APPM 2360: Final Exam

7.30 pm – 10.00 pm, May 3, 2008.

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section, (4) your instructor's name, and (5) a grading table. Show ALL of your work in your bluebook and box in your final answer. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Text books, class notes, and calculators are NOT permitted. A one-page crib sheet is allowed.

Problem 1: *(14 points, 2 each)* State whether the following statements are *always* "TRUE" or "FALSE" (meaning not *always* true). You MUST write the full word TRUE or FALSE. T/F or YES/NO will NOT be given any credit. You do not have to justify your answer for this question.

(a) If A and B are invertible matrices and X is a matrix such that $AX = B$, then X is also invertible.

(b) The transpose of a matrix exists if and only if the matrix is square.

(c) If A is a 3×3 matrix that is row equivalent to the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then A has rank 3.

(d) The system $\begin{cases} \dot{x} = & -y, \\ \dot{y} = & x \end{cases}$ has an asymptotically stable equilibrium point.

(e) If y_1 and y_2 are two different solutions of the equation $y' + y^2 = 1$, then the function $z = y_1 - y_2$ satisfies $z' + z^2 = 0$.

(f) If f, g, and h are continuously differentiable functions on $\mathbb{R}$ such that $f' = g$, then the function $y(t) = \displaystyle\int_0^t e^{-f(t)+f(s)}\, h(s)\, ds$ satisfies $y' + g\, y = h$.

(g) If the matrix A has eigenvalues 1 and 3 then the matrix $A - I$ has eigenvalues 0 and 2.

Problem 2: *(12 points, 6 each)* Consider the differential equation

$$y'' + 2\, y' + 5\, y = f(t)$$

(a) Construct the general solution of the equation when $f(t) = 0$.

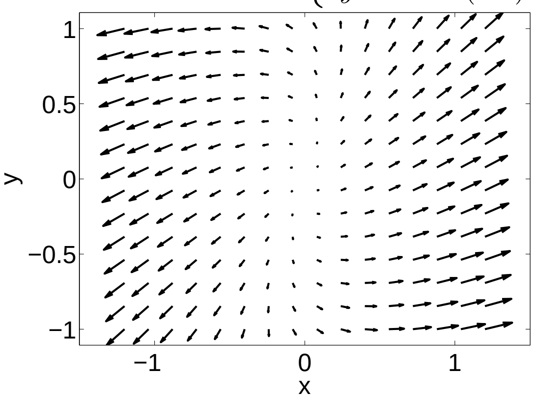
(b) Construct the solution of the equation when $f(t) = e^{-t}$ subject to the initial conditions $y(0) = 0$ and $y'(0) = 1$.

Problem 3: *(12 points, 2 each)* Match each linear system with the corresponding phase-plane. You do not have to justify your answer to this question.

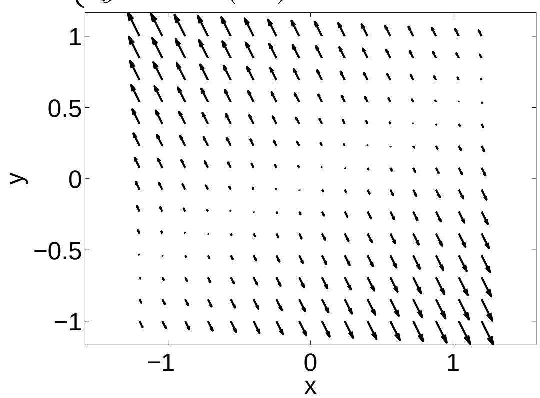
$$\textbf{(i)}\begin{cases} x' = 3\,x - 2\,y \\ y' = 4\,x - y \end{cases} \qquad \textbf{(ii)}\begin{cases} x' = -x + y \\ y' = -2\,x + y \end{cases}$$

$$\textbf{(iii)}\begin{cases} x' = 4\,x - y \\ y' = 2\,x + y \end{cases} \qquad \textbf{(iv)}\begin{cases} x' = x - 2\,y \\ y' = -2\,x + 4\,y \end{cases}$$

$$\textbf{(v)}\begin{cases} x' = 2\,y \\ y' = -\sin(\pi\,x) \end{cases} \qquad \textbf{(vi)}\begin{cases} x' = -2\,y \\ y' = -\sin(\pi\,x) \end{cases}$$



(a)



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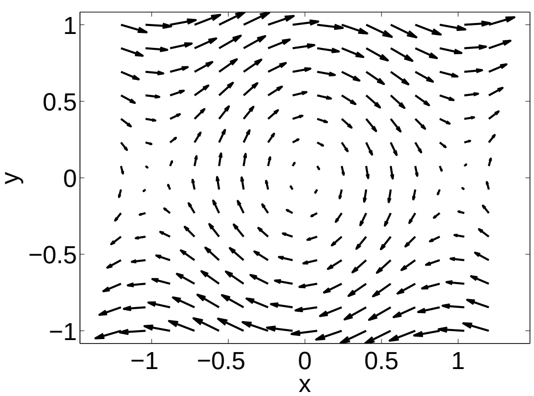

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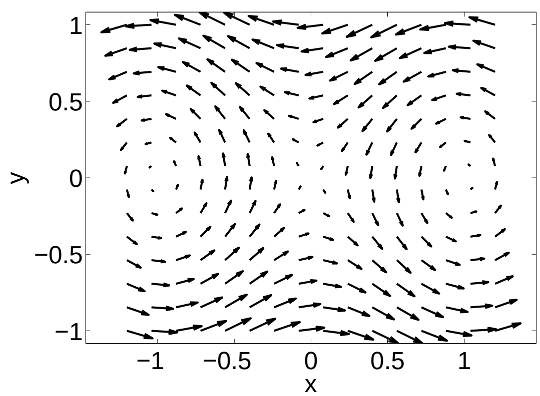

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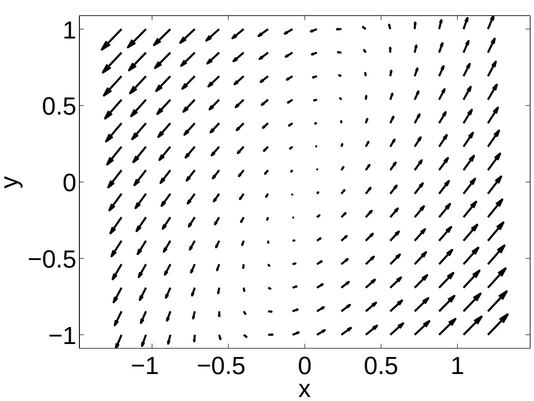

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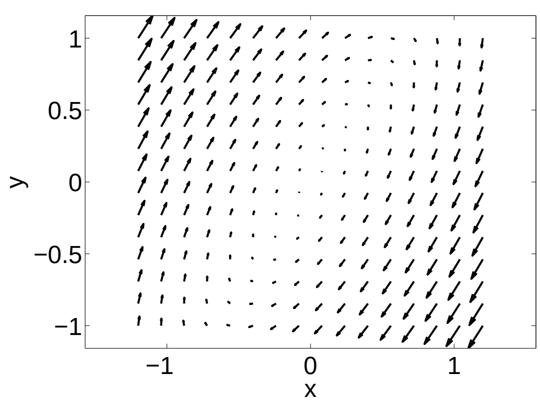

(f)

Problem 4: *(12 points, 4 each)* Let t_0 and y_0 be real numbers such that $t_0 > 0$ and $1 < y_0 < 2$, and consider for positive t the initial value problem (IVP)

$$y'(t) = \frac{1}{t}(y-1)(y-2)$$
$$y(t_0) = y_0.$$

(a) Show that for all positive values of t, $1 < y(t) < 2$.

(b) Show that if $t_1 < t_2$, then the solution of the IVP satisfies $y(t_1) > y(t_2)$.
Hint: $y(t_2) - y(t_1) = \int_{t_1}^{t_2} y'(t)\, dt$.

(c) Show that the solution of the IVP satisfies $\lim_{t \to 0} y(t) = 2$.

Problem 5: *(12 points, 4 each)* Consider the differential equation

$$w'(t) = w(t)^2 + P(t)\, w(t) + Q(t)$$

(a) Use the substitution $w = -\dfrac{y'}{y}$ to show that $y = y(t)$ satisfies $y'' - P(t)y' + Q(t)y = 0$.

(b) Take $P(t) = 2$, $Q(t) = 1$ and find the general solution of the equation for $w = w(t)$.

(c) For the same values of P and Q find the general solution of the equation for $y = y(t)$ and verify that the two solutions satisfy $y' + w\, y = 0$.

Problem 6: *(12 points, 4 each)* Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -5 \end{bmatrix},$$

and the matrix

$$A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \\ -3 & -1 & -5 \end{bmatrix}.$$

(a) Find all vectors $\mathbf{x} \in \mathbb{R}^3$ such that $A\,\mathbf{x} = 0$.

(b) Show that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not a basis for the linear subspace $\mathbb{V} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and provide a basis for $\mathbb{V}$.

(c) Determine whether the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ is in $\mathbb{V}$.

Problem 7: *(12 points, 6 each)* Consider the matrices A and B given by

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ -1 & -1 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -6 & 5 & t \\ s & -2 & 3 \\ -1 & 1 & -1 \end{bmatrix}.$$

(a) Find the real numbers s and t such that $A\,B = I$.

(b) Solve the following equation for $\mathbf{x}$:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ -1 & -1 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} \pi \\ 9 \\ \sqrt{5} \end{bmatrix}.$$

Problem 8: *(14 points)* Let h be a real number and consider the nonlinear system

$$\begin{cases} \dot{x} = h\,x + y - x\,(x^2 + y^2) \\ \dot{y} = -x + h\,y - y\,(x^2 + y^2). \end{cases}$$

(a) *(6 points)* Linearize this system about the origin and discuss the stability of the system at that point for different values of h.

(b) *(4 points)* Make the substitution $x = r\cos\theta$ and $y = r\sin\theta$ when $h \neq 0$ and show that the new functions $r = r(t)$ and $\theta = \theta(t)$ satisfy

$$\begin{cases} \dot{r} = r\,(h - r^2) \\ \dot{\theta} = -1 \end{cases}$$

You may use the following formulas:

$$r\dot{r} = x\dot{x} + y\dot{y} \quad \text{and} \quad \dot{\theta} = \frac{x\dot{y} - \dot{x}y}{x^2 + y^2}.$$

(c) *(4 points)* Prove that if $h \leq 0$, then any solution $r(t)$ satisfies $r(t) \to 0$ as $t \to \infty$, whereas if $h > 0$, then any solution except the zero solution itself satisfies $r(t) \to \sqrt{h}$ as $t \to \infty$.