Midterm exam for Numerical Analysis: Linear Algebra

9:00am – 10:45am, Oct. 28, 2021. Closed books.

Instructions:

- ***This is a closed books exam.*** No books, personal notes, or calculators are allowed.
- Try to answer questions within the space given. If you must use more space, you *can* attach extra pages, but try to avoid it. If you do need extra space, then please mark this very clearly on the exam.
- If a question does not specifically ask for a motivation, then the correct answer alone will give full points. You are welcome to enter a brief explanation of how you arrived at the answer — this might yield partial credit in case the given answer is incorrect. (An exception is question 1, where no motivations will be considered in the grading.)
- In this exam, the default is that for a vector $\mathbf{x}$, the notation $\|\mathbf{x}\|$ denotes the Euclidean norm. For a matrix $\mathbf{A}$, the notation $\|\mathbf{A}\|$ refers to the operator norm with respect to the Euclidean vector norm, and $\|\mathbf{A}\|_{\mathrm{F}}$ refers to the Frobenius norm.
- If $\mathbf{A}$ is a matrix, then $\mathbf{A}^*$ refers to the complex conjugate of the transpose of $\mathbf{A}$ (or merely the transpose when $\mathbf{A}$ is real).

Advice: Question 1 should not take much time — if a problem resists, then just move on and return to it later.

Name: ______________________________

Section: ______________

Question	Max points	Scored points
1	30	
2	15	
3	15	
4	15	
5	25	
Total:		

Question 1: (30p) For this question, please enter only the answer. (Motivations will not be considered in grading it.) 6 points per sub question.

(a) The 5×3 matrix $\mathbf{A}$ has the singular values $\{3, 2, 1\}$. Specify the spectral and Frobenius norms of $\mathbf{A}$ and its inverse:

$\|\mathbf{A}\| =$ $\qquad$ $\|\mathbf{A}\|_{\mathrm{F}} =$ $\qquad$ $\|\mathbf{A}^{-1}\| =$ $\qquad$ $\|\mathbf{A}^{-1}\|_{\mathrm{F}} =$

(b) The matrix $\mathbf{Q} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & \alpha & 2 \\ 2 & 0 & \beta \\ 0 & 2 & \gamma \\ -1 & 1 & 0 \end{bmatrix}$ is orthonormal (so that $\mathbf{Q}^*\mathbf{Q} = \mathbf{I}$). Specify the missing entries:

$\alpha =$ $\qquad$ $\beta =$ $\qquad$ $\gamma =$

(c) Let $\mathbf{A}$ be an $m \times n$ non-zero matrix, and let $\mathbf{b}$ be an $m \times 1$ vector. Specify the definition of the "least squares solution $\mathbf{x}$ to the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$". For full credit, you cannot assume that $\mathrm{rank}(\mathbf{A}) = n$.

(d) You are given a matrix $\mathbf{A}$ of size $m \times n$ and a vector $\mathbf{b}$ of size $m \times 1$. Let $\mathbf{x}$ denote the least squares solution to the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$. You know that $\|\mathbf{b}\| = 5$ and that $\|\mathbf{A}\mathbf{x}\| = 2$. Specify the length of the residual vector $\mathbf{r} = \mathbf{b} - \mathbf{A}\mathbf{x}$:

$$\|\mathbf{b} - \mathbf{A}\mathbf{x}\| =$$

(e) Let n be a positive integer, and let $\mathbf{I}_n$ denote the $n \times n$ identity matrix. Consider the matrix $\mathbf{A}_n = \begin{bmatrix} \mathbf{I}_n & \mathbf{I}_n \\ -\mathbf{I}_n & \mathbf{I}_n \end{bmatrix}$. Specify the spectral and Frobenius norms of $\mathbf{A}_n$:

$\|\mathbf{A}_n\|_{\mathrm{F}} =$ $\qquad$ $\|\mathbf{A}_n\| =$

Hint: To determine the spectral norm, you may find it useful to evaluate $\mathbf{A}_n^*\mathbf{A}_n$.

Question 2: (15p) Let $\mathbf{A}$ be an $m \times n$ matrix with the singular value decomposition

$$\underset{m\times n}{\mathbf{A}} = \underset{m\times m}{\mathbf{U}}\ \underset{m\times n}{\mathbf{D}}\ \underset{n\times n}{\mathbf{V}^*},$$

where $\mathbf{U}$ and $\mathbf{V}$ are unitary, and $\mathbf{D}$ is a diagonal matrix. Set $p = \max(m, n)$, and let $\{\sigma_j\}_{j=1}^{p}$ denote the diagonal entries of $\mathbf{D}$, ordered so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$, as usual.

(a) (3p) Prove that since $\mathbf{U}$ is unitary, $\|\mathbf{U}\mathbf{x}\| = \|\mathbf{x}\|$ for every $\mathbf{x} \in \mathbb{C}^{m\times 1}$.

(b) (6p) Prove that $\|\mathbf{A}\mathbf{x}\| \leq \sigma_1\|\mathbf{x}\|$ for all vectors $\mathbf{x}$.

(c) (6p) Prove that there exists a non-zero vector $\mathbf{x}$ such that $\|\mathbf{A}\mathbf{x}\| = \sigma_1\|\mathbf{x}\|$.

Question 3: (15p) Consider the function

$$f(x) = \frac{1 - \sqrt{1 - x^3}}{x^2},$$

defined for $x \in (0, 1]$. For questions (a) and (b), please provide brief motivations — merely answering "yes" or "no" will not give full credit!

(a) (4p) Is the function f well-conditioned on the interval $(0,\ 0.5]$? (See hint at bottom of page.)

(b) (4p) Is the function f well-conditioned on the interval $(0,\ 1]$? (See hint at bottom of page.)

(c) (4p) Estimate the number $y = f(10^{-7})$ in such a way that the first fifteen digits (beyond the first nonzero digit) are correct. Your answer should be an actual number (not a formula).

(d) (3p) Describe how you would evaluate $f(x)$ in Matlab (or on a scientific calculator) for small x to reduce the effect of round-off errors.

Hint: To answer (a) and (b), evaluate the "relative condition number" $\kappa_f(x) = \frac{|f'(x)|\,|x|}{|f(x)|}$.

Question 4: (15p) Consider the three matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 4 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \\ 0 & 3 & -3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 5 & 2.2 & 3.8 \\ 0 & 1 & 1 \\ 0 & 0.4 & 1.6 \\ 0 & 3 & -3 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 1 & 1 \\ -1 & -1 & 3 \\ -4 & 1 & -1 \end{bmatrix}.$$

Fully correct answers yield full credit without any motivation. In case there are errors in the answer, a brief description of how you computed it *might* result in partial credit.

(a) (6p) Specify a Householder reflector $\mathbf{H}$ such that $\mathbf{HA} = \mathbf{B}$.

(b) (6p) Specify a Householder reflector $\mathbf{G}$ such that $\mathbf{AG} = \mathbf{C}$.

(c) (3p) Specify a QR factorization $\mathbf{A}^* = \mathbf{QR}$. Please specify only the answer, no motivation.

$\mathbf{Q} =$

$\mathbf{R} =$

Hint: Note that part (c) is only worth 3p! Use your time wisely.

Question 5: (25p) Let ε be a positive number, and define

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & \varepsilon \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) (4p) Let $\mathbf{x}$ denote the least squares solution to the linear system $\mathbf{Mx} = \mathbf{b}$, and set $\mathbf{r} = \mathbf{b} - \mathbf{Mx}$. Specify $\mathbf{x}$ and $\mathbf{r}$. *Hint: Observe that* $\mathbf{M}$ *does not satisfy the assumption we often make in the lectures that the columns be linearly independent.*

(b) (4p) Specify the "normal equations" for the least squares problem $\mathbf{Ax} = \mathbf{b}$.

(c) (6p) Let $\mathbf{x}$ denote the least squares solution to the linear system $\mathbf{Ax} = \mathbf{b}$, and set $\mathbf{r} = \mathbf{b} - \mathbf{Ax}$. Specify $\mathbf{x}$ and $\mathbf{r}$.

Continued on the next page!

Question 5 continued:

(d) (4p) Suppose that $\varepsilon = 10^{-10}$ and that you work on a computer for which $\epsilon_{\text{mach}} = 10^{-15}$. Discuss what difficulties you may encounter if you use the normal equations you specified in (b) to compute the solution to (c).

(e) (7p) Let $\mathbf{A} = \mathbf{QR}$ be a QR factorization of $\mathbf{A}$. Specify $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{R}^{-1}$. Would you encounter difficulties if $\varepsilon = 10^{-10}$, and you use the QR factors to compute the least squares solution $\mathbf{x}$?