Midterm exam for Numerical Analysis: Linear Algebra
9:00am – 10:45am, Oct. 28, 2021. Closed books.

Instructions:

- ***This is a closed books exam.*** No personal notes or calculators are allowed.
- Try to answer questions within the space given. If you run out of space, then you can attach extra pages. If you do, then please mark this very clearly on the exam.
- If a question does not specifically ask for a motivation, then the correct answer alone will give full points. You are welcome to enter a brief explanation of how you arrived at the answer — this might yield partial credit in case the given answer is incorrect. (An exception is question 1, where no motivations will be considered in the grading.)
- In this exam, the default is that for a vector $\mathbf{x}$, the notation $\|\mathbf{x}\|$ denotes the Euclidean norm. For a matrix $\mathbf{A}$, the notation $\|\mathbf{A}\|$ refers to the operator norm with respect to the Euclidean vector norm, and $\|\mathbf{A}\|_{\mathrm{F}}$ refers to the Frobenius norm.
- A star next to a question means that you might find it slightly more challenging. Note that most of these questions are given only a small amount of points, so you might want to save them for last.
- If $\mathbf{A}$ is a matrix, then $\mathbf{A}^*$ refers to the complex conjugate of the transpose of $\mathbf{A}$ (or merely the transpose when $\mathbf{A}$ is real).

Advice: Question 1 should not take long — if a problem resists, then just move on and return to it later.

Name: ______________________________

Question	Max points	Scored points
1	30	
2	15	
3	15	
4	15	
5	25	
Total:		

Question 1: (30p) For this question, please enter only the answer. (Motivations will not be considered in grading it.) 6 points per sub question.

(a) The 5×3 matrix $\mathbf{A}$ has the singular values $\{3, 2, 1\}$. Specify the spectral and Frobenius norms of $\mathbf{A}$ and its inverse:

$\|\mathbf{A}\| = 3$ $\quad$ $\|\mathbf{A}\|_{\mathrm{F}} = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$ $\quad$ $\|\mathbf{A}^{-1}\| = 1$ $\quad$ $\|\mathbf{A}^{-1}\|_{\mathrm{F}} = \sqrt{1/3^2 + 1/2^2 + 1^2} = 7/6$

(b) The matrix $\mathbf{Q} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & \alpha & 2 \\ 2 & 0 & \beta \\ 0 & 2 & \gamma \\ -1 & 1 & 0 \end{bmatrix}$ is orthonormal (so that $\mathbf{Q}^*\mathbf{Q} = \mathbf{I}$). Specify the missing entries:

$\alpha = 1$ $\quad$ $\beta = -1$ $\quad$ $\gamma = -1$

To determine α, use that $\mathbf{q}_1 \cdot \mathbf{q}_2 = 0$. The determine β and γ, use that $\mathbf{q}_1 \cdot \mathbf{q}_3 = 0$ and that $\mathbf{q}_2 \cdot \mathbf{q}_3 = 0$.

(c) Let $\mathbf{A}$ be an $m \times n$ non-zero matrix, and let $\mathbf{b}$ be a vector of size $m \times 1$. Specify the definition of the "least square solution $\mathbf{x}$ to the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$". For full credit, you cannot assume that $\mathbf{A}$ has full column rank.

There are two conditions:
(1) The vector $\mathbf{x}$ should satisfy $\|\mathbf{A}\mathbf{x} - \mathbf{b}\| = \inf_{\mathbf{y}} \|\mathbf{A}\mathbf{y} - \mathbf{b}\|$.
(2) Among all vectors $\mathbf{x}$ that satisfy (1), the least square solution is the one with the smallest norm.

Answers that omitted part (2), or relied on the invertibility of the normal equations, got partial credit.

(d) You are given a matrix $\mathbf{A}$ of size $m \times n$ and a vector $\mathbf{b}$ of size $m \times 1$. Let $\mathbf{x}$ denote the least squares solution to the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$. You know that $\|\mathbf{b}\| = 5$ and that $\|\mathbf{A}\mathbf{x}\| = 2$. Specify the length of the residual vector $\mathbf{r} = \mathbf{b} - \mathbf{A}\mathbf{x}$:

Observe that $\mathbf{A}\mathbf{x} \perp \mathbf{r}$. In consequence:

$$\|\mathbf{b} - \mathbf{A}\mathbf{x}\| = \sqrt{\|\mathbf{b}\|^2 - \|\mathbf{A}\mathbf{x}\|^2} = \sqrt{5^2 - 2^2} = \sqrt{21}.$$

(e) Let n be a positive integer, and let $\mathbf{I}_n$ denote the $n \times n$ identity matrix. Consider the matrix $\mathbf{A}_n = \begin{bmatrix} \mathbf{I}_n & \mathbf{I}_n \\ -\mathbf{I}_n & \mathbf{I}_n \end{bmatrix}$. Specify the spectral and Frobenius norms of $\mathbf{A}_n$:

$\|\mathbf{A}_n\|_{\mathrm{F}} = \sqrt{4n} = 2\sqrt{n}$ $\quad$ $\|\mathbf{A}_n\| = \sqrt{\|\mathbf{A}_n^*\mathbf{A}_n\|} = \sqrt{\left\| \begin{bmatrix} 2\mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & 2\mathbf{I}_n \end{bmatrix} \right\|} = \sqrt{2\|\mathbf{I}_{2n}\|} = \sqrt{2}.$

Hint: To determine the spectral norm, you may find it useful to evaluate $\mathbf{A}_n^*\mathbf{A}_n$.

Question 2: (15p) Let $\mathbf{A}$ be an $m \times n$ matrix with the singular value decomposition

$$\underset{m\times n}{\mathbf{A}} = \underset{m\times m}{\mathbf{U}} \; \underset{m\times n}{\mathbf{D}} \; \underset{n\times n}{\mathbf{V}^*},$$

where $\mathbf{U}$ and $\mathbf{V}$ are unitary, and $\mathbf{D}$ is a diagonal matrix. Set $p = \max(m, n)$, and let $\{\sigma_j\}_{j=1}^{p}$ denote the diagonal entries of $\mathbf{D}$, ordered so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$, as usual.

(a) (3p) Prove that since $\mathbf{U}$ is unitary, $\|\mathbf{U}\mathbf{x}\| = \|\mathbf{x}\|$ for every $\mathbf{x} \in \mathbb{C}^{m\times 1}$.

(b) (6p) Prove that $\|\mathbf{A}\mathbf{x}\| \leq \sigma_1\|\mathbf{x}\|$ for all vectors $\mathbf{x}$.

(c) (6p) Prove that there exists a non-zero vector $\mathbf{x}$ such that $\|\mathbf{A}\mathbf{x}\| = \sigma_1\|\mathbf{x}\|$.

(a) For any vector $\mathbf{x}$, we have

$$\|\mathbf{U}\mathbf{x}\|^2 = (\mathbf{U}\mathbf{x})^* \cdot (\mathbf{U}\mathbf{x}) = \mathbf{x}^*\mathbf{U}^*\mathbf{U}\mathbf{x} = \{\mathbf{U}^*\mathbf{U} = \mathbf{I}\} = \mathbf{x}^*\mathbf{x} = \|\mathbf{x}\|^2.$$

(b) Fix a vector $\mathbf{x} \in \mathbf{C}^n$. Set $\mathbf{y} = \mathbf{V}^*\mathbf{x}$. Then

$$\begin{aligned}\|\mathbf{A}\mathbf{x}\|^2 = \|\mathbf{U}\mathbf{D}\mathbf{V}^*\mathbf{x}\|^2 = \{\text{Use (a)}\} = \|\mathbf{D}\mathbf{V}^*\mathbf{x}\|^2 = \|\mathbf{D}\mathbf{y}\|^2 = \sum_{j=1}^{p} |\sigma_j y_j|^2 \\ \leq \sigma_1^2 \sum_{j=1}^{p} |y_j|^2 \leq \sigma_1^2\|\mathbf{y}\|^2 = \{\text{Use (a)}\} = \sigma_1^2\|\mathbf{x}\|^2.\end{aligned}$$

(c) Pick $\mathbf{x} = \mathbf{v}_1$, where $\mathbf{v}_1 = \mathbf{V}(:,1)$ is the first column of $\mathbf{V}$. Then

$$\|\mathbf{A}\mathbf{x}\| = \|\mathbf{U}\mathbf{D}\mathbf{V}^*\mathbf{v}_1\| = \|\mathbf{U}\mathbf{D}\mathbf{e}_1\| = \|\sigma_1\mathbf{u}_1\| = \sigma_1,$$

where $\mathbf{e}_1 = [1, 0, 0, \ldots, 0]^*$ is the first canonical unit vector, and $\mathbf{u}_1 = \mathbf{U}(:,1)$.

Note: Many answers involved the argument $\|\mathbf{U}\mathbf{D}\mathbf{V}^*\mathbf{x}\| = \|\mathbf{D}\mathbf{V}^*\mathbf{x}\| = \|\mathbf{D}\mathbf{x}\|$ *with the argument being that* $\mathbf{U}$ *and* $\mathbf{V}$ *are unitary. The first equality is correct, but the second is emphatically not.*

Question 3: (15p) Consider the function

$$f(x) = \frac{1 - \sqrt{1 - x^3}}{x^2},$$

defined for $x \in (0, 1]$. For questions (a) and (b), please provide brief motivations — merely answering "yes" or "no" will not give full credit!

(a) (4p) Is the function f well-conditioned on the interval (0, 0.5]? (See hint at bottom of page.)

(b) (4p) Is the function f well-conditioned on the interval (0, 1]? (See hint at bottom of page.)

(c) (4p) Estimate the number $y = f(10^{-7})$ in such a way that the first fifteen digits (beyond the first nonzero digit) are correct. Your answer should be an actual number (not a formula).

(d) (3p) Describe how you would evaluate $f(x)$ in Matlab (or on a scientific calculator) for small x to reduce the effect of round-off errors.

Before we answer the questions, let us differentiate f:

$f'(x) = (-2)x^{-3}(1 - \sqrt{1 - x^3}) + x^{-2}(3x^2)(1/2)(1 - x^3)^{-1/2} = -2\,x^{-3}(1 - \sqrt{1 - x^3}) + (3/2)(1 - x^3)^{-1/2}.$

We see that $f'(x)$ blows up as $x \to 1$. The other potential trouble point is $x \to 0$. To see what happens there, let us do a Taylor expansion:

$$f(x) = x^{-2}(1 - (1 - \frac{1}{2}x^3 + O(x^6))) = x^{-2}(\frac{1}{2}x^3 + O(x^6)) = \frac{1}{2}x + O(x^4).$$

(a) The only potential trouble point is $x \to 0$. We find:

$$\kappa_f(x) = \frac{|f'(x)|\,|x|}{|f(x)|} = \frac{|(1/2) + O(x^3)|\,|x|}{|(1/2)x + O(x^4)|} = 1 + O(x^3).$$

So the function is well-conditioned for $x \in (0, 1/2]$.

Note: For a full 4p score, you needed the right answer, and an argument that clearly shows why the limit of $\kappa_f(x) \to 0$ is finite. 3p for an answer without the argument on the limit.

(b) As $x \to 1$, we have $|f'(x)| \to \infty$, while $|x| \to 1$ and $|f(x)| \to 1$. So κ blows up and the problem is ill-conditioned.

(c) Using the first term in the Taylor expansion, we get

$$f(x) \approx \frac{1}{2}x = 0.5 \cdot 10^{-7}.$$

(The question did not ask for a proof of the number of correct digits, but out of curiosity, let us try to estimate it: The error term is of the order x^4, so the relative error is of magnitude $x^4/x \sim x^3 \sim 10^{-21}$ so it appears that our approximation does have 15+ correct digits.)

(d) Let us rearrange f to a mathematically equivalent expression:

$$f(x) = \frac{1 - \sqrt{1 - x^3}}{x^2} \cdot \frac{1 + \sqrt{1 - x^3}}{1 + \sqrt{1 - x^3}} = \frac{1 - (1 - x^3)}{x^2(1 + \sqrt{1 - x^3})} = \frac{x^3}{x^2(1 + \sqrt{1 - x^3})} = \frac{x}{(1 + \sqrt{1 - x^3})}.$$

The last expression can be just typed in to matlab or a calculator, and will give a good answer for x close to zero. *The formula given can be used to answer (c)!*

Question 4: (15p) Consider the three matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 4 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \\ 0 & 3 & -3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 5.0 & 2.2 & 3.8 \\ 0 & 1 & 1 \\ 0 & 0.4 & 1.6 \\ 0 & 3 & -3 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 1 & 1 \\ -1 & -1 & 3 \\ -4 & 1 & -1 \end{bmatrix}.$$

Fully correct answers yield full score without any motivation. In case there are errors in the answer, a brief description of how you computed it *might* result in partial credit.

(a) (6p) Specify a Householder reflector $\mathbf{H}$ such that $\mathbf{HA} = \mathbf{B}$.

(b) (6p) Specify a Householder reflector $\mathbf{G}$ such that $\mathbf{AG} = \mathbf{C}$.

(c) (3p) Specify a QR factorization $\mathbf{A}^* = \mathbf{QR}$. Please specify only the answer, no motivation.

(a) Let $\mathbf{x} = [4, 0, 3, 0]^*$ denote the first column of $\mathbf{A}$. We want $\mathbf{Hx} = 5\,\mathbf{e}_1$. This is attained for $\mathbf{H} = \mathbf{I} - (2/\|\mathbf{u}\|^2)\mathbf{uu}^*$ where

$$\mathbf{u} = 5\mathbf{e}_1 - \mathbf{x} = [5, 0, 0, 0]^* - [4, 0, 3, 0]^* = [1, 0, -3, 0]^*.$$

We find

$$\mathbf{H} = \mathbf{I} - (2/10) \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \end{bmatrix} [1, 0, -3, 0] = \begin{bmatrix} 0.8 & 0 & 0.6 & 0 \\ 0 & 1 & 0 & 0 \\ 0.6 & 0 & -0.8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Let $\mathbf{y} = [4, 2, 4]^*$ denote the first row of $\mathbf{A}$. We want $\mathbf{Gy} = 6\,\mathbf{e}_3$. This is attained for $\mathbf{G} = \mathbf{I} - (2/\|\mathbf{v}\|^2)\mathbf{vv}^*$ where

$$\mathbf{v} = 6\mathbf{e}_3 - \mathbf{y} = [0, 0, 6]^* - [4, 2, 4]^* = [-4, -2, 2]^*.$$

We find

$$\mathbf{G} = \mathbf{I} - (2/24) \begin{bmatrix} -4 \\ -2 \\ 2 \end{bmatrix} [-4, -2, 2] = \frac{1}{3} \begin{bmatrix} -1 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

(c) From (b), we know that $\mathbf{AG} = \mathbf{C}$, so that

$$\mathbf{A}^* = \mathbf{GC}^*.$$

This is *almost* a QR factorization of $\mathbf{A}^*$, since $\mathbf{G}$ is unitary, and $\mathbf{C}^*$ is *lower* triangular. To convert $\mathbf{C}^*$ to *upper* triangular, all we need to do is to reorder its rows, and then of course reorder the columns of $\mathbf{G}$ accordingly. We find

$$\mathbf{A}^* = \underbrace{\frac{1}{3} \begin{bmatrix} 2 & -2 & -1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}}_{=\mathbf{Q}} \underbrace{\begin{bmatrix} 6 & 1 & 3 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -4 \end{bmatrix}}_{=\mathbf{R}}.$$

Question 5: (25p) Let ε be a positive number, and define

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & \varepsilon \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) (4p) Let $\mathbf{x}$ denote the least square solution to the linear system $\mathbf{Mx} = \mathbf{b}$, and set $\mathbf{r} = \mathbf{b} - \mathbf{Mx}$. Specify $\mathbf{x}$ and $\mathbf{r}$. *Hint: Observe that* $\mathbf{M}$ *does not satisfy the assumption we often make in the lectures that the columns be linearly independent.*

(b) (4p) Specify the normal equations for the least square problem $\mathbf{Ax} = \mathbf{b}$.

(c) (5p) Let $\mathbf{x}$ denote the least square solution to the linear system $\mathbf{Ax} = \mathbf{b}$, and set $\mathbf{r} = \mathbf{b} - \mathbf{Ax}$. Specify $\mathbf{x}$ and $\mathbf{r}$.

(a) The column space of $\mathbf{M}$ is the set of all vectors of the form $[t, t, 0]^*$ for $t \in \mathbb{R}$. The orthogonal projection of $\mathbf{b}$ onto this space is the vector $\mathbf{b}' = [1, 1, 0]^*$. So the solution $\mathbf{x} = [x_1, x_2, x_3]^*$ must satisfy

$$x_1 + x_2 = 1, \qquad x_1 + x_2 = 1, \qquad 0 = 0.$$

The solution set consists of all vectors of the form $\mathbf{x} = [t, 1-t, 0]^*$ for $t \in \mathbb{R}$. The solution that is closest to the origin is $\mathbf{x} = [0.5, 0.5, 0]^*$. So

$$\mathbf{x} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}, \qquad \mathbf{r} = \mathbf{b} - \mathbf{b}' = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) The normal equations are $\mathbf{A}^*\mathbf{Ax} = \mathbf{A}^*\mathbf{b}$, which in this case works out to:

$$\begin{bmatrix} 2 & 2 \\ 2 & 2+\varepsilon^2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 2+\varepsilon \end{bmatrix}.$$

(c) Solving the normal equation we derived in (b), we find that

$$\mathbf{x} = \begin{bmatrix} 2 & 2 \\ 2 & 2+\varepsilon^2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2+\varepsilon \end{bmatrix} = \frac{1}{2\varepsilon^2} \begin{bmatrix} 2+\varepsilon^2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2+\varepsilon \end{bmatrix} = \frac{1}{2\varepsilon^2} \begin{bmatrix} 2\varepsilon^2 - 2\varepsilon \\ 2\varepsilon \end{bmatrix} = \begin{bmatrix} 1 - 1/\varepsilon \\ 1/\varepsilon \end{bmatrix}$$

For the residual, we get

$$\mathbf{r} = \mathbf{Ax} - \mathbf{b} = \begin{bmatrix} (1-1/\varepsilon) + 1/\varepsilon \\ (1-1/\varepsilon) + 1/\varepsilon \\ \varepsilon(1/\varepsilon) \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}.$$

(d) (5p) Suppose that $\varepsilon = 10^{-10}$ and that you work on a computer for which $\epsilon_{\text{mach}} = 10^{-15}$. Discuss what difficulties you may encounter if you use the normal equations you specified in (b) to compute the solution to (c).

(e) (5p) Let $\mathbf{A} = \mathbf{QR}$ be the QR factorization of $\mathbf{A}$. Specify $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{R}^{-1}$. Would you encounter difficulties if $\varepsilon = 10^{-10}$, and you use QR to compute the least square solution $\mathbf{x}$?

(d) If $\varepsilon = 10^{-10}$, then when we evaluate the normal equations in floating point arithmetic, the entry $2 + \varepsilon^2$ will be rounded to 2. So the normal equations we will actually obtain involve the singular coefficient matrix $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

(e) We perform Gram-Schmidt on the columns of $\mathbf{A}$ to find that

$$\begin{aligned}
r_{11} &= \|\mathbf{a}_1\| = \sqrt{2}, \\
\mathbf{q}_1 &= \mathbf{a}_1/r_{11} = [1/\sqrt{2}, 1/\sqrt{2}, 0]^*, \\
r_{12} &= \mathbf{q}_1 \cdot \mathbf{a}_2 = \sqrt{2}, \\
\mathbf{a}_2' &= \mathbf{a}_2 - r_{12}\mathbf{q}_1 = [0, 0, \varepsilon]^*, \\
r_{22} &= \|\mathbf{a}_2'\| = \varepsilon, \\
\mathbf{q}_2 &= \mathbf{a}_2'/r_{22} = [0, 0, 1]^*.
\end{aligned}$$

So

$$\mathbf{A} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}}_{=\mathbf{Q}} \underbrace{\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \varepsilon \end{bmatrix}}_{=\mathbf{R}}.$$

We then get

$$\mathbf{R}^{-1} = \frac{1}{\sqrt{2}\varepsilon} \begin{bmatrix} \varepsilon & -\sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\varepsilon \\ 0 & 1/\varepsilon \end{bmatrix}.$$

Using the QR factors to compute the least squares solution, we get

$$\mathbf{x} = \mathbf{R}^{-1}\mathbf{Q}^*\mathbf{b} = \begin{bmatrix} 1/\sqrt{2} & -1/\varepsilon \\ 0 & 1/\varepsilon \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\varepsilon \\ 0 & 1/\varepsilon \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - 1/\varepsilon \\ 1/\varepsilon \end{bmatrix}.$$

All these entries can be evaluated accurately. (Well, to be precise, to about 5 accurate digits, which is the best we can hope for since in this case, $\kappa(\mathbf{A}) \approx 2\sqrt{2}/\varepsilon \sim 10^{10}$.)