APPM 2360: Final exam

7:30am – 10:00am, May 6, 2009.

- ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) recitation section (4) your instructor's name, and (5) a grading table.
- Text books, class notes, and calculators are NOT permitted.
- A one-page double-sided crib sheet is allowed.
- For questions 1 — 6, motivate your answers.
- For question 7, brief answers with no motivation are sufficient.

Problem 1: (30 points) Consider the matrix

$$A = \begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix}.$$

(a) (20 points) Find the eigenvalues and eigenvectors of A.

(b) (10 points) Find the general solution to the system $\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$.

Problem 2: (30 points)

(a) (12 points) Determine the general solution to

$$y' = t^2\, y^2.$$

(b) (12 points) Determine the general solution to

$$(t^2 + 1)\, y' + 2\, t\, y = 0. \tag{1}$$

(c) (6 points) Find the solution of (1) that satisfies $y(1) = 1$.

Problem 3: (30 points)

(a) (15 points) Find the general solution to

$$y'' - 4\, y' + 13\, y = 0.$$

(b) (15 points) Find the general solution to

$$y'' - 4\, y' + 13\, y = t\, e^t.$$

Problem 4: (30 points) Consider the matrix and the vector

$$A = \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

(a) (20 points) Find the general solution to the equation $A\,\mathbf{x} = \mathbf{b}$.

(b) (10 points) Find the general solution to the equation $A\,\mathbf{x} = \mathbf{0}$.

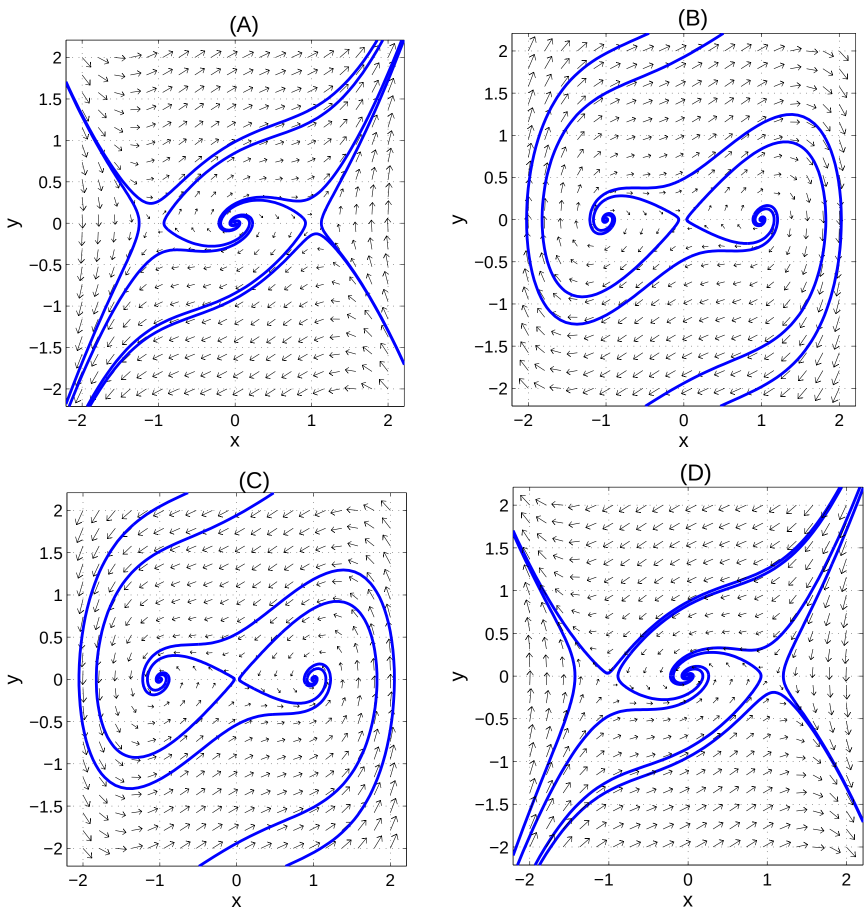
Problem 5: (30 points) Consider the nonlinear system

$$\begin{cases} x' = 2\,y \\ y' = y + x - x^3 \end{cases} \tag{2}$$

(a) (10 points) Identify all equilibrium points of the system (2).

(b) (15 points) Compute the Jacobian matrix at each equilibrium. Determine the geometry type (for example: saddle, spiral, center, star, *etc*) and stability of each equilibrium.

(c) (5 points) Which graph is the direction field of the system (2).



Note: *No problems on this page should require lengthy calculations. The starred problems (6c, 6d, and 7f) are intentionally slightly different from the homework problems; unless you quickly "see" how to solve them you may want to save them for last.*

Problem 6: (20 points) Consider the matrices $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Given that $A^{-1} = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, determine which of the following equations are solvable, and solve the ones that are.

(a) (5 points) $A\,\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ with the 3×1 vector $\mathbf{x}$ as unknown.

(b) (5 points) $B\,\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ with the 3×1 vector $\mathbf{y}$ as unknown.

(c) (*5 points) $Z\,A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$ with the 2×3 matrix Z as unknown.

(d) (*5 points) $W\,A + B\,W\,A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with the 3×3 matrix W as unknown.

Problem 7: (30 points) Give a brief answer to each question. Box your answer.
No work given for this question will be graded.

(a) (5 points) Give the determinant of the matrix $A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$.

(b) (5 points) Determine a function y such that $y' = 2\,y$ and $y(0) = 3$.

(c) (5 points) Which of the following equations have stable equilibrium points at the origin:

(1) $y' = -y$ (2) $y' = -y^2$ (3) $y' = -y^4$

(d) (5 points) Consider the system $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.
For which values of a is it the case that all solutions satisfy $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$?

(e) (5 points) A sample of a single radioactive material weighs 1 ounce on Jan. 1 of 1990. On Jan. 1 of 2000, the sample weighs 0.1 ounce. How much does it weigh on Jan. 1 of 2010?

(f) (*5 points) Let a be a real number and consider the following equations for $y = y(t)$:

$$y'' + 2\,y' + 3\,y = 0, \qquad y(0) = 1, \qquad y'(0) = 2, \qquad y''(0) = a.$$

For which values of a does there exist a function y that satisfies all conditions?