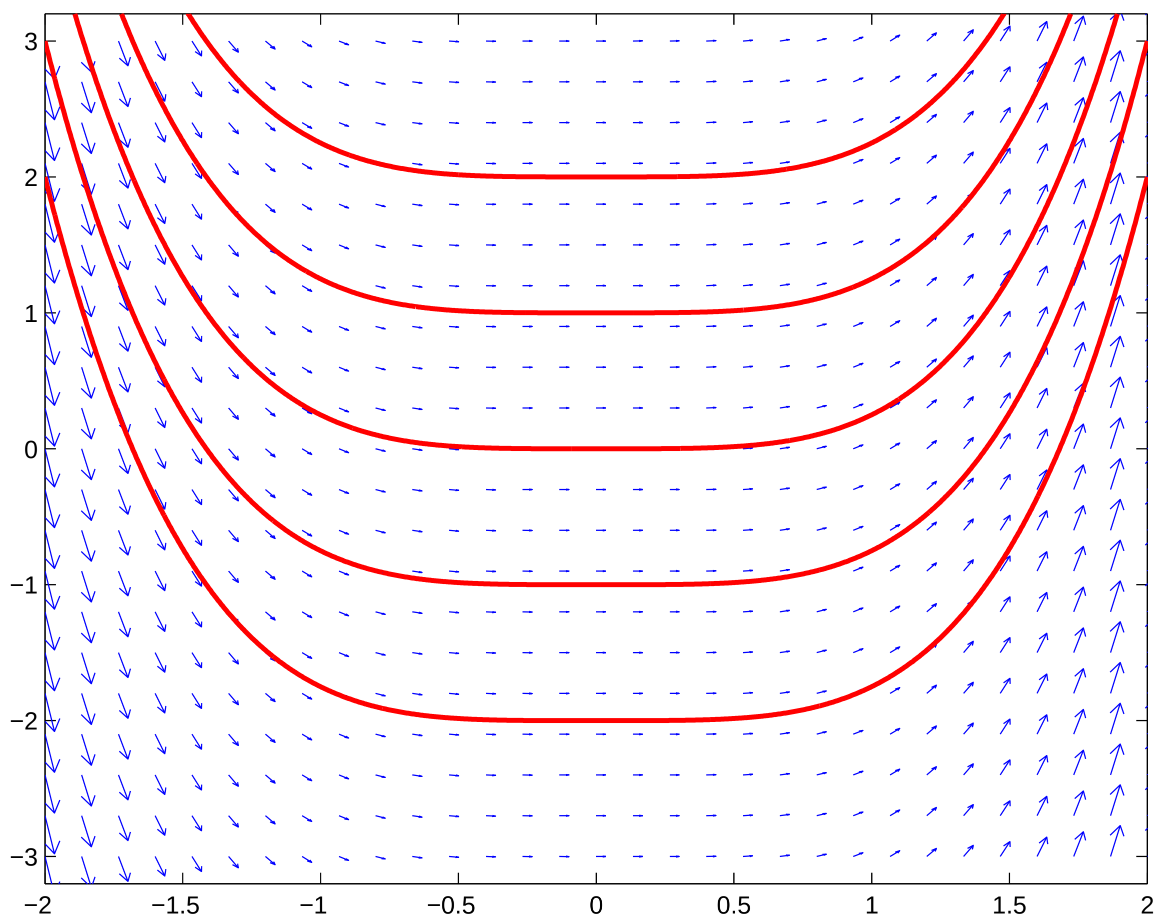
Recall that an exact solution of the equation takes the form $y(t) = \dfrac{1}{4}t^4 + C$.

Below are the plots for $C = -2, -1, 0, 1, 2$.



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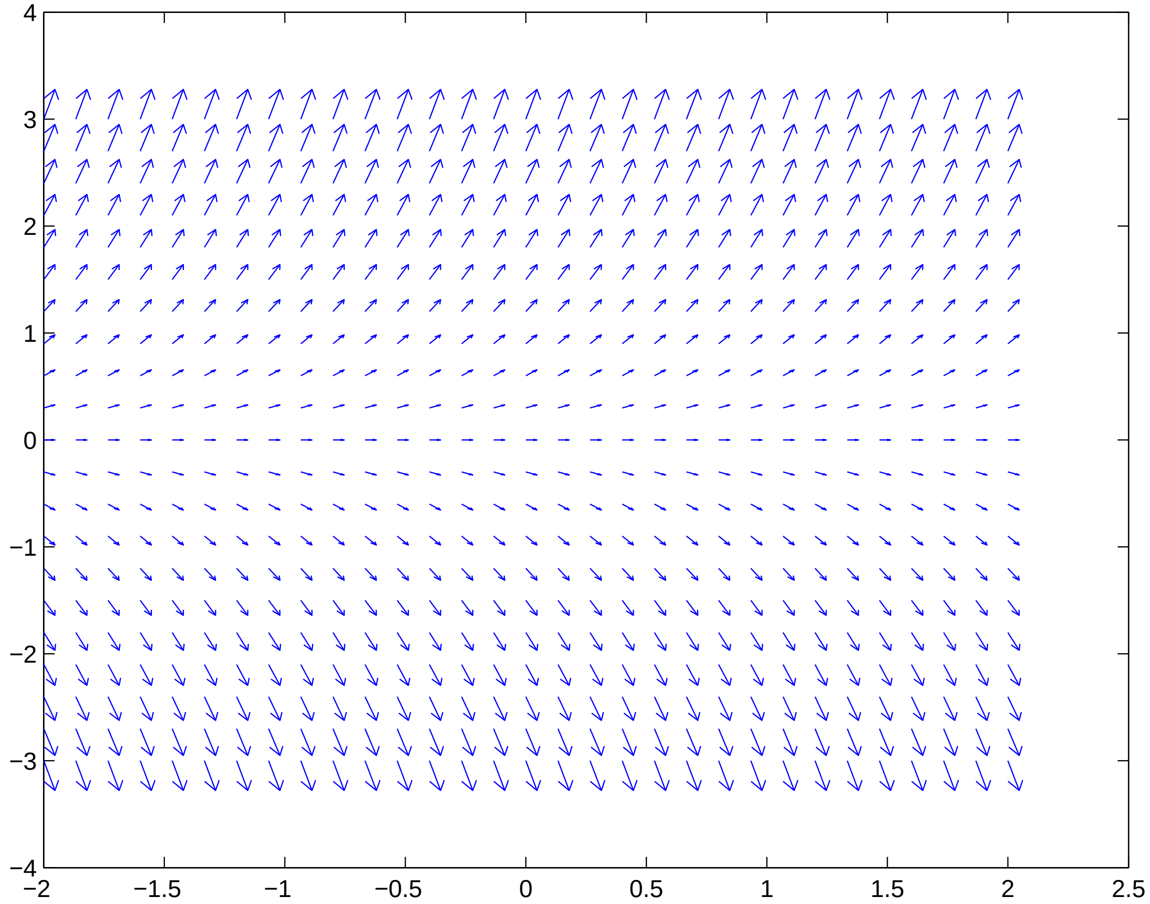

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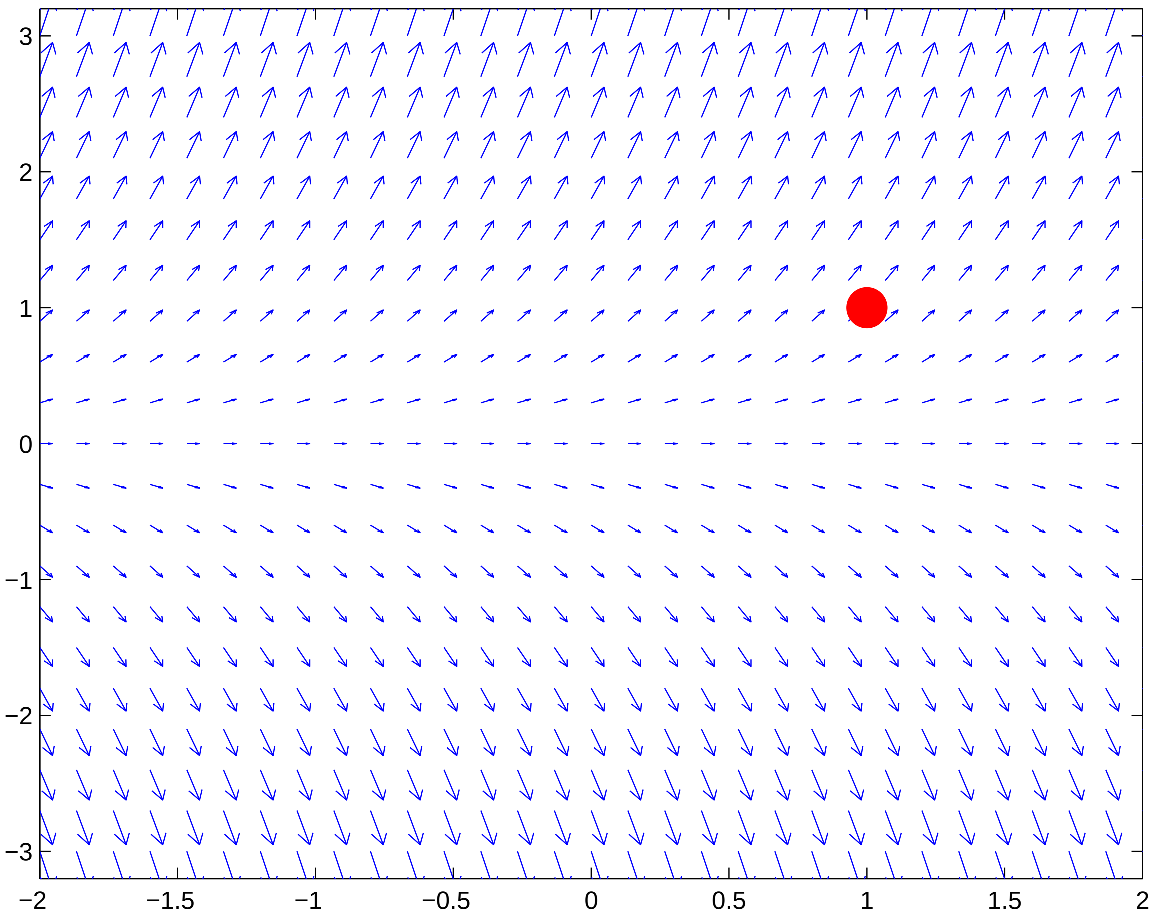
We can graphically sketch out the solution that passes through any given point. Say $t_0 = 1$ and $y_0 = 1$.



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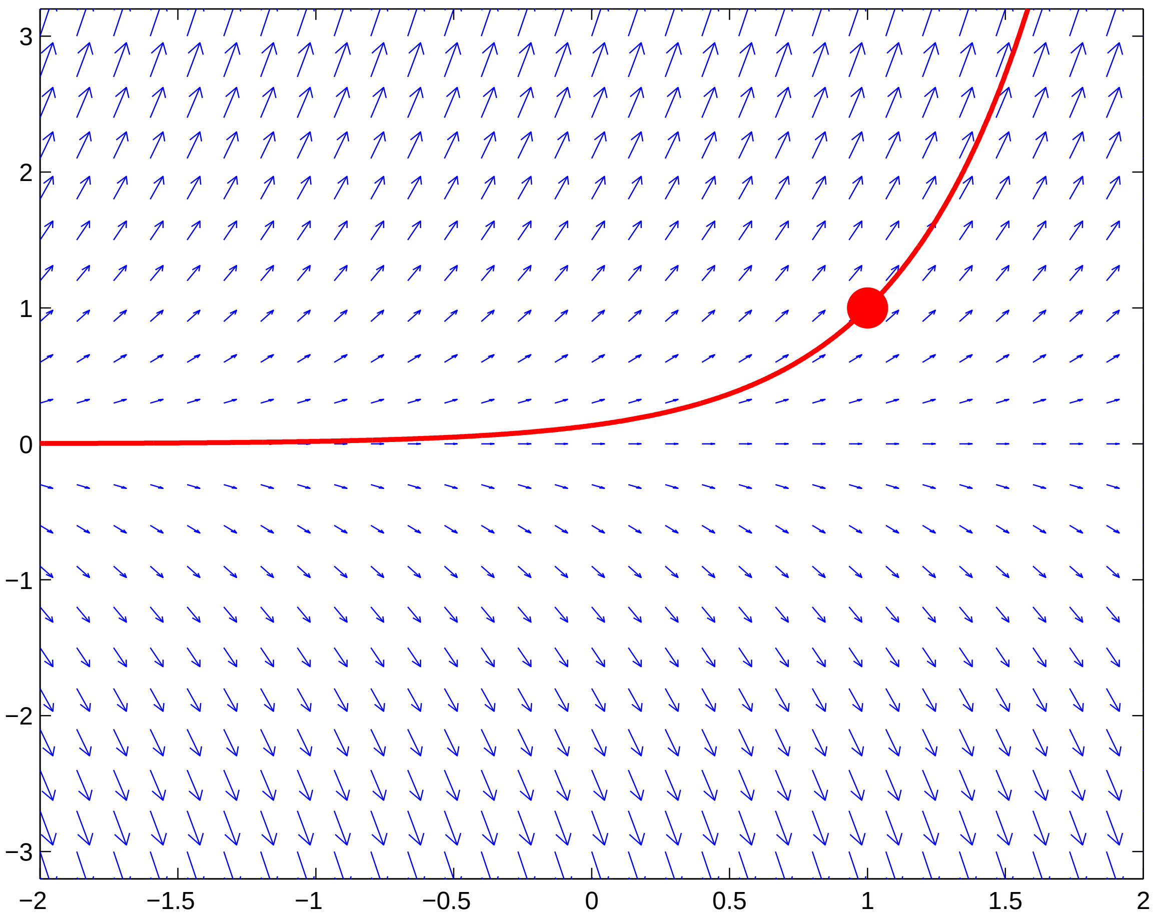

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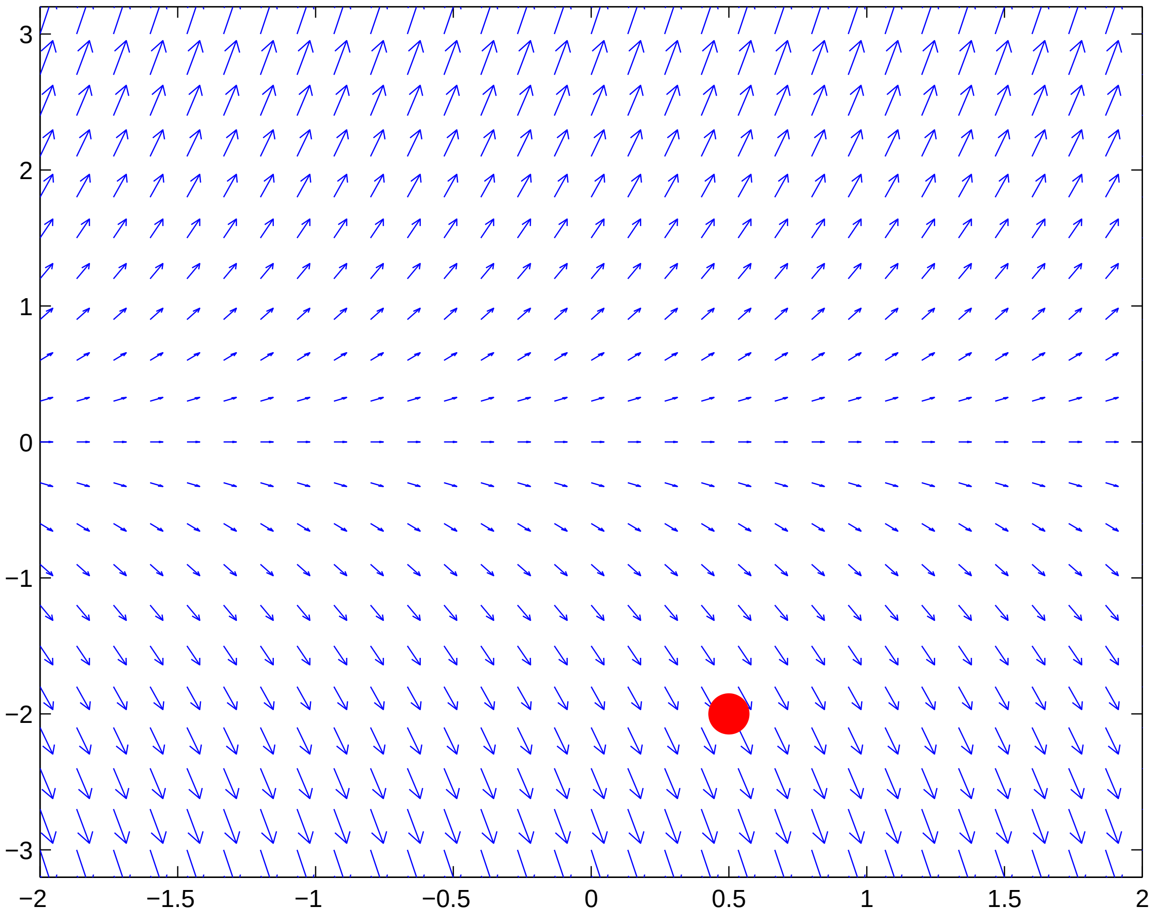
Let us take a different initial value: $t_0 = 0.5$ and $y_0 = -2$.



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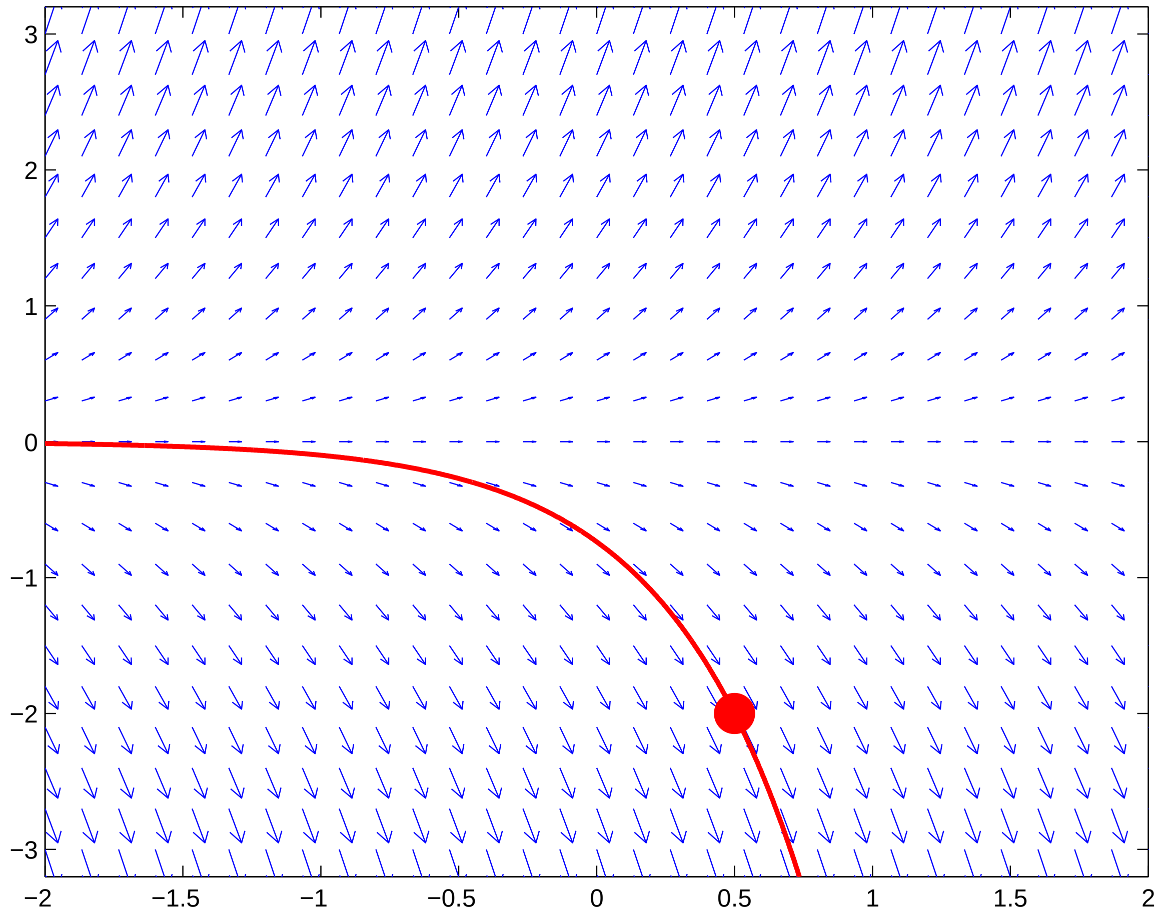

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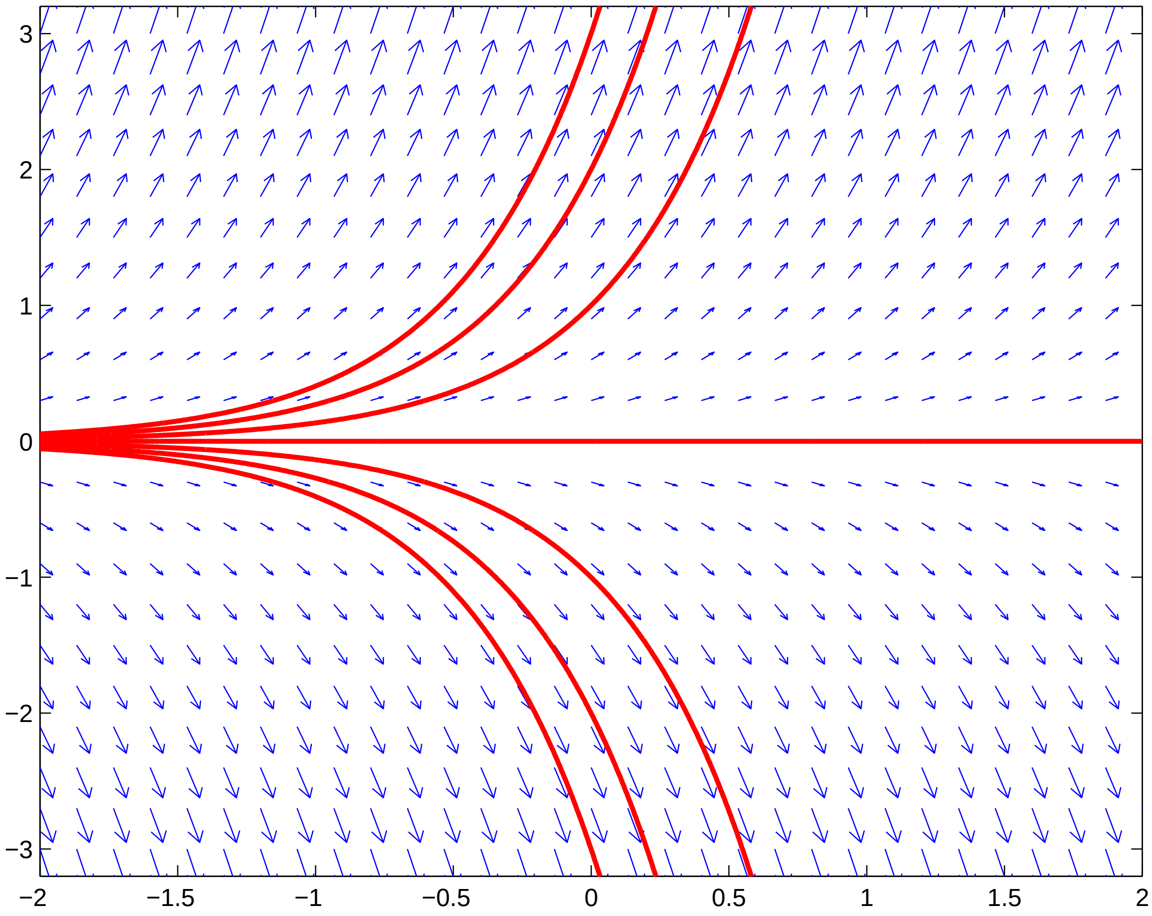
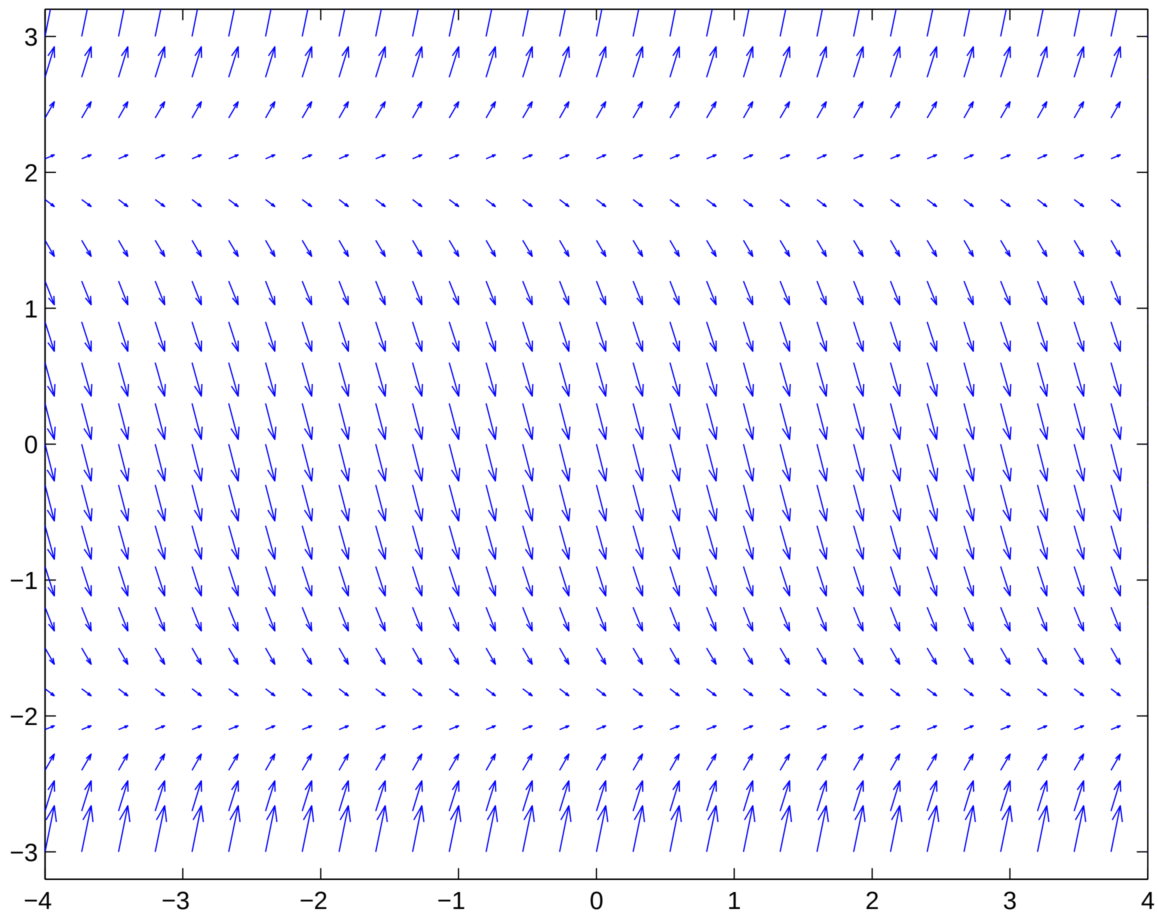


The figure shows $C = -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3.$

Yet another example: $\begin{cases} y' = & y^2 - 4 \\ y(t_0) = & y_0 \end{cases}$



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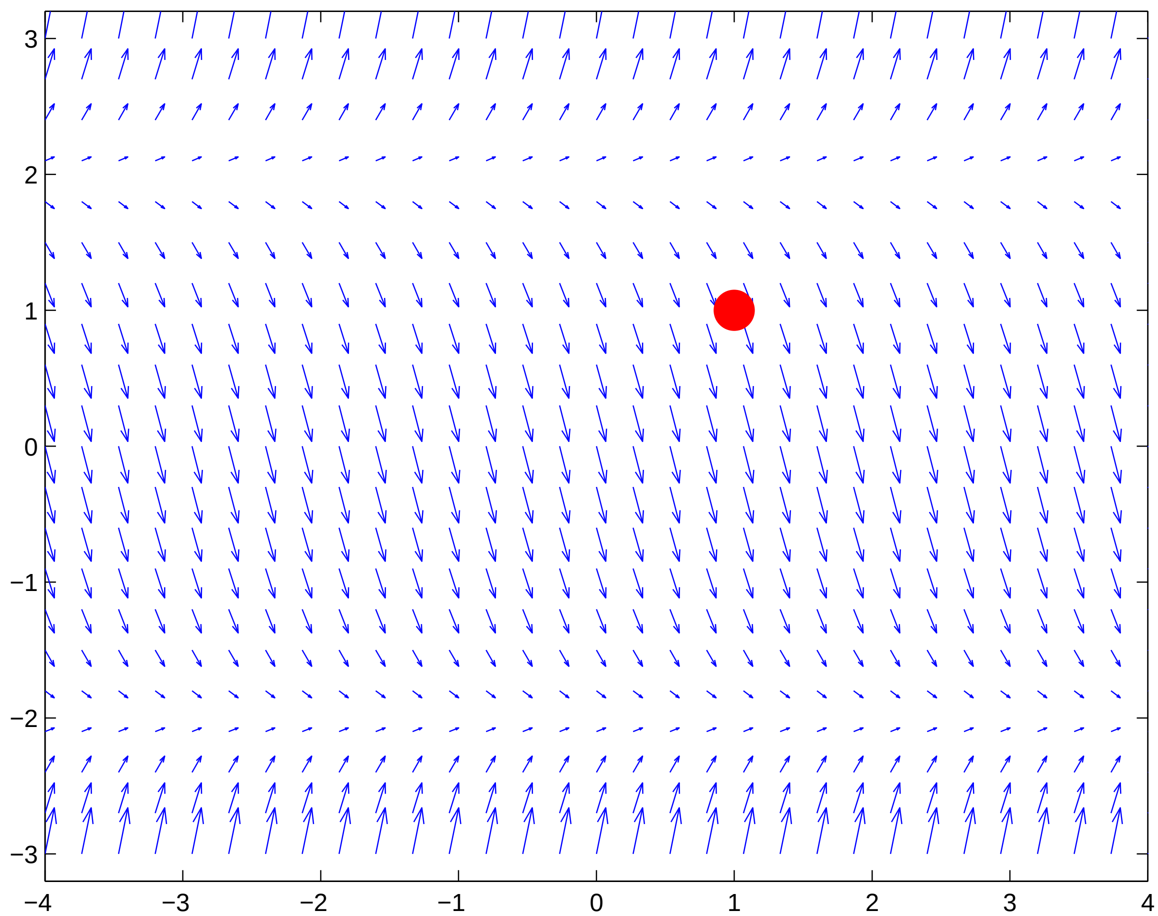
Set $t_0 = 1$ and $y_0 = 1$.



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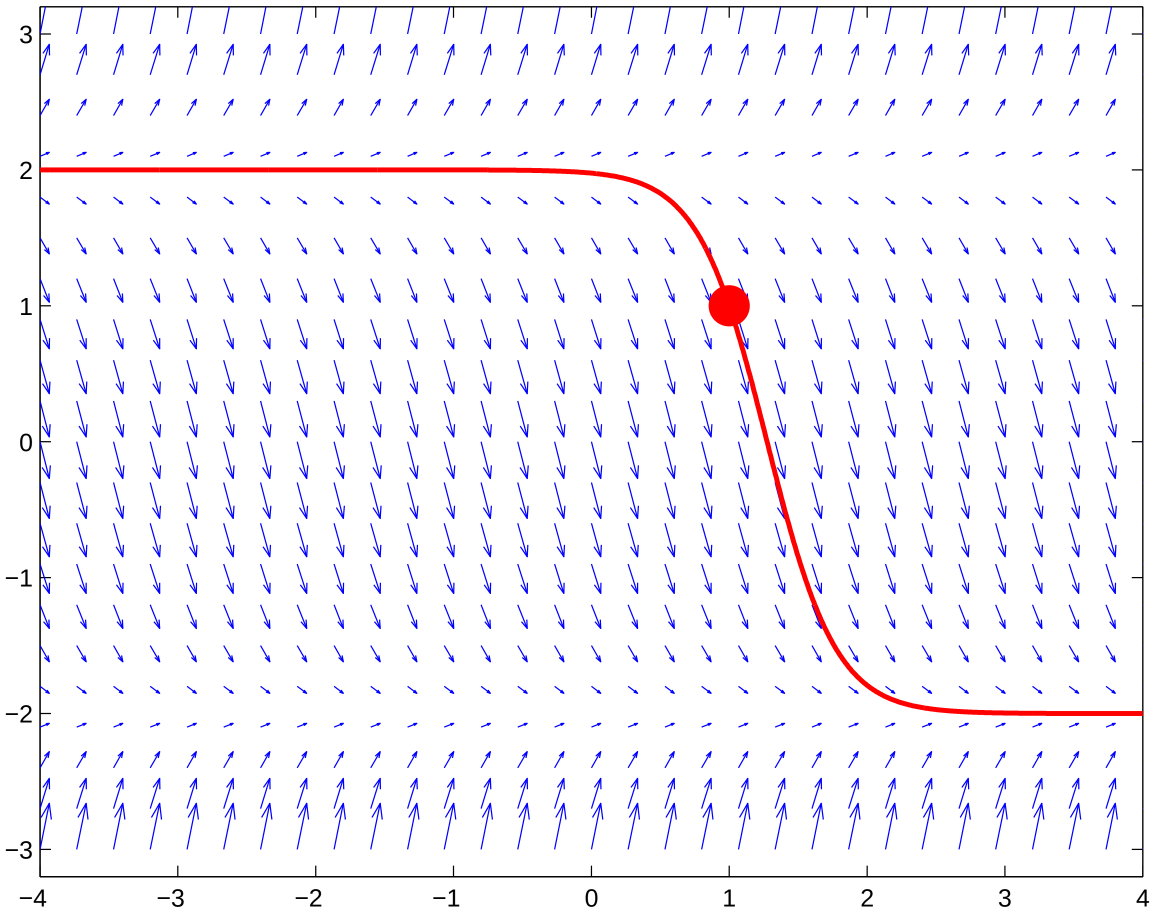

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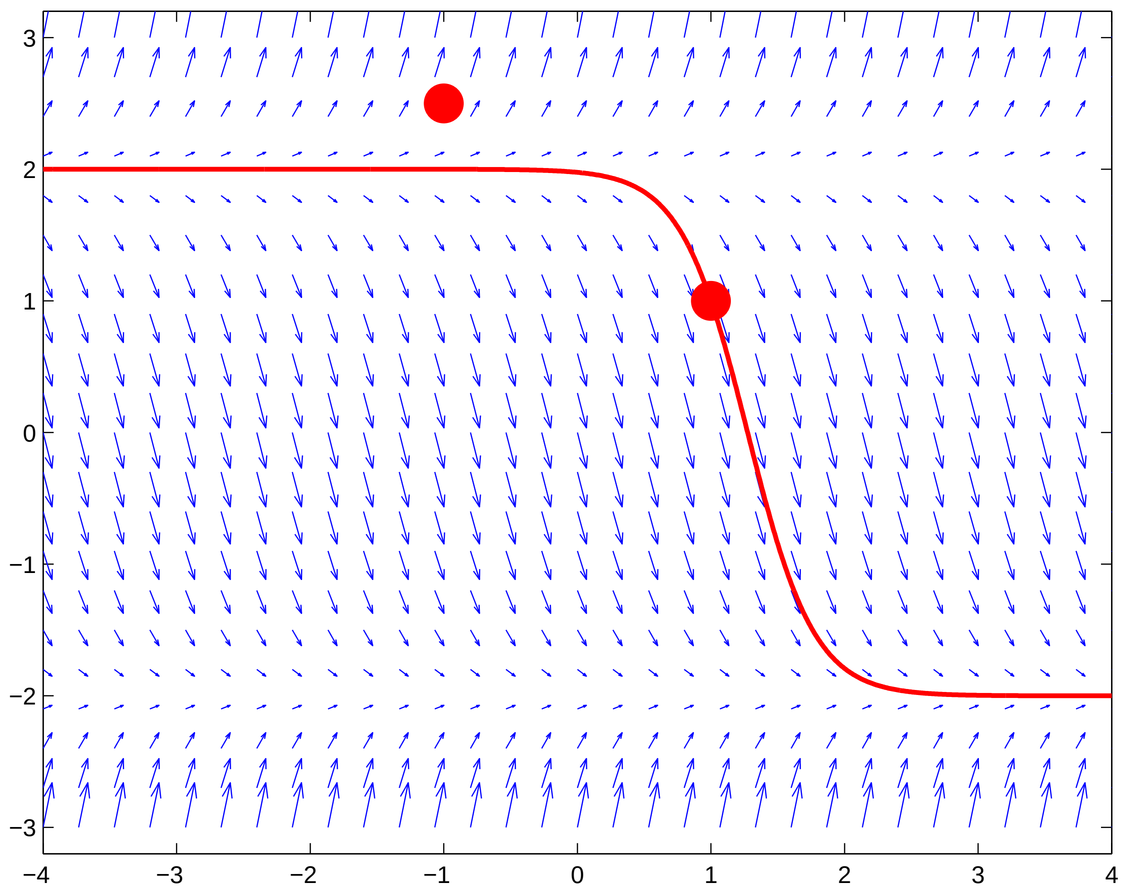

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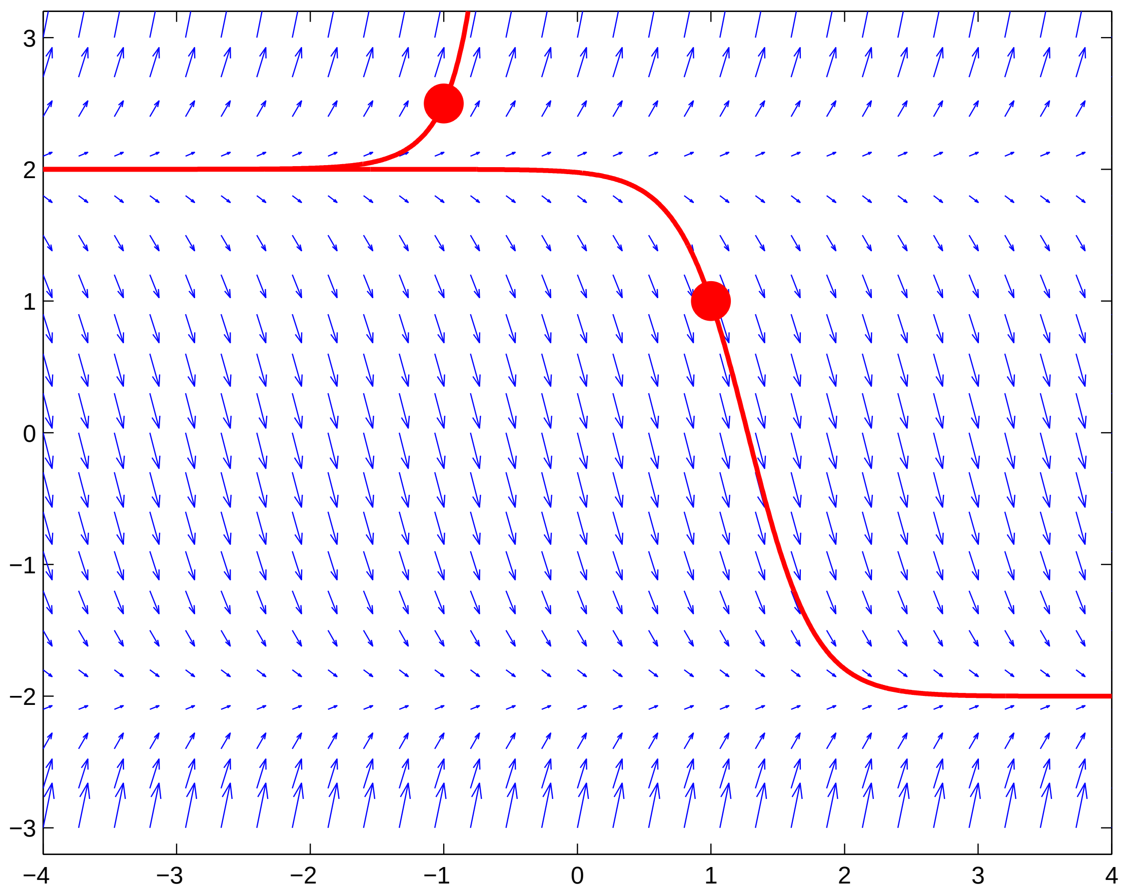


Given an equation

$$y'(t) = f(t, y)$$

it is of interest to determine any points $\hat{y}$ such that

$$f(t, \hat{y}) = 0.$$

These values $\hat{y}$ are the constant solutions of the ODE.

Note that if we set

$$y(t) \equiv \hat{y},$$

then

$$y'(t) = f(t, y) = f(t, \hat{y}) = 0.$$

And since y is constant, this is a solution!

Let us see how it works for the equation $\begin{cases} y' = & y^2 - 4 \\ y(t_0) = & y_0 \end{cases}$

The equation

$$f(t, \hat{y}) = 0$$

takes the form

$$\hat{y}^2 - 4 = 0,$$

which gives

$$\hat{y}^2 = 4,$$

which has solutions

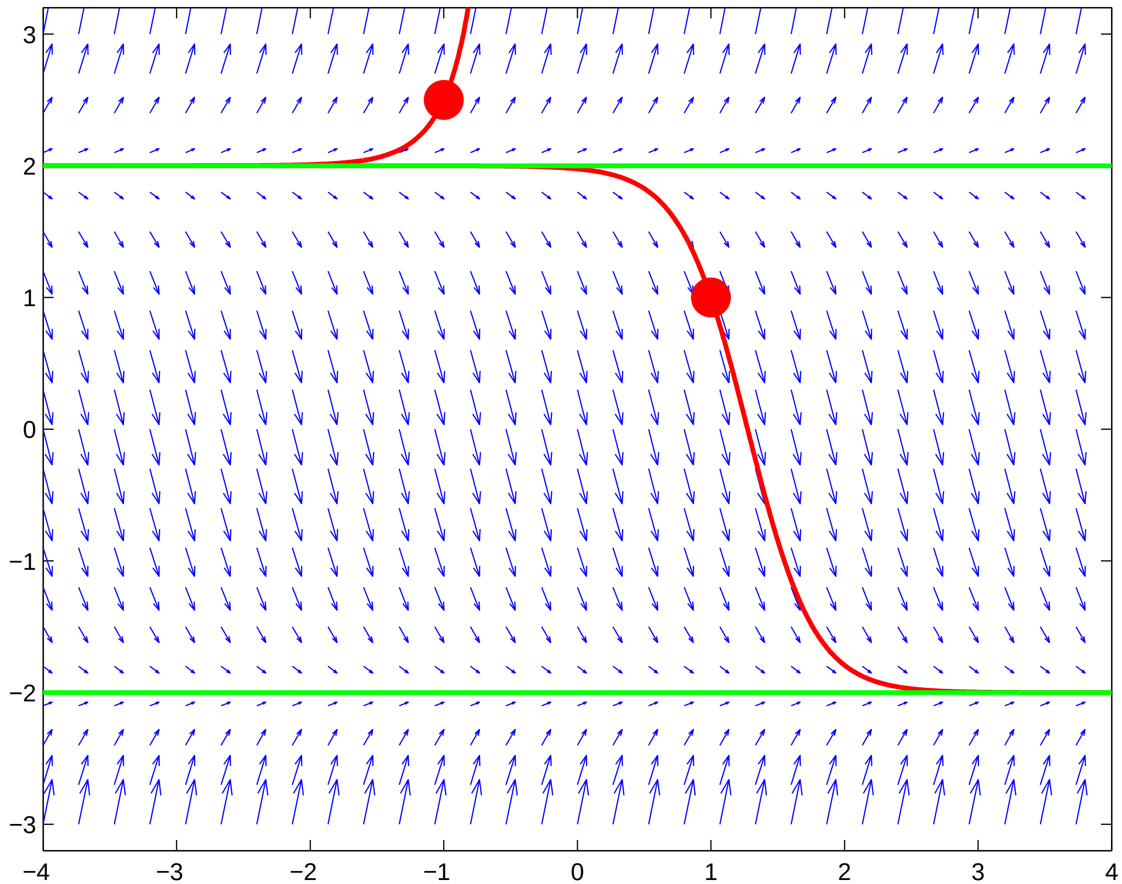
$$\hat{y} = 2$$

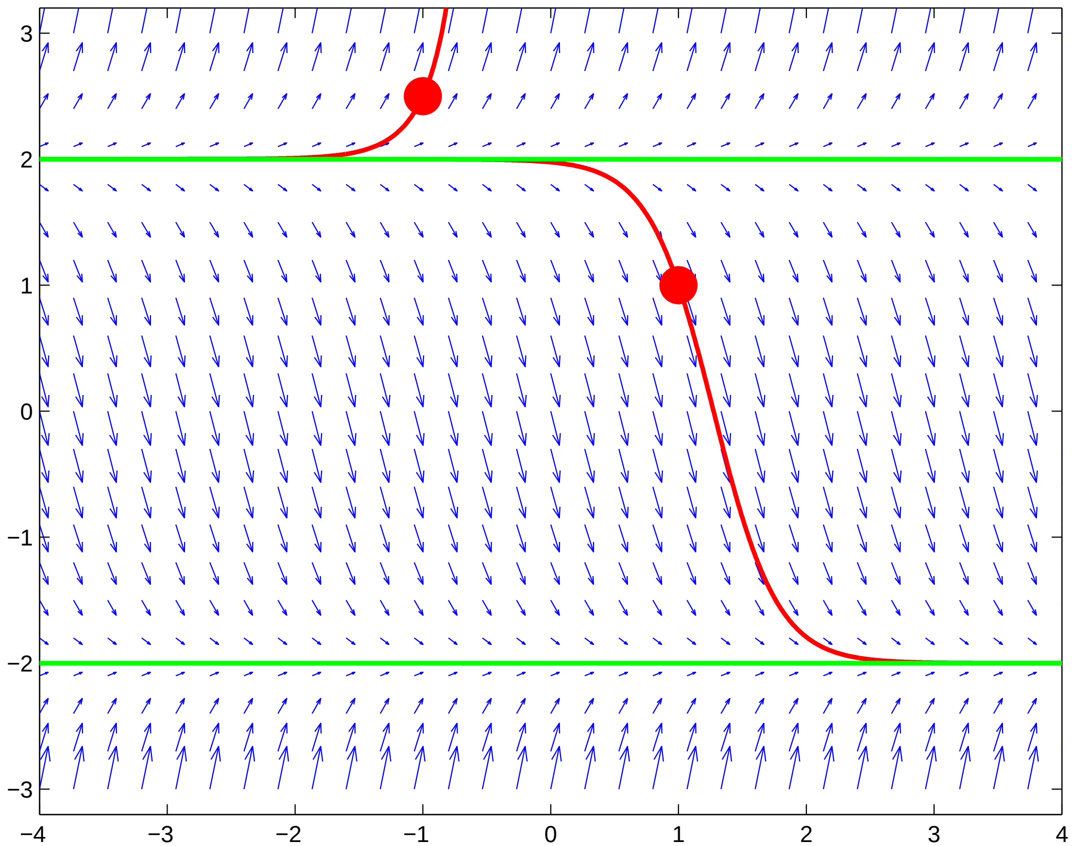
and

$$\hat{y} = -2.$$

Recall: $\begin{cases} y' = & y^2 - 4 \\ y(t_0) = & y_0 \end{cases}$

We mark these lines in the flow field:





Note that any solutions that start near the $y = 2$ line *move away from it,* while any solutions that start near the $y = -2$ line *move away towards it.*

Definition: Let $\hat{y}$ be an equilibrium point of an ODE $y'(t) = f(t, y)$. We say that

- $\hat{y}$ is *stable* if solutions near it tend towards it as $t \to \infty$,
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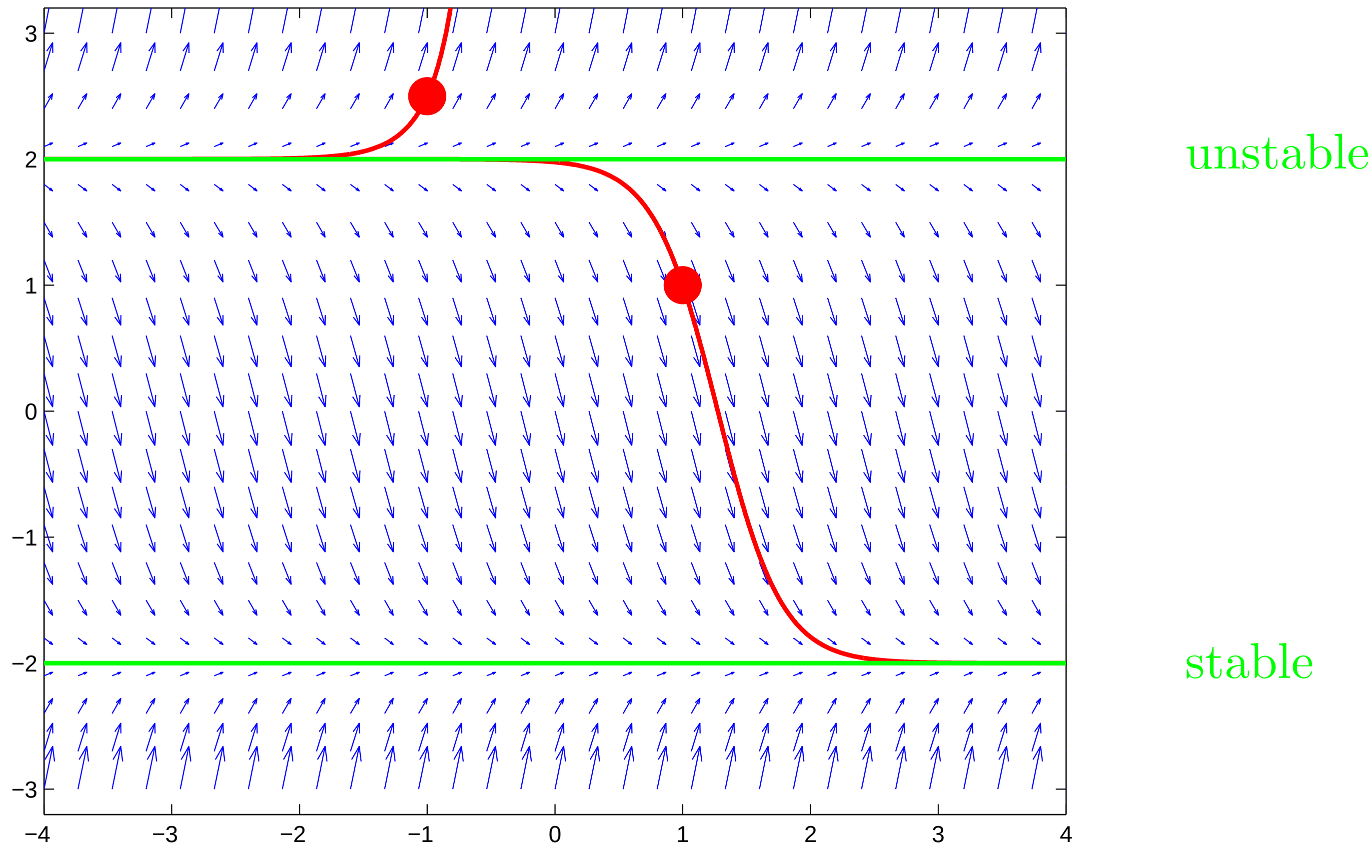

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This type of analysis can do much more than just identifying equilibrium points.

As an example, let us return to the equation
$$\begin{cases} y' = & y^2 - 4 \\ y(1) = & 1 \end{cases}$$

Question: Is the solution $y(t)$ convex near $t = 1$?

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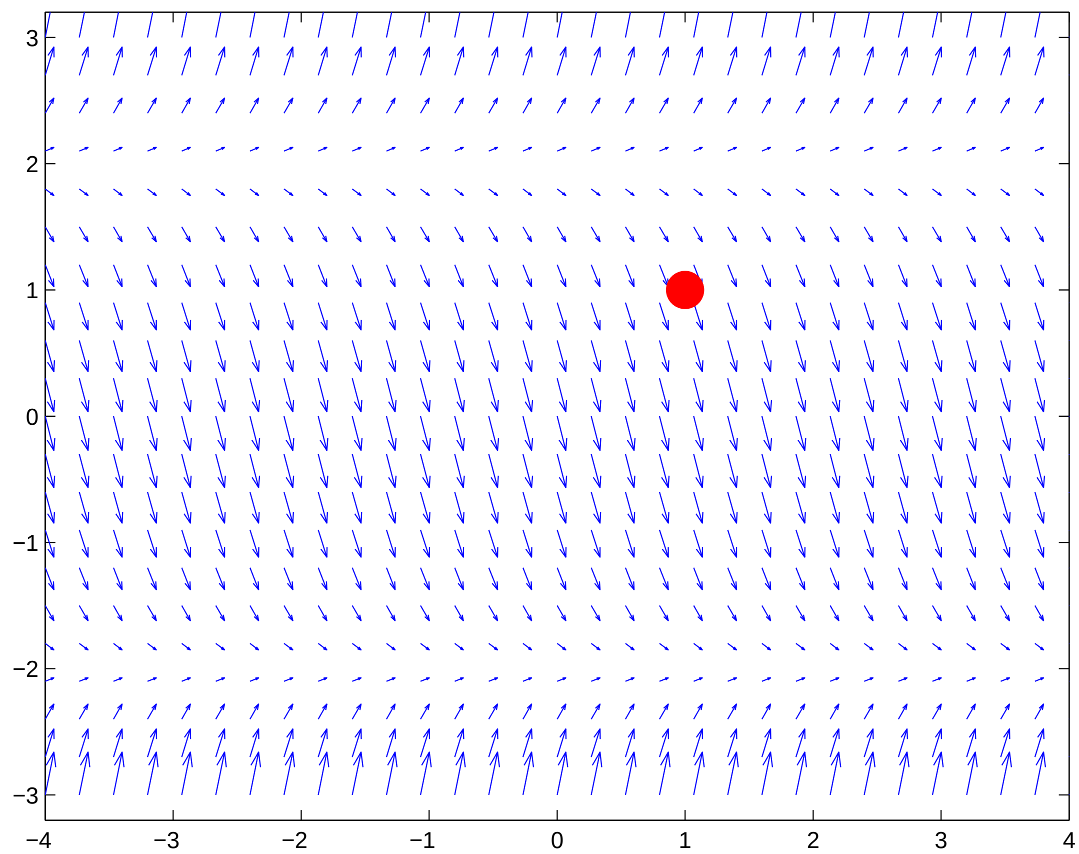
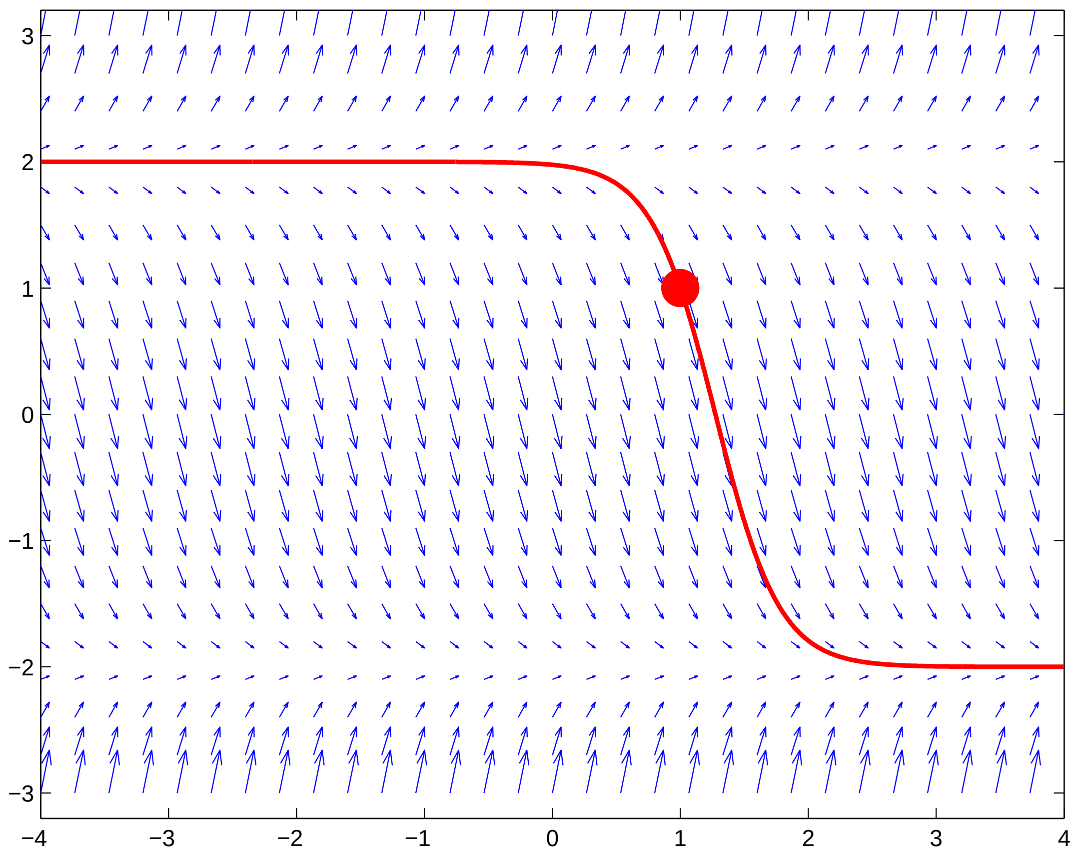


The phase diagram does not help much!

If we could somehow construct the full solution, that would help:



It appears to be concave.

But constructing the solution is hard — sometimes impossible!!

Can we manage without doing this?

Recall the equation: $\begin{cases} y' = & y^2 - 4 \\ y(1) = & 1 \end{cases}$

Now also recall that a function $y(t)$ is

- convex at t if $y''(t) > 0$, and
- concave at t if $y''(t) < 0$.
- (If $y''(t) = 0$ you cannot tell.)

We have

$$y''(t) = \frac{d}{dt} y'(t) = \frac{d}{dt}\big(y(t)^2 - 4\big) = y'(t)\,\frac{d}{dy}(y^2 - 4) = y'(t)\,2\,y(t) = (y^2 - 4)\,2\,y.$$

Inserting $y = 1$ we get

$$y''(t) = (1^2 - 4) \cdot 2 \cdot 1 = -6.$$

We conclude that $y(t)$ is concave at the point $(t, y) = (1, 1)$.

We did not need to solve the equation!

Key concept — geometrical solutions:

$$y'(t) = f(t, y).$$

- Gives "quantitative" information about the solution without the need to actually solve the equation.
- It is very useful to determine "equilibrium points".
  These are points $\hat{y}$ such that $f(t, \hat{y}) = 0$.
  Then the constant function $y(t) = \hat{y}$ is a solution of the ODE.
- Classification of equilibrium points: they can be stable or unstable.
- We saw how to determine whether the solution is convex of concave without solving the equation. There are many other similar things that can be done.