APPM 2360: Final exam — Solutions

7:30am – 10:00am, May 6, 2009.

Problem 1: (30 points) Consider the matrix

$$A = \begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix}.$$

(a) (20 points) Find the eigenvalues and eigenvectors of A.

(b) (10 points) Find the general solution to the system $\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$.

Solution:

(a) The characteristic equation is $0 = (-4-\lambda)(1-\lambda) - 6 = \lambda^2 + 3\lambda - 10$.

The roots are $\lambda_{1,2} = -3/2 \pm \sqrt{9/4 + 40/4} = -3/2 \pm 7/2$.

Analyze $\lambda_1 = -5$: We solve $(A + 5I)\mathbf{v} = \mathbf{0}$ to find $\mathbf{v}_1$:

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so we pick, for instance, $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Analyze $\lambda_2 = 2$: We solve $(A - 2I)\mathbf{v} = \mathbf{0}$ to find $\mathbf{v}_2$:

$$\left[\begin{array}{cc|c} -6 & 3 & 0 \\ 2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so we pick, for instance, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

To summarize:

$$\lambda_1 = -5, \qquad \mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 2, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(b) The general solution is $\mathbf{x} = c_1 e^{-5t} \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Alternatively,

$$\begin{aligned} x(t) &= -3c_1 e^{-5t} + c_2 e^{2t}, \\ y(t) &= c_1 e^{-5t} + 2c_2 e^{2t} \end{aligned}$$

Problem 2: (30 points)

(a) (12 points) Determine the general solution to

$$y' = t^2\, y^2.$$

(b) (12 points) Determine the general solution to

$$(t^2 + 1)\, y' + 2\, t\, y = 0. \tag{1}$$

(c) (6 points) Find the solution of (1) that satisfies $y(1) = 1$.

Solution:

(a) The equation is separable. Assuming $y \neq 0$ we find

$$\frac{dy}{y^2} = t^2\, dt \qquad \Rightarrow \qquad -\frac{1}{y} = \frac{1}{3}\, t^3 - C.$$

Then

$$y = \frac{1}{C - \frac{1}{3}\, t^3}.$$

Finally observe that

$$y = 0$$

is also a solution.

(b) This equation can also be solved via separation of variables. Alternatively, we can simply observe that it is already written in "integrating factor form":

$$\frac{d}{dt}\left[(1 + t^2)\, y\right] = 0 \qquad \Rightarrow \qquad (1 + t^2)\, y = C \qquad \Rightarrow \qquad y = \frac{C}{1 + t^2}.$$

(c) Simply insert the condition $y(1) = 1$

$$1 = \frac{C}{1 + 1^2} \qquad \Rightarrow \qquad C = 2 \qquad \Rightarrow \qquad y = \frac{2}{1 + t^2}.$$

Problem 3: (30 points)

(a) (15 points) Find the general solution to

$$y'' - 4\,y' + 13\,y = 0.$$

(b) (15 points) Find the general solution to

$$y'' - 4\,y' + 13\,y = t\,e^{t}.$$

Solution:

(a) The roots of $r^2 - 4\,r + 13 = 0$ are $r_1 = 2 + 3\,i$ and $r_2 = 2 - 3\,i$.

Either $\boxed{y = b_1\,e^{(2+3\,i)\,t} + b_2\,e^{(2-3\,i)\,t}}$ or $\boxed{y = c_1\,e^{2\,t}\,\cos(3\,t) + c_2\,e^{2\,t}\,\sin(3\,t)}$

(b) We make the "guess" $y_{\mathrm{p}} = (A + B\,t)\,e^{t}$. Inserting this into the equation we get

$$\begin{aligned} y_{\mathrm{p}}'' - 4\,y_{\mathrm{p}}' + 13\,y_{\mathrm{p}} &= \big(A + 2\,B + B\,t\big)\,e^{t} - 4\,\big(A + B + B\,t\big)\,e^{t} + 13\,\big(A + B\,t\big)\,e^{t} \\ &= (10\,A - 2\,B)\,e^{t} + 10\,B\,t\,e^{t}. \end{aligned}$$

We must have $10\,B = 1$ which gives $B = 1/10$. Then $10\,A - 2\,B = 0$ gives $A = B/5 = 1/50$.

Adding the homogeneous solution we get $\boxed{y = c_1\,e^{2\,t}\,\cos(3\,t) + c_2\,e^{2\,t}\,\sin(3\,t) + \big(\tfrac{1}{50} + \tfrac{t}{10}\big)\,e^{t}.}$

Problem 4: (30 points) Consider the matrix and the vector

$$A = \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

(a) (20 points) Find the general solution to the equation $A\mathbf{x} = \mathbf{b}$.

(b) (10 points) Find the general solution to the equation $A\mathbf{x} = \mathbf{0}$.

Solution: We first derive the RREF of $[A|\mathbf{b}]$:

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 1 \\ 0 & 1 & 2 & -1 & 1 \\ 1 & 1 & -1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 1 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 0 & 2 \end{array}\right]$$
$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 1 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

There is one free variable, x_3. We set $x_3 = t$. The general solution is then

$$\begin{aligned} x_1 &= 3\,x_3 = 3\,t \\ x_2 &= 2 - 2\,x_3 = 2 - 2\,t \\ x_3 &= t \\ x_4 &= 1 \end{aligned}$$

which could also be written

$$\mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} t.$$

(b) From the solution in (a), we immediately get

$$\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} t.$$

Problem 5: (30 points) Consider the nonlinear system

$$\begin{cases} x' = 2\,y \\ y' = y + x - x^3 \end{cases} \tag{2}$$

(a) (10 points) Identify all equilibrium points of the system (2).

(b) (15 points) Compute the Jacobian matrix at each equilibrium. Determine the geometry type (for example: saddle, spiral, center, star, *etc*) and stability of each equilibrium.

(c) (5 points) Which graph is the direction field of the system (2).

Solution:

(a) The condition $x' = 0$ immediately yields that $y = 0$. Then $y' = 0$ yields $0 = x - x^3$ which is true if $x = 0$ or $x = \pm 1$. Thus, we have three equilibrium points:

$$\mathbf{x}_1 = (0, 0), \qquad \mathbf{x}_2 = (1, 0), \qquad \mathbf{x}_3 = (-1, 0).$$

(b) The general Jacobian is $J = \begin{bmatrix} 0 & 2 \\ 1 - 3\,x^2 & 1 \end{bmatrix}$.

Point 1: $J = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$. The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$.

Saddle point. Unstable.

Point 2: $J = \begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix}$. The eigenvalues are $\lambda_{1,2} = 1/2 \pm i\,\sqrt{15}/2$.

Unstable spiral point.

Point 3: $J = \begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix}$. The eigenvalues are $\lambda_{1,2} = 1/2 \pm i\,\sqrt{15}/2$.

Unstable spiral point.

(c) The correct graph is B.

Problem 6: (20 points) Consider the matrices $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Given that $A^{-1} = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, determine which of the following equations are solvable, and solve the ones that are.

(a) (5 points) $A\,\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ with the 3×1 vector $\mathbf{x}$ as unknown.

(b) (5 points) $B\,\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ with the 3×1 vector $\mathbf{y}$ as unknown.

(c) (*5 points) $Z\,A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$ with the 2×3 matrix Z as unknown.

(d) (*5 points) $W\,A + B\,W\,A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with the 3×3 matrix W as unknown.

Solution:

(a) Left multiplying the equation by A^{-1} we get $\mathbf{x} = A^{-1} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

(b) The last entry of the vector $B\,\mathbf{y}$ must be zero for any vector $\mathbf{y}$. Therefore, the equation cannot have a solution.

(c) Multiply from the right by A^{-1}. Then $Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 1 & -3 \\ 0 & -2 & 2 \end{bmatrix}$.

(d) Setting $C = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ the given equation can be written $(I + B)\,W\,A = C$. Now $I + B = A$ so the equation is $A\,W\,A = C$. Then $W = A^{-1}\,C\,A^{-1} = \begin{bmatrix} 1 & 1 & -3 \\ 1 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$.

Problem 7: (30 points) Give a brief answer to each question. Box your answer.
No work given for this question will be graded.

(a) (5 points) Give the determinant of the matrix $A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$.

(b) (5 points) Determine a function y such that $y' = 2\,y$ and $y(0) = 3$.

(c) (5 points) Which of the following equations have stable equilibrium points at the origin:

(1) $y' = -y$ (2) $y' = -y^2$ (3) $y' = -y^4$

(d) (5 points) Consider the system $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.
For which values of a is it the case that all solutions satisfy $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$?

(e) (5 points) A sample of a single radioactive material weighs 1 ounce on Jan. 1 of 1990. On Jan. 1 of 2000, the sample weighs 0.1 ounce. How much does it weigh on Jan. 1 of 2010?

(f) (*5 points) Let a be a real number and consider the following equations for $y = y(t)$:

$$y'' + 2\,y' + 3\,y = 0, \qquad y(0) = 1, \qquad y'(0) = 2, \qquad y''(0) = a.$$

For which values of a does there exist a function y that satisfies all conditions?

Solution:

(a) -2

(b) $y = 3\,e^{2\,t}$

(c) Only (1).

(d) $a < 0$

(e) 0.01 ounces.

(f) $a = -7$

Comments:

(c) Note that for the equations (2) and (3), any solution that starts slightly negative will move away from the equilibrium point at $y = 0$ and tend to $-\infty$.

(d) The eigenvalues of the system matrix are $\lambda_1 = -1$ and $\lambda_2 = a$. All solutions tend to zero if and only if both eigenvalues are negative.

(e) Simply note that the sample loses 90% of its weight every 10 years. (The formula for the amount left is $y(t) = 1\text{oz} \cdot 10^{-(t-1990)/10} = 1\text{oz} \cdot e^{-(t-1990)\,\log(10)/10}$ where t is the year. This is making things unnecessarily complicated, though.)

(f) Note that at $t = 0$ we must have $y''(0) + 2\,y'(0) + 3\,y(0) = 0$. It then follows that

$$a = y''(0) = -2\,y'(0) - 3\,y(0) = -2 \cdot 2 - 3 \cdot 1 = -7.$$