Homework set 1 — CSE 383C / CS 383C / M 383E / ME 397

P.G. Martinsson, UT-Austin, Sep. 2024

From the book: 1.1, 2.2, 2.3, 2.6, 3.3.

If you found the topics in the initial lectures challenging, then I recommend that you do ***all*** *the homeworks for chapters 1, 2, and 3. Understanding this material well will help tremendously in what follows.*

Problem 1: Consider the vector space $X = \mathbb{C}^m$ with the ℓ^∞-norm $\|\mathbf{x}\| = \max|x_i|$. What is the corresponding induced norm of a matrix? *Hint: We solved this problem in class for the ℓ^1 norm. The argument for the ℓ^∞ norm is very similar.*

Problem 2: Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

Compute a rough *estimate* to the norm on $\mathbf{A}$ induced by the norm ℓ^p for $p = 1.3$. Explain your methodology.

Problem 3 – 5 are ***optional*** *and you are not asked to hand in solutions. They consist of the proof that $\|\cdot\|_p$ is a norm for $p \in [1,\infty]$.*

Problem 3: Let λ be a real number such that $\lambda \in (0,1)$, and let a and b be two non-negative real numbers. Prove that

$$a^\lambda\, b^{1-\lambda} \leq \lambda\, a + (1-\lambda)\, b, \tag{1}$$

with equality iff $a = b$. *Hint: Consider $b = 0$ first. When $b \neq 0$, change variables to $t = a/b$.*

Problem 4: [Hölder's inequality] Suppose that p is a real number such that $1 < p < \infty$, and let q be such that $p^{-1} + q^{-1} = 1$. Let $\mathbf{f}, \mathbf{g} \in \mathbb{C}^m$ and prove that

$$|\mathbf{f}\cdot\mathbf{g}| \leq \|\mathbf{f}\|_p\, \|\mathbf{g}\|_q. \tag{2}$$

Prove that equality holds iff $\alpha|\mathbf{f}(i)|^p = \beta|\mathbf{g}(i)|^q$ for some α, β such that $\alpha\beta \neq 1$.

Hint: Consider first the case where $\|\mathbf{f}\|_p = 0$ or $\|\mathbf{g}\|_q = 0$. For the case $\|\mathbf{f}\|_p\, \|\mathbf{g}\|_q \neq 0$, use (1) with

$$a = \left|\frac{\mathbf{f}(i)}{\|\mathbf{f}\|_p}\right|^p, \qquad b = \left|\frac{\mathbf{g}(i)}{\|\mathbf{g}\|_q}\right|^q, \qquad \lambda = \frac{1}{p}.$$

Problem 3: [Minkowski's inequality] Prove that for $p \in [1,\infty]$, and for $\mathbf{f}, \mathbf{g} \in \mathbb{C}^m$, we have

$$\|\mathbf{f} + \mathbf{g}\|_p \leq \|\mathbf{f}\|_p + \|\mathbf{g}\|_p.$$

Hint: Consider the cases $p = 1, \infty$ separately. For $p \in (1,\infty)$, note that

$$|\mathbf{f}(i) + \mathbf{g}(i)|^p \leq \big(|\mathbf{f}(i)| + |\mathbf{g}(i)|\big)\, |\mathbf{f}(i) + \mathbf{g}(i)|^{p-1}. \tag{3}$$

Then sum both sides of (3) and apply (2) to the right hand side.