CSE 383C: Numerical Analysis: Linear Algebra – Final Exam
10:30am – 12:30pm, December 13, 2024.

Instructions:

- ***This is a closed books exam.*** No books or calculators are allowed.
  However, you are allowed to bring one hand written, single sided formula sheet.
- Try to answer the questions within the space given. If you must use more space, you *can* attach extra pages, but try to avoid it. If you do need extra space, then please mark this very clearly on the exam.
- If a question does not specifically ask for a motivation, then the correct answer alone will give full points. You are welcome to enter a brief explanation of how you arrived at the answer — this might yield partial credit in case the given answer is incorrect.
- Very few of the questions require non-trivial computations. Questions 4b and 4c both involve short matrix computations. *Question 5b does involve some computations – you may want to save that one for last.*
- In this exam, the default is that for a vector $\mathbf{x}$, the notation $\|\mathbf{x}\|$ denotes the Euclidean norm. For a matrix $\mathbf{A}$, the notation $\|\mathbf{A}\|$ refers to the operator norm with respect to the Euclidean vector norm, and $\|\mathbf{A}\|_{\mathrm{F}}$ refers to the Frobenius norm. If $\mathbf{A}$ is a matrix, then $\mathbf{A}^*$ refers to the complex conjugate of the transpose of $\mathbf{A}$ (or merely the transpose when $\mathbf{A}$ is real).

Name: ______________________________

Section: ________________

Question	Max points	Scored points
1	20	
2	10	
3	20	
4	20	
5	20	
6	10	
Total:		

Question 1: (20p) Please provide answers only. Motivations will not be graded.

(a) (5p) Mark as true or false:

	TRUE	FALSE
If $\mathbf{A}$ is tridiagonal and positive definite, then its LU factors are bidiagonal.		
If $\mathbf{A}$ is tridiagonal and invertible, then $\mathbf{A}^{-1}$ is tridiagonal.		
Every square matrix $\mathbf{A}$ admits a factorization $\mathbf{A} = \mathbf{QTQ}^*$ where $\mathbf{Q}$ is unitary and $\mathbf{T}$ is upper triangular.		

(b) (5p) Given a square $m \times m$ matrix $\mathbf{A}$, an $m \times 1$ vector $\mathbf{b}$, and a positive integer n, give the definition of the *Krylov space* $\mathcal{K}_n(\mathbf{A}, \mathbf{b})$:

(c) (5p) Given the function $f(x) = 2x^2 + x^3$, specify its *relative* condition number at $x = 2$:

$$\kappa_f(2) =$$

(d) (5p) The matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ has the QR factorization $\mathbf{A} = \mathbf{QR}$. Specify $\mathbf{Q}$ and $\mathbf{R}$

$\mathbf{Q} =$ $\qquad\qquad$ $\mathbf{R} =$

Question 2: (10p) Let $\mathbf{A}$ be symmetric and positive definite, and suppose that it has the Cholesky factorization $\mathbf{A} = \mathbf{R}^*\mathbf{R}$, where $\mathbf{R}$ is upper triangular. Set $\mathbf{B} = \mathbf{RR}^*$. Do $\mathbf{A}$ and $\mathbf{B}$ necessarily have the same eigenvalues? If you answer yes, then briefly motivate why. If you answer no, then provide a counter example.

Question 3: (20p)

(a) (10p) For a positive integer n, define

$$s_n = \sum_{j=1}^{n} \frac{1}{j^2} \qquad \text{and} \qquad s = \sum_{j=1}^{\infty} \frac{1}{j^2}.$$

It is known that $s = \pi^2/6$, and we clearly have $s_n \to s$ as $n \to \infty$. Moreover, as we showed in class, $s - s_n \approx \int_{n+1}^{\infty} x^{-2}\,dx = (n+1)^{-1}$. For $n = 10^{10}$, we would expect $s - s_n \approx 10^{-10}$, yet the code

```
nmax = 1e10;
s    = 0;
for j = 1:nmax
  s = s + 1/(j*j);
end
fprintf(1,'Error          = %20.16f\n',pi*pi/6-s)
fprintf(1,'Expected error = %20.16f\n',1/(nmax+1))
```

produces the output:

```
Error          =   0.0000000090136514
Expected error =   0.0000000001000000
```

How would you explain the discrepancy? How would you compute s_n more accurately?

(b) (10p) Consider the function

$$f(x) = 1 - e^{-x^2}.$$

Suppose that for some number x such that $10^{-11} \leq x \leq 10^{-10}$ you want to evaluate $f(x)$ with several correct digits in *relative* precision. How would you proceed if you worked in an environment such as Matlab, and you could only use standard double precision arithmetic?

Question 4: (20p) The matrix $\mathbf{A}$ has the LU factorization

$$\underbrace{\begin{bmatrix} 3 & 1 & -2 \\ -3 & -2 & a_{23} \\ 6 & a_{32} & -1 \end{bmatrix}}_{=\mathbf{A}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}}_{=\mathbf{L}} \underbrace{\begin{bmatrix} 3 & 1 & u_{13} \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{=\mathbf{U}}.$$

For this problem, no motivation is required. Just give the answers.

(a) (5p) Specify the missing entries

$a_{23} =$ $\qquad$ $a_{32} =$ $\qquad$ $u_{13} =$

(b) (5p) Specify the determinant of $\mathbf{A}$:

$$\det(\mathbf{A}) =$$

(c) (5p) Specify the solution $\mathbf{x}$ to the linear system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$:

$$\mathbf{x} =$$

(d) (5p) Set $\mathbf{V} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $\mathbf{B} = \mathbf{AV}$. Suppose that $\mathbf{B}$ has the LU factorization $\mathbf{B} = \mathbf{L}'\mathbf{U}'$. Specify $\mathbf{L}'$ and $\mathbf{U}'$.

$\mathbf{L}' =$

$\mathbf{U}' =$

Question 5: (20p) Consider an $m \times n$ matrix $\mathbf{A}$, and an $m \times 1$ vector $\mathbf{b}$.

(a) (10p) Let $\mathbf{x}_\star$ denote the *least squares solution* to the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$. Mark the following statements as true or false:

	TRUE	FALSE
If $\operatorname{rank}(\mathbf{A}) = n$, then $\mathbf{x}_\star = \left(\mathbf{A}^*\mathbf{A}\right)^{-1}\mathbf{A}^*\mathbf{b}$.		
Regardless of the rank of $\mathbf{A}$, we have $\mathbf{A}^*\mathbf{A}\mathbf{x}_\star = \mathbf{A}^*\mathbf{b}$.		
$\|\mathbf{A}\mathbf{x}_\star - \mathbf{b}\| = \inf\{\|\mathbf{b} - \mathbf{y}\| : \mathbf{y} \in \operatorname{col}(\mathbf{A})\}$.		
If $\operatorname{rank}(\mathbf{A}) = m$, then the minimization problem $\inf\{\|\mathbf{A}\mathbf{y} - \mathbf{b}\| : \mathbf{y} \in \mathbb{R}^n\}$ has a unique solution.		
It is always the case that $\mathbf{x}_\star \in \operatorname{null}(\mathbf{A})^\perp$.		

(b) (10p) Consider the matrix $\mathbf{A}$ and the vector $\mathbf{b}$ defined by

$$\mathbf{A} = \underbrace{\begin{bmatrix} -2 & -2 \\ 2 & 4 \\ -2 & -2 \\ 2 & 4 \end{bmatrix}}_{=\mathbf{A}} = \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}}_{=\mathbf{W}} \underbrace{\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}}_{=\mathbf{S}} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} -1 \\ 5 \\ -1 \\ 1 \end{bmatrix}$$

Specify the least squares solution $\mathbf{x}_\star$ to $\mathbf{A}\mathbf{x} = \mathbf{b}$. Please motivate your answer. *If you run out of time, then just indicate what steps you would take! Observe that the columns of the matrix* $\mathbf{W}$ *are orthogonal.*

Question 6: (10p) Let $\mathbf{A}$ be a symmetric and positive definite matrix of size $n \times n$, with eigenvalues $\{\lambda_j\}_{j=1}^n$ that are ordered by modulus, so that

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_n.$$

You know that

$$\lambda_1 = 1 \qquad \text{and} \qquad \lambda_n = 10^5.$$

Given a positive integer p and a vector $\mathbf{b}$, we seek to solve the linear system

$$\mathbf{A}^p \mathbf{x} = \mathbf{b}.$$

We work in Matlab, using standard double precision floating point arithmetic. Our first attempt is:

```
T = A;
for i = 1:(p-1)
  T = T*A;
end
x = inv(T)*b;
```

We find that for $p = 2$, the code works fine. We next try $p = 5$, and find that we then run into problems, and get no accurate digits in the answer.

Explain what problem occurred, and then describe how you would proceed instead. For maximal credit, pay attention to computational efficiency in the case where p and n are large.