Section exam 1 Numerical Analysis: Linear Algebra
9:30am – 10:45am, Sep. 26, 2024.

Instructions:

- ***This is a closed books exam.*** No books or calculators are allowed. However, you are allowed to bring one hand written, single sided formula sheet.
- Try to answer the questions within the space given. If you must use more space, you *can* attach extra pages, but try to avoid it. If you do need extra space, then please mark this very clearly on the exam.
- If a question does not specifically ask for a motivation, then the correct answer alone will give full points. You are welcome to enter a brief explanation of how you arrived at the answer — this might yield partial credit in case the given answer is incorrect.
- Use your time strategically. Questions 1 – 3 should be fast. Questions 5 and 7 are the only ones that require more than minimal computations. Questions 6 and 7 are only worth 10 points each, it may be wise to save these for last.
- In this exam, the default is that for a vector $\mathbf{x}$, the notation $\|\mathbf{x}\|$ denotes the Euclidean norm. For a matrix $\mathbf{A}$, the notation $\|\mathbf{A}\|$ refers to the operator norm with respect to the Euclidean vector norm, and $\|\mathbf{A}\|_{\mathrm{F}}$ refers to the Frobenius norm. If $\mathbf{A}$ is a matrix, then $\mathbf{A}^*$ refers to the complex conjugate of the transpose of $\mathbf{A}$ (or merely the transpose when $\mathbf{A}$ is real).

Name: ____

Section: ____________________

Question	Max points	Scored points
1	5	
2	5	
3	20	
4	25	
5	25	
6	10	
7	10	
Total:		

Question 1: (5p) Let $\mathbf{u} = [1,\, 2,\, 3]^*$ and $\mathbf{v} = [2,\, t,\, 2]$ be two vectors in $\mathbb{R}^3$. For which value(s) of t are $\mathbf{u}$ and $\mathbf{v}$ orthogonal?

Question 2: (5p) Recall that a *Householder reflector* $\mathbf{F}$ is a special unitary matrix that you construct in order to map a given vector $\mathbf{x}$ to the vector $\mathbf{F}\mathbf{x} = \|\mathbf{x}\|\, \mathbf{e}_1 = [\|\mathbf{x}\|,\, 0,\, 0,\, \dots,\, 0]^*$. Give a formula for $\mathbf{F}$, expressed in terms of the vector $\mathbf{x}$.

Question 3: (20p) Let $\mathbf{A}$ be a 5×5 matrix with the singular value decomposition

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^*,$$

where

$$\mathbf{D} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and where the matrices

$$\mathbf{U} = \big[\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \mathbf{u}_4,\, \mathbf{u}_5\big], \qquad \mathbf{V} = \big[\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \mathbf{v}_4,\, \mathbf{v}_5\big]$$

are unitary. *Please give only the answers. Motivations will not be graded.*

(a) (2p) Specify the spectral norm of $\mathbf{A}$: $\quad \|\mathbf{A}\| =$

(b) (2p) Specify the Frobenius norm of $\mathbf{A}$: $\quad \|\mathbf{A}\|_{\mathrm{F}} =$

(c) (2p) Specify the rank of $\mathbf{A}$: $\quad \mathrm{rank}(\mathbf{A}) =$

(d) (2p) Specify an orthonormal basis for the column space of $\mathbf{A}$:

(e) (2p) Specify an orthonormal basis for the row space of $\mathbf{A}$:

(f) (2p) Specify an orthonormal basis for the null space of $\mathbf{A}$:

(g) (2p) Specify an orthonormal basis for the null space of $\mathbf{A}^*$:

(h) (6p) Specify how well $\mathbf{A}$ can be approximated by a matrix of rank 2:

$$\inf\{\|\mathbf{A} - \mathbf{B}\| \,:\, \mathbf{B} \text{ has rank at most } 2\} =$$

Question 4: (25p) Let $\mathbf{Q}$ be an $n \times n$ unitary matrix, and define $\mathbf{A} = \begin{bmatrix} \mathbf{Q} \\ \mathbf{Q} \end{bmatrix}$.

(a) (5p) Define what it means for a matrix to be *orthonormal.*

(b) (10p) Prove that there exists a real number t for which the matrix $\mathbf{B} = t\mathbf{A}$ is orthonormal.

(c) (10p) Specify an "economy size" singular value decomposition of $\mathbf{A}$. In other words, specify a $2n \times n$ orthonormal matrix $\mathbf{U}$, an $n \times n$ diagonal matrix $\mathbf{D}$, and an $n \times n$ unitary matrix $\mathbf{V}$ such that $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^*$.

Question 5: (25p) Compute the QR factorization of $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$. Use the convention that the diagonal entries of $\mathbf{R}$ should be non-negative.

Question 6: (10p) Suppose that $\mathbf{A}$ is a 2×2 matrix with the QR factorization

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}.$$

Define the 3×3 matrix $\mathbf{B}$ via

$$\mathbf{B} = \begin{bmatrix} 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \\ 1 & 0 & 0 \end{bmatrix}.$$

Specify the QR factorization of $\mathbf{B}$ (expressed in terms of the QR factorization of $\mathbf{A}$).

Hint: There exists a very simple unitary matrix $\mathbf{U}$ *such that* $\mathbf{UB} = \begin{bmatrix} 1 & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix}$.

Question 7: (10p) Compute the singular values of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$. Briefly motivate your answer.

Hint: It might be helpful to compute the eigenvalues of $\mathbf{A}^*\mathbf{A}$.