4.5

#5. $y'' + y = \sec t \tan t$.

$y'' + y = 0$

$r^2 + 1 = 0$

$r = \pm i$

$y_h = C_1 \cos t + C_2 \sin t$.

let $y_1 = \cos t \quad y_2 = \sin t$.

$f = \sec t \tan t$.

$\cancel{v_1'(t) = -y_2}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$$

$$V_1'(t) = \frac{-y_2 f}{W} = -\tan^2 t.$$

$$V_2'(t) = \frac{y_1 f}{W} = \tan t.$$

$$V_1(t) = \int v_1'(t)dt = \int -\tan^2 t\, dt = \int -(\sec^2 t - 1)\,dt$$

$$= \int 1 - \sec^2 t\,dt = t - \tan t.$$

$$V_2(t) = \int v_2'(t)\,dt = \int \tan t\,dt = \ln|\sec t|.$$

$$y_p = V_1 y_1 + V_2 y_2 = (t - \tan t)\cos t + \ln|\sec t|\sin t.$$

$$= t\cos t - \sin t + \ln|\sec t|\sin t.$$

$$\boxed{y = y_h + y_p = C_1\cos t + C_2 \sin t + t\cos t + \ln|\sec t|\sin t.}$$

#7. $y'' - 3y' + 2y = \frac{1}{1+e^{-t}}$.

$y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$r = 2 \quad r = 1$

$\Rightarrow \; y_h = C_1 e^{t} + C_2 e^{2t}$.

let. $y_1 = e^t \qquad y_2 = e^{2t}$.

$f = \frac{1}{1+e^{-t}}$.

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = e^{3t}$.

$v_1'(t) = \frac{-y_2 f}{W} = \frac{-e^{-t}}{1+e^{-t}}$.

$v_2'(t) = \frac{y_1 f}{W} = \frac{e^{-2t}}{1+e^{-t}}$

$v_1(t) = \int v_1'(t)\,dt = \int \frac{-e^{-t}}{1+e^{-t}}\,dt$ let $u = 1+e^{-t}$. $du = -e^{-t}dt$

$= \int \frac{1}{u}\,du = \ln(1+e^{-t})$.

$v_2(t) = \int v_2'(t) = \int \frac{e^{-2t}}{1+e^{-t}}\,dt = \int \frac{e^{-t}(1+e^{-t}) - e^{-t}}{1+e^{-t}}\,dt$

$= \int e^{-t}\,dt - \int \frac{e^{-t}}{1+e^{-t}}\,dt = -e^{-t} + \ln(1+e^{-t})$

$y_p = (e^t + e^{2t})\ln(1+e^{-t}) - e^{-t}$

$\boxed{y = y_h + y_p = C_1 e^t + C_2 e^{2t} + (e^t + e^{2t})\ln(1+e^{-t})}$

#12. $$y'' - y = \frac{e^t}{t}$$

$$y'' - y = 0$$

$$r^2 - 1 = 0$$

$$r = 1 \qquad r = -1$$

$$\Rightarrow y_h = C_1 e^t + C_2 e^{-t}$$

Let $y_1 = e^t \quad y_2 = e^{-t} \quad f = \frac{e^t}{t}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -2$$

$$v_1'(t) = \frac{-y_2 f}{W} = \frac{-e^{-t} \cdot \frac{e^t}{t}}{-2} = \frac{1}{2t}$$

$$v_2'(t) = \frac{y_1 f}{W} = \frac{e^t \cdot \frac{e^t}{t}}{-2} = -\frac{e^{2t}}{2t}$$

$$v_1(t) = \int v_1'(t)\, dt = \int \frac{1}{2t}\, dt = \frac{1}{2} \ln|t|$$

$$v_2(t) = \int v_2'(t)\, dt = \int_{t_0}^{t} -\frac{e^{2s}}{2s}\, ds$$

$$y_p = v_1 y_1 + v_2 y_2 = e^t \cdot \left(\frac{1}{2}\ln|t|\right) + e^{-t}\left(\int_{t_0}^{t} -\frac{e^{2s}}{2s}\, ds\right)$$

$$\boxed{y = y_h + y_p = C_1 e^t + C_2 e^{-t} + \frac{1}{2} e^t \ln|t| - \frac{1}{2} \int_{t_0}^{t} \frac{e^{2s}}{s}\, ds\, e^{-t}}$$

#15. $(1-t)y''+ty'-y=2(t-1)^2e^{-t}$.

$y_1(t)=t$. $y_2(t)=e^t$.

$(1-t)y_1''+ty_1'-y_1$

$=(1-t)0+t-t=0$

$(1-t)y_2''+ty_2'-y_2$

$=(1-t)e^t+te^t-e^t=0$

$\Rightarrow$ y_1 y_2 are solution of homogeneous equation.

Rewrite.

$y''+\frac{t}{1-t}y' \cancel{+} -\frac{y}{1-t}=2(1-t)e^{-t}$.

Let $f=2(1-t)e^{-t}$. $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} t & e^t \\ 1 & e^t \end{vmatrix}=(t-1)e^t$

$v_1'(t)=\frac{-y_2f}{W}=\frac{-e^t\cdot 2(1-t)e^{-t}}{(t-1)e^t}=2e^{-t}$.

$v_2'(t)=\frac{y_1f}{W}=\frac{t\cdot 2(1-t)e^{-t}}{e^t(t-1)}=-2te^{-2t}$.

$v_1(t)=\int v_1'(t)dt=\int 2e^{-t}dt=-2e^{-t}$

$v_2(t)=\int v_2'(t)dt=\int -2te^{-2t}dt=\int t\,de^{-2t}$

$=te^{-2t}-\int e^{-2t}dt=te^{-2t}+\frac{1}{2}e^{-2t}$.

$y_p=v_1y_1+v_2y_2=-2te^{-t}+te^{-t}+\frac{1}{2}e^{-t}$

$=e^{-t}(\frac{1}{2}-t)$

$\Rightarrow$ $\boxed{y=y_h+y_p=C_1t+C_2e^t+e^{-t}(\frac{1}{2}-t)}$

4.6

#2. $x'' + 2x' + 3x = \cos 3t$.

$x'' + 2x' + 3x = 0$.

$\Rightarrow r^2 + 2r + 3 = 0$

$r = -1 \pm \sqrt{2} i$.

$\Rightarrow \cancel{y_h = e^{-t}}$

$x_h = e^{-t}(c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}t)$.

Let $x_p = A\cos 3t + B \sin 3t$.

$x_p'' + 2x_p' + 3x_p = \cos 3t$

$\Rightarrow (-6A + 6B)\cos 3t + (-6A - 6B)\sin 3t = \cos 3t$

$$\Rightarrow \begin{cases} -6A + 6B = 1 \\ -6A - 6B = 0 \end{cases}$$

$\Rightarrow A = -\frac{1}{12} \quad B = \frac{1}{12}$

$$\Rightarrow \boxed{\begin{aligned} x_{ss} = x_p &= -\tfrac{1}{12}\cos 3t + \tfrac{1}{12}\sin 3t \\ &= \tfrac{\sqrt{2}}{12}\cos\left(3t - \tfrac{3\pi}{4}\right) \\ \Rightarrow \text{amplitude} &= \tfrac{\sqrt{2}}{12} \quad \text{phase shift} = \tfrac{\pi}{4} \end{aligned}}$$

$$\boxed{\begin{aligned} x = x_h + x_p &= e^{-t}(c_1\cos\sqrt{2}t + c_2 \sin\sqrt{2}t) \\ &\quad - \tfrac{1}{12}\cos 3t + \tfrac{1}{12}\sin 3t \end{aligned}}$$

#5. $x'' + 2x' + 2x = 2\cos t$.

$x'' + 2x' + 2x = 0$

$\Rightarrow r^2 + 2r + 2 = 0$

$r = -1 \pm i$

$\Rightarrow X_h = e^{-t}(C_1 \cos t + C_2 \sin t)$.

let $X_p = A\cos t + B\sin t$.

$X_p'' + 2X_p' + 2X_p = 2\cos t$

$(A+2B)\cos t + (-2A+B)\sin t = 2\cos t$

$\Rightarrow \begin{cases} A+2B = 2 \\ -2A+B = 0 \end{cases}$

$\Rightarrow A = \frac{2}{5} \quad B = \frac{4}{5}$

$X_{ss} = X_p = \frac{2}{5}\cos t + \frac{4}{5}\sin t$

$= \frac{2}{\sqrt{5}}\cos(t - 1.1)$

Amplitude $= \frac{2}{\sqrt{5}}$ phase shift $= 1.1$.

$X = X_p + X_h = e^{-t}(C_1\cos t + C_2 \sin t) + \frac{2}{5}\cos t + \frac{4}{5}\sin t$.

#12. $\ddot{x} + 8\dot{x} + 36x = 72\cos 6t$.

Let $x_p = A\cos 6t + B\sin 6t$

$x_p'' + 8x_p' + 36x_p = 72\cos 6t$.

$\Rightarrow$ $\cancel{48B\cos 6t - 48A}$

$48B\cos 6t - 48A\sin 6t = 72\cos 6t$.

$\Rightarrow$ $B = \frac{3}{2}$ $A = 0$

$$\boxed{X_{ss} = X_p = \frac{3}{2}\sin 6t}$$

#19. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\Rightarrow$ $\sin(A+B) - \sin(A-B) = 2\cos A\sin B$. ①

we want to have

$\sin 3t - \sin t$

let $A+B = 3t$. $A - B = t$

$\Rightarrow$ ~~sin~~ Based on ①

~~sin~~($\sin 3t - \sin t = 2\cos A\sin B$.

where $\begin{cases} A+B = 3t \\ A-B = t \end{cases}$ $\Rightarrow$ $\begin{cases} A = 2t \\ B = t \end{cases}$.

$\Rightarrow$ $\boxed{\sin 3t - \sin t = 2\cos 2t\sin t}$