Section exam 1 Numerical Analysis: Linear Algebra — Solutions

9:30am – 10:45am, Sep. 26, 2024.

Question 1: (5p) Let $\mathbf{u} = [1,\, 2,\, 3]^*$ and $\mathbf{v} = [2,\, t,\, 2]$ be two vectors in $\mathbb{R}^3$. For which value(s) of t are $\mathbf{u}$ and $\mathbf{v}$ orthogonal?

Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal iff $\mathbf{u} \cdot \mathbf{v} = 0$. We find

$$0 = \mathbf{u} \cdot \mathbf{v} = 1 \cdot 2 + 2 \cdot t + 3 \cdot 2 = 8 + 2t.$$

The only solution is $t = -4$.

Question 2: (5p) Recall that a *Householder reflector* $\mathbf{F}$ is a special unitary matrix that you construct in order to map a given vector $\mathbf{x}$ to the vector $\mathbf{F}\mathbf{x} = \|\mathbf{x}\|\,\mathbf{e}_1 = [\|\mathbf{x}\|,\, 0,\, 0,\, \dots,\, 0]^*$. Give a formula for $\mathbf{F}$, expressed in terms of the vector $\mathbf{x}$.

Set $\mathbf{v} = \|\mathbf{x}\|\,\mathbf{e}_1 - \mathbf{x}$. Then

$$\mathbf{F} = \mathbf{I} - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^*.$$

(Alternatively, you could set $\mathbf{v} = \theta\|\mathbf{x}\|\,\mathbf{e}_1 - \mathbf{x}$ with $|\theta| = 1$.)

Question 3: (20p) Let $\mathbf{A}$ be a 5×5 matrix with the singular value decomposition

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^*,$$

where

$$\mathbf{D} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and where the matrices

$$\mathbf{U} = \big[\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \mathbf{u}_4,\, \mathbf{u}_5\big], \qquad \mathbf{V} = \big[\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \mathbf{v}_4,\, \mathbf{v}_5\big]$$

are unitary. *Please give only the answers. Motivations will not be graded.*

(a) (2p) Specify the spectral norm of $\mathbf{A}$: $\|\mathbf{A}\| = 4$

(b) (2p) Specify the Frobenius norm of $\mathbf{A}$: $\|\mathbf{A}\|_{\mathrm{F}} = \sqrt{4^2 + 3^2 + 2^2} = \sqrt{29}$

(c) (2p) Specify the rank of $\mathbf{A}$: $\mathrm{rank}(\mathbf{A}) = 3$

(d) (2p) Specify an orthonormal basis for the column space of $\mathbf{A}$: $\{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3\}$

(e) (2p) Specify an orthonormal basis for the row space of $\mathbf{A}$: $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}$

(f) (2p) Specify an orthonormal basis for the null space of $\mathbf{A}$: $\{\mathbf{v}_4,\, \mathbf{v}_5\}$

(g) (2p) Specify an orthonormal basis for the null space of $\mathbf{A}^*$: $\{\mathbf{u}_4,\, \mathbf{u}_5\}$

(h) (6p) Specify how well $\mathbf{A}$ can be approximated by a matrix of rank 2:

$$\inf\{\|\mathbf{A} - \mathbf{B}\| : \mathbf{B} \text{ has rank at most } 2\} = 2$$

Question 4: (25p) Let $\mathbf{Q}$ be an $n \times n$ unitary matrix, and define $\mathbf{A} = \begin{bmatrix} \mathbf{Q} \\ \mathbf{Q} \end{bmatrix}$.

(a) (5p) Define what it means for a matrix to be *orthonormal*.

(b) (10p) Prove that there exists a real number t for which the matrix $\mathbf{B} = t\mathbf{A}$ is orthonormal.

(c) (10p) Specify an "economy size" singular value decomposition of $\mathbf{A}$. In other words, specify a $2n \times n$ orthonormal matrix $\mathbf{U}$, an $n \times n$ diagonal matrix $\mathbf{D}$, and an $n \times n$ unitary matrix $\mathbf{V}$ such that $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^*$.

Solution:

(a) *A matrix* $\mathbf{B}$ *is* ***orthonormal*** *iff* $\mathbf{B}^*\mathbf{B} = \mathbf{I}$. *(In other words, its columns form an orthonormal set.)*

(b) We find that

$$\mathbf{B}^*\mathbf{B} = (t\mathbf{A})^*(t\mathbf{A}) = \begin{bmatrix} t\mathbf{Q}^* & t\mathbf{Q}^* \end{bmatrix} \begin{bmatrix} t\mathbf{Q} \\ t\mathbf{Q} \end{bmatrix} = t\mathbf{Q}^*t\mathbf{Q} + t\mathbf{Q}^*t\mathbf{Q} = 2t^2\mathbf{Q}^*\mathbf{Q} = 2t^2\mathbf{I},$$

where in the last step we used that $\mathbf{Q}$ is orthonormal, so that $\mathbf{Q}^*\mathbf{Q} = \mathbf{I}$.

Now observe that if we pick $t = 1/\sqrt{2}$, then $\mathbf{B}$ will be unitary. ($t = -1/\sqrt{2}$ works too, of course.)

(c) Set $\mathbf{U} = \frac{1}{\sqrt{2}}\mathbf{A}$, so that $\mathbf{U}$ is unitary. Then

$$\mathbf{A} = \sqrt{2}\mathbf{U} = \{\text{Set } \mathbf{D} = \sqrt{2}\mathbf{I}\} = \mathbf{U}\mathbf{D} = \{\text{Set } \mathbf{V} = \mathbf{I}\} = \mathbf{U}\mathbf{D}\mathbf{V}^*.$$

To summarize, $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^*$, with

$$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{Q} \\ \mathbf{Q} \end{bmatrix}, \qquad \mathbf{D} = \sqrt{2}\mathbf{I}, \qquad \mathbf{V} = \mathbf{I}.$$

Alternative solution: It is also the case that $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^*$, with

$$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I} \\ \mathbf{I} \end{bmatrix}, \qquad \mathbf{D} = \sqrt{2}\mathbf{I}, \qquad \mathbf{V} = \mathbf{Q}^*.$$

Question 5: (25p) Compute the QR factorization of $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$. Use the convention that the diagonal entries of $\mathbf{R}$ should be non-negative.

Solution: We perform Gram-Schmidt on the columns of $\mathbf{A}$

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

in order to compute the QR factorization.

We first normalize $\mathbf{a}_1$ to build an ON basis for $\langle \mathbf{a}_1 \rangle$:

$$r_{11} = \|\mathbf{a}_1\| = \sqrt{5}.$$

$$\mathbf{q}_1 = \frac{1}{r_{11}}\mathbf{a}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

Then project $\mathbf{a}_2$ onto the orthogonal complement of $\langle \mathbf{q}_1 \rangle$. First, we compute

$$r_{12} = \mathbf{q}_1^* \mathbf{a}_2 = \frac{1}{\sqrt{5}}[1, -2] \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{5}}(1-6) = -\sqrt{5}.$$

Then

$$\mathbf{a}_2' = \mathbf{a}_2 - r_{12}\mathbf{q}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} - (-\sqrt{5})\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Finally,

$$r_{22} = \|\mathbf{a}_2'\| = \sqrt{5},$$

and

$$\mathbf{q}_2 = \frac{1}{r_{22}}\mathbf{a}_2' = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Putting everything together:

$$\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2] = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{R} = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = \sqrt{5} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Question 6: (10p) Suppose that $\mathbf{A}$ is a 2×2 matrix with the QR factorization

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}.$$

Define the 3×3 matrix $\mathbf{B}$ via

$$\mathbf{B} = \begin{bmatrix} 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \\ 1 & 0 & 0 \end{bmatrix}.$$

Specify the QR factorization of $\mathbf{B}$ (expressed in terms of the QR factorization of $\mathbf{A}$).

Solution: Set $\mathbf{U} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and observe that $\mathbf{UB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{12} & a_{22} \end{bmatrix}$.

Now use the given QR factorization of $\mathbf{A}$

$$\mathbf{UB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{12} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & q_{11} & q_{12} \\ 0 & q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & r_{11} & r_{12} \\ 0 & 0 & r_{22} \end{bmatrix}.$$

Finally move $\mathbf{U}$ over to the right by multiplying both sides by $\mathbf{U}^*$:

$$\mathbf{B} = \underbrace{\mathbf{U}^* \begin{bmatrix} 1 & 0 & 0 \\ 0 & q_{11} & q_{12} \\ 0 & q_{12} & q_{22} \end{bmatrix}}_{=:\mathbf{Q}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & r_{11} & r_{12} \\ 0 & 0 & r_{22} \end{bmatrix}}_{=:\mathbf{R}}.$$

In other words, $\mathbf{B} = \mathbf{QR}$ where

$$\mathbf{Q} = \begin{bmatrix} 0 & q_{11} & q_{12} \\ 0 & q_{12} & q_{22} \\ 1 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r_{11} & r_{12} \\ 0 & 0 & r_{22} \end{bmatrix}.$$

Question 7: (10p) Compute the singular values of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$. Briefly motivate your answer.

Solution: Set $\mathbf{B} = \mathbf{A}^*\mathbf{A} = \begin{bmatrix} 10 & 5 \\ 5 & 5 \end{bmatrix}$. Recall that the eigenvalues of $\mathbf{B}$ are the singular values of $\mathbf{A}$ squared. So all we need to do is to compute the eigenvalues of $\mathbf{B}$.

$$p_{\mathbf{B}}(\lambda) = \det(\lambda\mathbf{I} - \mathbf{B}) = \det \begin{bmatrix} \lambda - 10 & -5 \\ -5 & \lambda - 5 \end{bmatrix} = \lambda^2 - 15\lambda + 25.$$

We find that

$$\lambda_{1,2} = \frac{15}{2} \pm \sqrt{\frac{15^2}{2^2} - 25} = \frac{15}{2} \pm \sqrt{\frac{225}{4} - \frac{100}{4}} = \frac{15}{2} \pm \frac{\sqrt{125}}{2} = \frac{5}{2}(3 \pm \sqrt{5}).$$

Taking square roots, we find

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{\frac{5}{2}(3 + \sqrt{5})}, \qquad \text{and} \qquad \sigma_2 = \sqrt{\lambda_2} = \sqrt{\frac{5}{2}(3 - \sqrt{5})}.$$