# DISCRETISATION OF CERTAIN EVOLUTION EQUATIONS WITH MEMORY USING CONVOLUTION QUADRATURE

PER-GUNNAR MARTINSSON

ABSTRACT. We discuss techniques for discretising a Volterra type integro-differential equation, in which the integral operator is a convolution with an exponentially decreasing scalar positive definite function multiplied by a positive definite operator, such as an elliptic differential operator.

We start by describing how one can use Laplace tranforms to derive first and second order quadrature rules for convolution integrals. We then prove error bounds for two classes of kernels of relevance for our evolution equations.

It is known that the solutions to the integro-differential equations described above decrease exponentially. We show that by using the convolution quadrature rules derived in the first part of the thesis to discretise the integral term we obtain methods that are first and second order accurate in time, and for which we can derive error bounds with an exponentially decreasing factor. The results are applied to fully discrete methods obtained by discretization in space by standard Galerkin finite element methods.

1991 *Mathematics Subject Classification.* 45K05, 65D32, 65R20, 65M60.

*Key words and phrases.* Partial integro-differential equation, memory term, convolution quadrature.

## Contents

## 0. Acknowledgements

I am deeply indebted to my supervisor, Professor Vidar Thomée, for all his support throughout the process of creating this thesis. He was always there when I needed him, showing me all these little tricks of the trade, encouraging me when I lost heart, bringing me down from the skies when I got carried away and last but not least spending an immense amount of time reading the text as it slowly evolved during numerous cycles of reviews and corrections. I am also indebted to Dr William McLean at the University of New South Wales in Sydney for providing many useful pointers in the early stages of the work and for hosting me during a visit to UNSW in the autumn of 1997.

A large amount of the work presented was done during the spring of 1998, when I was given the opportunity to spend five months at the Mittag-Leffler Institute in Stockholm during their year on numerical methods for PDEs. In the course of the spring I got to meet an impressive number of outstanding scholars in the field and it would be hard to imagine a more stimulating environment for conducting mathematical research. I am indebted to the organisers of the program and to my supervisor for arranging this visit.

I would also like to thank my parents for encouraging me to learn for as long as I can remember, Mats Larson for being my mentor and guide on things mathematical and worldly these last seven years and finally Emma for making this year so very much more joyous.

## 1. Introduction

1.1. **The continuous problem.** In this thesis we will discuss discretisation techniques for a Volterra type integro-differential equation of the form,

$$(1.1)\qquad \begin{aligned} &u_t(t) + \int_0^t \beta(t-s)Au(s)\,ds = 0, \quad t > 0,\\ &u(0) = v, \end{aligned}$$

set in a Hilbert space $H$. We assume that $A$ is a positive definite, self-adjoint operator with a compact inverse, typically an elliptic differential operator with some boundary conditions. The function $\beta$ is a scalar function which we refer to to as the *integration kernel* or simply *kernel* of the equation. We will assume throughout the thesis that it is *positive definite*, by which we mean that

$$(1.2)\qquad \int_0^T \int_0^t \beta(t-s)\,\varphi(t)\,\varphi(s)\,ds\,dt \geq 0, \quad \forall\, T \geq 0, \quad \varphi \in C([0,\,T]).$$

A straightforward energy argument shows that positivity, in combination with the positive definiteness of $A$ guarantees stability, $||u(t)|| \leq ||v||$.

The mathematical properties of (1.1) depend in an essential way on the kernel $\beta$. For instance, if $\beta(t) = e^{-\alpha t}$, one can differentiate (1.1) and find that it is an integrated form of the damped wave equation,

$$(1.3)\qquad \begin{aligned} &u_{tt} + \alpha u_t + Au = 0, \quad t > 0,\\ &u(0) = v, \quad u_t(0) = 0. \end{aligned}$$

This equation is hyperbolic in character and has exponentially decaying solutions. On the other hand, if $\beta(t) = t^{\mu-1}$ for some $\mu \in (0,\,1)$, then the equation displays certain features of parabolic equations. It was shown in Lubich, Sloan and Thomée [4] that the solutions are smoothed as follows,

$$(1.4)\qquad |u^{(m)}(t)|_{\gamma+\nu} \leq Ct^{-(\mu+1)\gamma-m}|v|_\nu, \quad \text{where} \quad |v|_\nu = ||A^{\nu/2}v||,$$

for $\nu \geq 0$, for $|\gamma| \leq 1$ if $m > 0$, and for $0 \leq \gamma \leq 1$ if $m = 0$. These two cases are intermediate between the extreme cases, $\beta(t) \equiv 1$ and $\beta(t) \equiv \delta(t)$, the Dirac $\delta$-function, in which cases (1.1) turn into the wave equation and the heat equation, respectively. For further analysis of the mathematical properties we refer to MacCamy and Wong [7], Londen [2], Dafermos and Nohel [1] and Staffans [11].

Equations of the form (1.1), and non linear versions thereof, appear when modelling physical phenomena with a memory effect, for instance heat conduction in a material with changing permeability or visco-elasticity. See MacCamy [5, 6].

1.2. **Discretisation techniques.** Even though the core results in this thesis concern time discretisation techniques valid for a general equation of the form (1.1) our main purpose is to derive and analyse techniques for space and time discretisation when $H$ is some function space over a subset of $\mathbb{R}^n$ and $A$ is an elliptic differential operator. We view the discretisation of such an equation as a two step process. First we discretise in space using a standard Galerkin method. For $S_h$ a finite element space we define a spatially discrete solution $u_h(t) \in S_h$ by,

$$(1.5)\quad \begin{aligned} &(u_{h,t}(t), \chi) + \int_0^t \beta(t-s)\, A(u_h(s), \chi)\, ds = 0, \quad \forall \chi \in S_h, \quad t > 0, \\ &u_h(0) = v_h, \end{aligned}$$

where $A(\cdot, \cdot)$ is the energy inner product, $A(u, v) := (Au, v)$. Note that after introducing a basis for $S_h$, equation (1.5) can be viewed as a system of ordinary integro-differential equations. By defining $A_h$ as an approximation to $A$ defined on $S_h$,

$$(A_h\varphi, \psi) = A(\varphi, \psi), \quad \forall\ \varphi, \psi \in S_h,$$

we can write (1.5) as

$$(1.6)\quad u_{h,t}(t) + \int_0^t \beta(t-s) A_h u_h(s)\, ds = 0, \quad \text{for } t > 0, \quad u_h(0) = v_h.$$

Under standard assumptions on $S_h$ it was shown in [8] that

$$|u_h(t) - u(t)|_j \le C|v - v_h|_j + Ch^{r-j}\big(|v|_r + \int_0^t |u_t(s)|_r\, ds\big), \quad \text{for} \quad j = 0, 1.$$

The second step is to discretise (1.6) in time. We introduce a time step $k$, set $t_n = nk$, and let $U^n$ denote an approximation to $u_h(t_n)$. The derivative in (1.6) is approximated by a one or two step backward difference operator $\bar\partial$ and for the integral we use a quadrature rule

$$q_n(\varphi) = \sum_{j=0}^{n} \omega_{nj}\varphi^j \approx \int_0^{t_n} \beta(t_n - s)\varphi(s)\, ds,$$

where $\omega_{nj}$ are some quadrature weights. This gives the fully discrete equation

$$(1.7)\quad \bar\partial U^n + q_n(A_h U) = 0.$$

In order to ensure that a numerical method defined by (1.7) in conjunction with suitable initial data is stable one must use a quadrature rule that satisfies some discrete analogue of (1.2). Ideally one would want that for any sequence $\{\varphi^n\}_{n=0}^N$,

$$(1.8)\quad Q_N(\varphi) := k\sum_{n=1}^{N} q_n(\varphi)\varphi^n = k\sum_{n=1}^{N}\sum_{j=0}^{n} \omega_{nj}\varphi^j\varphi^n \ge 0.$$

We say that a rule satisfying this condition is *positive.* Note that for a quadrature rule to be positive we must have $\omega_{n0} = 0$ for all $n$. Relaxing this condition we say that a rule is *weakly positive* if it satisfies (1.8) for any sequence $\{\varphi^n\}$ with $\varphi^0 = 0$.

In McLean and Thomée [8] the authors show that if a one step backward difference operator is combined with any first order accurate positive quadrature rule then the resulting method is stable, first order accurate in time and optimal order accurate in space. Similarly, a stable and optimal order accurate method is obtained by combining a two step backward difference operator with any second order accurate weakly positive quadrature rule, provided initial conditions are chosen appropriately. In McLean, Thomée and Wahlbin [10] some of the results of [8] were extended to discretisations with variable time steps.

1.3. **Convolution quadrature.** In Lubich [3] the author introduces a class of quadrature rules that are especially designed for approximating convolution integrals. The derivation of these rules involves Laplace transforms and is rather complicated but they have the advantage that they are highly proficient at handling kernels that are rapidly oscillating or have singularities. It turns out out that by using these quadratures when discretising integro-differential equations of type (1.1) one obtains schemes that are amenable for a theoretical analysis. This technique was first used in [4], where the authors consider the particular case $\beta(t) = t^{\mu-1}$ for some $\mu \in (0, 1)$. As mentioned above, equation (1.1) then has the smoothing effect (1.4). By using the new quadrature rules, the authors constructed first and second order methods that satisfy the following nonsmooth data estimate

$$||U^n - u_h(t_n)|| \leq Ck^r t_n^{-r} ||v||,$$

for $r = 1, 2$, respectively. The technique was also used in McLean and Thomée [9] where the authors proposed a first order scheme based on a convolution quadrature to discretise (1.1) in a situation where the solution decays exponentially.

In Section 2 we present a detailed analysis of the two particular rules that were used in [4] and [9]. These rules are inspired by the general class of rules presented in [3] but are not exactly covered by the analysis done there. We therefore propose a new way to analyse the rules that also allows slightly sharper bounds for the quadrature error than those presented in [3]. Some of the analysis is entirely new since the assumptions in [3] do not fit the analysis in [9] and Section 3 of this thesis. We also extend some results regarding positivity for these rules given in [4].

In Appendix A we present some numerical examples that illustrate how the convolution quadrature rules behave in practice for a certain class of convolution kernels.

1.4. **Equations with exponentially decaying kernels.** In [9] the authors assume that $\beta$ is three times continuously differentiable, that there exist positive constants $C$ and $\alpha$ such that

$$|\beta^{(j)}(t)| \leq Ce^{-\alpha t}, \quad \text{for} \quad j = 0, 1, 2, 3,$$

and further that $e^{\bar{\alpha}t}\beta(t)$ is positive definite for any $\bar{\alpha} \in [0, \alpha)$. Under these assumptions they demonstrate that the solution to (1.1) is exponentially decaying. This is proved by estimating an integral representation of the solution obtained by taking the Laplace transform of (1.1) and then applying the inversion formula to the resulting expression for $\hat{u}(z)$. Then they proceed to demonstrate that when convolution quadrature is used to discretise the integral term, the discrete solution $U^n$ can be represented by an integral that is similar in form to the integral that represents $u(t_n)$. By estimating this integral representation it is proved that $U^n$ decays exponentially, and by estimating the difference between the two integral representations an error bound containing an exponentially decreasing factor is obtained,

$$||U^n - u(t_n)|| \leq Cke^{-\gamma t_n}||Av||. \tag{1.9}$$

The paper also treats the fully discrete method obtained by combining the time discretisation technique described with a second order accurate standard Galerkin finite element method in space. It is demonstrated that the error in the space semi-discretisation satisifes

$$||u_h(t) - u(t)|| \leq Ch^2 e^{-\gamma t}||A^{3/2}v||.$$

Combining this with (1.9), applied to equation (1.6), it is then shown that

$$||U^n - u(t_n)|| \leq Ce^{-\gamma t_n}(h^2||A^{3/2}v|| + k||Av||).$$

In Section 3 we use somewhat modified techniques to show results which are marginally sharper than those in [9]. Our techniques may be applied to the discretisation obtained by using a two step backward difference operator together with a second order accurate convolution quadrature rule. For this method we demonstrate that

$$||U^n - u(t_n)|| \leq Ck^2 e^{-\gamma t_n}(\log k^{-1}||Av|| + ||A^{3/2}v||).$$

We also prove the corresponding estimate for the fully discrete method,

$$||U^n - u(t_n)|| \leq C(h^2 + k^2 \log k^{-1})e^{-\gamma t_n}||A^{3/2}v||.$$

It appears that the positivity condition used for the kernel $\beta$ is insufficient to derive the exponential decay properties of our above results by the energy method in a similar way to what is possible for the heat equation.

In Appendix B we give some concrete examples to illustrate how the numerical schemes we study behave in practice. We further show that when applied to a model problem, some simple numerical schemes not covered in the theoretical analysis also capture the exponential decay rate of the exact solution very accurately.

## 2. Convolution Quadrature Rules

2.1. **Introduction.** In this section we will investigate ways to approximate the convolution integral in (1.1). For any function $\varphi(t)$ we set

$$I(t) := \int_0^t \beta(t-s)\varphi(s)\,ds = \int_0^t \beta(s)\varphi(t-s)\,ds, \tag{2.1}$$

In Lubich [3] the author proposes approximations to $I(t_n)$ of the form,

$$I_k^n = \sum_{j=0}^{n} \beta_{n-j}\varphi^j,$$

where $\varphi^j = \varphi(t_j)$ but where the quadrature weights $\beta_n$ are derived from the Laplace transform of $\beta(t)$, defined by

$$\hat{\beta}(z) = \int_0^\infty e^{-zt}\,\beta(t)\,dt, \tag{2.2}$$

rather than from pointwise values.

In Section 2.2 we first demonstrate how to derive a general class of quadrature rules inspired by Lubich [3]. Then we specialise and derive the specific first and second order rules used in [4, 9] and Section 3 of this thesis.

In Section 2.3 we perform an error analysis for the rules under consideration when $\beta$ satisfies the conditions used in Lubich [3], namely:

**B1:** *Assume that there exist constants $\alpha \geq 0$ and $\theta \in (0, \pi/2)$ such that $\hat{\beta}(z)$ is analytic in the region $\mathcal{P}_\theta^\alpha := \{z\colon |\arg(z+\alpha)| < \pi - \theta\}$. Assume further that there exists a $\mu \in (0,1]$ and a constant $M$ such that*

$$|\hat{\beta}(z)| \leq \frac{M}{|z+\alpha|^\mu}, \quad \forall z \in \mathcal{P}_\theta^\alpha.$$

A typical example of a kernel satisfying B1 is the function $\beta(t) = e^{-\alpha t}t^{\mu-1}$.

In Section 2.4 we assume that the kernel satisfies

**B2:** *With $r$ denoting the accuracy order of the quadrature, assume that $\beta(t) \in C^{r+2}([0,\infty))$, and that there exist positive constants $\alpha$ and $C$ such that*

$$|\beta^{(j)}(t)| \leq Ce^{-\alpha t}, \quad \textit{for} \quad j = 0, 1, \ldots, r+2.$$

This is the condition used in McLean and Thomée [9] but there the authors do not give a bound for the quadrature error explicitly. A typical example of a kernel satisfying B2 is $\beta(t) = e^{-\alpha t}$. Since the assumptions are very strong we can give sharper bounds for the quadrature error in this section than those presented in Section 2.3. Under these conditions we also demonstrate that we can prove error bounds for some quadrature rules that were not allowed in Lubich [3].

In Section 2.5 we verify that the new first and second order rules are positive and weakly positive respectively.

2.2. **Derivation of the rules.** As mentioned above we will compute the quadrature weights from the Laplace transform $\hat{\beta}(z)$, rather than from pointwise values. Both B1 and B2 admit the Laplace inversion formula,

$$\beta(s) = \frac{1}{2\pi i}\int_\Gamma e^{zs}\,\hat{\beta}(z)\,dz,$$

for a suitable contour $\Gamma$. Inserting this into (2.1) and interchanging the order of integration we find

$$I(t) = \frac{1}{2\pi i}\int_\Gamma \hat{\beta}(z) y_z(t)\,dz, \tag{2.3}$$

where

$$y_z(t) := \int_0^t e^{zs}\,\varphi(t-s)\,ds. \tag{2.4}$$

We can alternatively define $y_z(t)$ as the solution to

$$y_z'(t) - z\,y_z(t) = \varphi(t), \quad y_z(0) = 0, \tag{2.5}$$

where the prime denotes differentiation with respect to time. We will demonstrate that we get an approximation to $I(t_n)$ of the desired form by replacing $y_z(t_n)$ in equation (2.3) by the solution $Y_z^n$ to a $p$-step time stepping scheme,

$$k^{-1}(a_0 Y_z^n + \cdots + a_p Y_z^{n-p}) - z\,(b_0 Y_z^n + \cdots + b_p Y_z^{n-p}) \\ = (b_0\varphi^n + \cdots + b_p\varphi^{n-p}).$$

By choosing the initial condition judicially we can ensure that the generating function, $\tilde{Y}_z(\xi) = \sum_{n=0}^\infty \xi^n Y_z^n$, takes the form

$$\tilde{Y}_z(\xi) = \frac{\tilde{\varphi}(\xi) + q(\xi)}{\delta_k(\xi) - z}, \tag{2.6}$$

where $q(\xi)$ is a low order polynomial and

$$\delta_k(\xi) = \frac{1}{k}\frac{a_0 + \cdots + a_p\xi^p}{b_0 + \cdots + b_p\xi^p},$$

is the *symbol* of the difference operator. For an $r$'th order scheme we must have $k\delta_k(e^{-z}) = z + O(z^{r+1})$.

Replacing $y_z(t_n)$ by $Y_z^n$ in equation (2.3) we find

$$I_k^n := \frac{1}{2\pi i}\int_\Gamma \hat{\beta}(z) Y_z^n\,dz. \tag{2.7}$$

Taking generating functions and using (2.6) this gives

$$\tilde{I}_k(\xi) = \frac{1}{2\pi i}\int_\Gamma \hat{\beta}(z)\tilde{Y}_z(\xi)\,dz = \frac{1}{2\pi i}\int_\Gamma \hat{\beta}(z)\frac{\tilde{\varphi}(\xi) + q(\xi)}{\delta_k(\xi) - z}\,dz. \tag{2.8}$$

Assuming that the functions involved and the contour satisfy the conditions in the residue theorem, (2.8) gives $\tilde{I}_k(\xi) = \hat{\beta}(\delta_k(\xi))(\tilde{\varphi}(\xi) + q(\xi))$. By *defining* the quadrature weights $\beta_n$ so that $\tilde{\beta}(\xi) \equiv \sum_{n=0}^\infty \beta_n \xi^n := \hat{\beta}(\delta_k(\xi))$ we obtain

$$\tilde{I}_k(\xi) = \tilde{\beta}(\xi)\big(\tilde{\varphi}(\xi) + q(\xi)\big),$$

which implies

$$I_k^n = \sum_{j=0}^{n} \beta_{n-j}\varphi^n + \cdots ,$$

where the dots represent a few correction terms due to $q(\xi)$. For instance, if $q(\xi) = c_0 + c_1\xi$, then the dots represent $c_0\beta_n + c_1\beta_{n-1}$.

By combining (2.1) and (2.7) we get an integral representation for the quadrature error,

$$I_k^n - I(t_n) = \frac{1}{2\pi i}\int_\Gamma \hat\beta(z)\bigl(Y_z^n - y_z(t_n)\bigr)\,dz. \tag{2.9}$$

Both B1 and B2 give information on how $\hat\beta(z)$ behaves for large $z$. In Sections 2.3 and 2.4 we combine this information with certain bounds for $|Y_z^n - y_z(t_n)|$ in order to derive bounds for $|I_k^n - I(t_n)|$.

2.2.1. *A first order rule.* In order to get a first order accurate quadrature rule we use the Backward Euler method to discretise (2.5),

$$\bar\partial_1 Y_z^n - zY_z^n = \varphi^n, \quad \text{when} \quad n \ge 1, \quad \text{and} \quad Y_z^0 = 0, \tag{2.10}$$

where $\bar\partial_1$ is a one step backward difference operator, $\bar\partial_1 v^n = k^{-1}(v^n - v^{n-1})$. In order to find the quadrature rule resulting from this scheme we need to find the generating function $\tilde Y(\xi)$ and then perform the integration in (2.8). Multiplying (2.10) by $\xi^n$ and summing from 1 to infinity we find

$$k^{-1}\bigl(\tilde Y_z(\xi) - Y_z^0 - \xi\tilde Y_z(\xi)\bigr) - z\bigl(\tilde Y_z(\xi) - Y_z^0\bigr) = \tilde\varphi(\xi) - \varphi^0.$$

Since $Y_z^0 = 0$ this shows that

$$\tilde Y_z(\xi) = \frac{\tilde\varphi(\xi) - \varphi^0}{\delta_k(\xi) - z}, \quad \text{where} \quad \delta_k(\xi) = k^{-1}(1-\xi). \tag{2.11}$$

This is of the form (2.6), with $q(\xi) = -\varphi^0$. If $\operatorname{Re}\xi < 1 + k\nu$ then $\operatorname{Re}\delta_k(\xi) > -\nu$. The residue theorem then applies and (2.8) yields the quadrature rule

$$I_k^n = \sum_{j=1}^{n} \beta_{n-j}\varphi^j, \tag{2.12}$$

where the quadrature weights are defined by $\tilde\beta(\xi) = \hat\beta(\delta_k(\xi))$.

2.2.2. *A second order rule.* In order to achieve higher accuracy we can use a second order approximation $\bar\partial_2$ of the time derivative in (2.5),

$$\bar\partial_2 v^n := k^{-1}\bigl(\tfrac32 v^n - 2v^{n-1} + \tfrac12 v^{n-2}\bigr).$$

Introducing the auxiliary value $Y_z^{-1}$ we define a numerical scheme by

$$\begin{aligned} &\bar\partial_2 Y_z^n - zY_z^n = \varphi^n, \quad n \ge 1, \\ &Y_z^0 = 0, \quad Y_z^{-1} = -k\varphi^0. \end{aligned} \tag{2.13}$$

Multiplying the equation by $\xi^n$ and summing from 1 to infinity, keeping in mind that $Y_z^0 = 0$, we get

$$k^{-1}\big(\tfrac{3}{2}\tilde{Y}_z(\xi) - 2\xi\tilde{Y}_z(\xi) + \tfrac{1}{2}(\xi^2\tilde{Y}_z(\xi) + \xi Y_z^{-1})\big) - z\tilde{Y}_z(\xi) = \tilde{\varphi}(\xi) - \varphi^0.$$

With $\delta_k(\xi) = k^{-1}(\frac{3}{2} - 2\xi + \frac{1}{2}\xi^2)$ this simplifies to

$$\big(\delta_k(\xi) - z\big)\tilde{Y}_z(\xi) = -\tfrac{1}{2}k^{-1}\xi Y_z^{-1} + \tilde{\varphi}(\xi) - \varphi^0.$$

Using our seemingly *ad hoc* choice of initial conditions we now get an expression for $\tilde{Y}_z(\xi)$ of the form (2.6), with $q(\xi) = -\varphi^0 + \frac{1}{2}\xi\varphi^0$,

$$\tilde{Y}_z(\xi) = \frac{\tilde{\varphi}(\xi) - \varphi^0 + \frac{1}{2}\xi\varphi^0}{\delta_k(\xi) - z}. \tag{2.14}$$

Choosing the quadrature weights so that $\tilde{\beta}(\xi) = \hat{\beta}(\delta_k(\xi))$ we end up with the quadrature rule

$$I_k^n = \sum_{j=1}^{n} \beta_{n-j}\varphi^j + \tfrac{1}{2}\beta_{n-1}\varphi^0. \tag{2.15}$$

2.3. **Error bounds for non-regular kernels.** Throughout this section we assume that the kernel satisfies assumption B1. For the rules derived in Subsections 2.2.1 and 2.2.2 this will allow us to prove that for any $\gamma < 1$ there exists a constant $C$ such that

$$|I_k^n - I(t_n)| \le CMk^r e^{-\gamma\alpha t_n}\big(\sum_{j=0}^{r-1} t_n^{\mu-r+j}|\varphi^{(j)}(0)| + t_n^{\mu}||e^{\gamma\alpha t}\varphi^{(r)}(t)||_{t_n}\big),$$

where we used the shorthand $||f||_{t_n} = ||f||_{L^\infty(0,\,t_n)}$. The proofs will be based on the general error representation (2.9). We will use a contour $\Gamma$ that is almost wholly contained in the set $\mathcal{N}_\theta^\alpha := (\mathcal{P}_\theta^\alpha)^c$ for some constant $\theta \in (\theta', \pi/2)$, with $\theta'$ as in assumption B1. Since $\hat{\beta}(z)$ itself is not integrable along that path we will need to prove that the error $Y_z^n - y_z(t_n)$ is both $O(k^r)$ and decays when $|z| \to \infty$ in $\mathcal{N}_\theta^\alpha$.

The bounds we prove in this section should be compared to the sharper bound derived in [9] for the particular case $\beta(t) = t^{\mu-1}$,

$$|I_k^n - I(t_n)| \le CMk^r\big(\sum_{j=0}^{r-1} t_n^{\mu-r+j}|\varphi^{(j)}(0)| + \int_0^{t_n}(t_{n+1} - s)^{\mu-1}|\varphi^{(r)}(s)|\,ds\big).$$

2.3.1. *The first order rule.* This section is devoted to proving the following estimate for the quadrature rule derived in Subsection 2.2.1.

**Theorem 2.1.** *Let $\beta$ satisfy condition B1 and let $I(t_n)$ and $I_k^n$ be given by (2.1) and (2.12) respectively. Then, for any $\gamma < 1$,*

$$|I_k^n - I(t_n)| \le CMke^{-\gamma\alpha t_n}\big(t_n^{\mu-1}|\varphi(0)| + t_n^{\mu}||e^{\gamma\alpha t}\varphi'(t)||_{L^\infty(0,\,t_n)}\big).$$

When proving the theorem we use the error representation (2.9) with the contour $\Gamma = \Gamma_\alpha^1 \cup \Gamma_\alpha^2$, where

$$\Gamma_\alpha^1 := \{z \colon |\arg(z+\alpha)| = \pi - \theta \text{ and } |z+\alpha| \geq t_n^{-1}\},$$
$$\Gamma_\alpha^2 := \{z \colon |\arg(z+\alpha)| \leq \pi - \theta \text{ and } |z+\alpha| = t_n^{-1}\},$$

and where $\theta$ is any number in $(\theta',\, \pi/2)$. Throughout this paper we always assume that contours such as $\Gamma$ are oriented upwards. The following two lemmas provide the required bounds for $|Y_z^n - y_z(t_n)|$ on $\Gamma_\alpha^1$ and $\Gamma_\alpha^2$, respectively.

**Lemma 2.1.** *Let $y_z(t)$ and $Y_z^n$ be given by (2.5) and (2.10) respectively and let $z \in \mathcal{N}_\theta^\alpha$ for some $\theta \in (0,\, \pi/2)$ and $\alpha \geq 0$. Then for any $\gamma < 1$, there exist constants $c > 0$ and $\rho \in (0,\, 1)$ such that, with $x = -\mathrm{Re}(z+\alpha)$,*

$$|Y_z^n - y_z(t_n)| \leq Cke^{-\gamma\alpha t_n}\big(|\varphi(0)|R^n + ||e^{\gamma\alpha t}\varphi'(t)||_{L^\infty(0,\,t_n)}x^{-1}\big),$$

*where*

$$R^n = \begin{cases} e^{-\gamma x t_n} & \text{when } |zk| \leq c, \\ (xk)^{-1}\rho^n & \text{when } |zk| > c. \end{cases}$$

**Lemma 2.2.** *Let $y_z(t)$ and $Y_z^n$ be given by (2.5) and (2.10) respectively. Then, for any $\gamma < 1$,*

$$|Y_z^n - y_z(t_n)| \leq Cke^{-\gamma\alpha t_n}\big(|\varphi(0)| + t_n||e^{\gamma\alpha t}\varphi'(t)||_{L^\infty(0,\,t_n)}\big),$$

*when $|z+\alpha| \leq t_n^{-1}$.*

In order to make the ideas of the proof as transparent as possible we start by carrying out the proofs for the case $\alpha = 0$. Once this is done we show what modifications are needed for the general case.

Note that for $z \in \mathcal{N}_\theta^0$,

$$x := -\mathrm{Re}\, z, \quad \text{whence} \quad x \leq |z| \leq \frac{x}{\cos\theta}.$$

This commensurability will be used frequently, without further reference.

**Proof of Lemma 2.1 for $\alpha = 0$.** Setting $e_z^n = Y_z^n - y_z(t_n)$, we find, for $n \geq 1$,

$$\bar\partial_1 e_z^n - z\, e_z^n = y_z'(t_n) - \bar\partial_1 y_z^n = k^{-1}\int_{I_n}(s - t_{n-1})\, y_z''(s)\, ds =: \tau^n,$$

where $I_n = (t_{n-1},\, t_n)$. Since $e_z^0 = 0$ we then get the error representation,

$$e_z^n = k\sum_{j=1}^{n} r(zk)^{n+1-j}\,\tau^j, \quad \text{where} \quad r(zk) := (1-zk)^{-1}. \tag{2.16}$$

We will use that $|\tau^j| \leq ||y_z''||_{L^1(I_j)} \leq k||y_z''||_{L^\infty(I_j)}$. In order to bound these quantities we need an expression for $y_z''$. Differentiating (2.5) we find $y_z''(t) =$

$zy_z'(t) + \varphi'(t)$. Inserting the expression for $y_z'(t)$ obtained by differentiating (2.4) this gives

$$y_z''(t) = ze^{zt}\varphi(0) + \varphi'(t) + z\int_0^t e^{zs}\varphi'(t-s)\,ds. \tag{2.17}$$

We start by proving the lemma under the assumption that $\varphi(0) = 0$. Then $|\tau^j| \leq k||\varphi'||_{L^\infty(0,\,t_n)}$ and noting that $|r(zk)| \leq (1+xk)^{-1}$ we obtain

$$|e_z^n| \leq k\sum_{j=1}^n (1+xk)^{n+1-j}\,Ck||\varphi'||_{L^\infty(0,t_n)} \leq Ck||\varphi'||_{L^\infty(0,t_n)}x^{-1}.$$

By linearity the proof will be complete if we can prove the claim for a constant $\varphi$. We therefore assume that $\varphi(t) \equiv 1$. Pick any $\gamma' \in (\gamma,\,1)$. Then, since $r(zk) = e^{zk} + O((zk)^2)$, we can find a constant $c > 0$ such that $|r(zk)| \leq e^{-\gamma' xk}$ when $|zk| \leq c$. Using also that $|\tau^j| \leq Cxke^{-xk(j-1)}$ we find

$$|e_z^n| \leq k\sum_{j=1}^n e^{-\gamma' xk(n+1-j)}\,Cxke^{-xk(j-1)} \leq Cxk^2ne^{-\gamma' xkn} \leq Cke^{-\gamma x t_n}.$$

When $|zk| \geq c$ we use that $|r(zk)| \leq (1+xk)^{-1}$ and that $|\tau^j| \leq ||y_z''||_{L^1(I_j)} \leq Ce^{-xk(j-1)}$ to obtain

$$\begin{aligned}
|e_z^n| &\leq k\sum_{j=1}^n (1+xk)^{-(n+1-j)}\,Ce^{-xk(j-1)} \\
&\leq Ck(1+xk)^{-n}\sum_{j=1}^n \big((1+xk)e^{-xk}\big)^{j-1} \\
&\leq Ck(1+xk)^{-1}(1+c\,\cos\theta)^{-(n-1)}n \leq Cx^{-1}\rho^n,
\end{aligned}$$

where $\rho$ is any number in the range $((1+c\,\cos\theta)^{-1},\,1)$. ☐

**Proof of Lemma 2.2 for $\alpha = 0$.** We set $\varepsilon = t_n^{-1}$ and use the same notation as in the proof of Lemma 2.1. Since $|\tau^j| \leq k||y_z''||_{L^\infty(I_j)}$ we have, by (2.17),

$$|\tau^j| \leq k\big(\varepsilon e^{\varepsilon t_n}|\varphi(0)| + ||\varphi'||_{t_n} + \varepsilon t_n e^{\varepsilon t_n}||\varphi'||_{t_n}\big) \leq Ck(\varepsilon|\varphi(0)| + ||\varphi'||_{t_n}).$$

When $n \geq 2$ we have $\varepsilon k \leq 1/2$ and then $|r(zk)| \leq (1-\varepsilon k)^{-1} \leq e^{2\varepsilon k}$. Consequently $k\sum_{j=1}^n |r(zk)|^j \leq t_n e^{2\varepsilon t_n}$. Inserting these estimates in the error representation (2.16) gives the desired bound. ☐

**Proof of Theorem 2.1 for $\alpha = 0$.** Choosing any $\theta \in (\theta',\,\pi/2)$ we use equation (2.9) with $\Gamma = \Gamma_0^1 \cup \Gamma_0^2$ to get

$$|I_k^n - I(t_n)| \leq \frac{1}{2\pi}\int_{\Gamma_0^1} |\hat\beta(z)e_z^n|\,|dz| + \frac{1}{2\pi}\int_{\Gamma_0^2} |\hat\beta(z)e_z^n|\,|dz| =: \Upsilon_1 + \Upsilon_2.$$

Using B1 and Lemma 2.1 we find, with $\bar{c} = c/\cos\alpha$,

$$\begin{aligned}\Upsilon_1 \leq & \frac{1}{2\pi}\int_{t_n^{-1}}^{\infty} \frac{M}{x^{\mu}} Ck||\varphi'||_{t_n}\frac{1}{x}\,dx + \frac{1}{2\pi}\int_{t_n^{-1}}^{\bar{c}/k} \frac{M}{x^{\mu}} Ck|\varphi(0)|e^{-\gamma x t_n}\,dx \\ & + \frac{1}{2\pi}\int_{\bar{c}/k}^{\infty} \frac{M}{x^{\mu}} C|\varphi(0)|\rho^n \frac{1}{x}\,dx =: J_1 + J_2 + J_3.\end{aligned}$$

The first term, $J_1$, is clearly bounded as stated. In order to bound $J_2$ we change variables so that $y = xt_n$. Then

$$J_2 \leq CMk|\varphi(0)|\int_1^{\bar{c}t_n/k} \frac{t_n^{\mu}}{y^{\mu}} e^{-\gamma y} t_n^{-1}\,dy \leq CMk|\varphi(0)|t_n^{\mu-1}.$$

Using in the third term that $\rho^n \leq Cn^{\mu-1}$ we obtain

$$\begin{aligned}J_3 &= CM|\varphi(0)|\rho^n \int_{\bar{c}/k}^{\infty} x^{-(\mu+1)}\,dx \leq CM|\varphi(0)|\rho^n k^{\mu} \\ &\leq CM|\varphi(0)|(t_n/k)^{\mu-1}k^{\mu}.\end{aligned}$$

Thus $\Upsilon_1$ is bounded as specified.

Using B1 and Lemma 2.2 we find,

$$\Upsilon_2 \leq \frac{1}{2\pi}\int_{\Gamma_0^2} \frac{M}{(1/t_n)^{\mu}} Ck\big(|\varphi(0)| + t_n||\varphi'||_{L^{\infty}(0,\,t_n)}\big)\,|dz|.$$

Since $|\Gamma_0^2| \leq 2\pi t_n^{-1}$ this is also bounded as specified. □

We next turn to the proof of Theorem 2.1 for a general $\alpha$.

**Proof of Lemma 2.1.** We use the same notation as we did in the proof for the case $\alpha = 0$, except that now $x = -\mathrm{Re}(z+\alpha)$.

First we suppose that $\varphi(0) = 0$. Then for any $\gamma < 1$ we have, by (2.17),

$$y_z''(t) = \varphi'(t) + ze^{-\gamma\alpha t}\int_0^t e^{(z+\gamma\alpha)s}e^{\gamma\alpha(t-s)}\varphi'(t-s)\,ds. \tag{2.18}$$

Recalling that $|\tau^j| \leq k||y_z''||_{L^{\infty}(I_j)}$ we find

$$\begin{aligned}|\tau^j| &\leq Cke^{-\gamma\alpha t_{j-1}}||e^{\gamma\alpha t}\varphi'(t)||_{L^{\infty}(0,t_n)}\big(1 + |z|\int_0^{\infty} e^{-(x+\alpha-\gamma\alpha)s}\,ds\big) \\ &\leq Cke^{-\gamma\alpha t_{j-1}}||e^{\gamma\alpha t}\varphi'(t)||_{L^{\infty}(0,t_n)}.\end{aligned}$$

Since, for small $k$,

$$|r(zk)| \leq \frac{1}{1+(\alpha+x)k} = \frac{(1+\alpha k)^{-1}}{1+(1+\alpha k)^{-1}xk} \leq \frac{e^{-\gamma\alpha k}}{1+xk/2}, \tag{2.19}$$

we obtain, using (2.16),

$$
\begin{aligned}
|e_z^n| &\le k \sum_{j=1}^{n} \frac{e^{-\gamma\alpha k(n+1-j)}}{(1+xk/2)^{n+1-j}}\, Cke^{-\gamma\alpha k(j-1)} ||e^{\gamma\alpha t}\varphi'(t)||_{L^\infty(0,t_n)} \\
&\le Cke^{-\gamma\alpha t_n}||e^{\gamma\alpha t}\varphi'(t)||_{L^\infty(0,t_n)} k \sum_{j=1}^{n} (1+xk/2)^{-j}.
\end{aligned}
$$

Since the sum is bounded by $(xk)^{-1}$ this completes the proof when $\varphi(0) = 0$.

By linearity the proof will be complete if we can prove the claim for a constant $\varphi$. We therefore assume that $\varphi(t) \equiv 1$. Pick first a $\gamma' \in (\gamma, 1)$ and a positive $c$ such that $|r(zk)| \le e^{-\gamma'(\alpha+x)k}$ when $|zk| \le c$. Using (2.17) we find $|\tau^j| \le k||ze^{zt}||_{L^\infty(I_j)} \le Ck(\alpha + x)e^{-(\alpha+x)k(j-1)}$ and thus, by (2.16),

$$
\begin{aligned}
|e_z^n| &\le k \sum_{j=1}^{n} e^{-\gamma'(\alpha+x)k(n+1-j)} Ck(\alpha+x)e^{-\gamma'(\alpha+x)k(j-1)} \\
&\le Ck^2 n(\alpha+x)e^{-\gamma'(\alpha+x)t_n} \le Cke^{-\gamma(\alpha+x)t_n}.
\end{aligned}
$$

When $|zk| \ge c$ use $|\tau^j| \le ||ze^{zt}||_{L^1(I_j)} \le Ce^{-(\alpha+x)k(j-1)}$ and (2.19),

$$
\begin{aligned}
|e_z^n| &\le k \sum_{j=1}^{n} \frac{e^{-\gamma\alpha k(n+1-j)}}{(1+xk/2)^{n+1-j}}\, Ce^{-(\alpha+x)k(j-1)} \\
&\le Ck \frac{e^{-\gamma\alpha t_n}}{(1+xk/2)^n} \sum_{j=1}^{n} \big((1+xk/2)e^{-xk}\big)^{j-1}.
\end{aligned}
$$

Since $xk \ge \cos\theta |z+\alpha| \ge \cos\theta\,(c - \alpha k)$ and $(1 + xk/2)e^{-xk} \le 1$, we get, for small $k$,

$$
|e_z^n| \le Ce^{-\gamma\alpha t_n} \frac{k}{1+xk/2} n(1+\cos\theta\,(c/3))^{-(n-1)} \le Ce^{-\gamma\alpha t_n} x^{-1}\rho^n,
$$

for any $\rho \in ((1 + \cos\theta\,(c/3))^{-1}, 1)$. □

**Proof of Lemma 2.2.** We use the same notation as in the proof of Lemma 2.1 with $\varepsilon = t_n^{-1}$. Consider first the case $\varphi(0) = 0$. Then (2.18) gives,

$$
y_z''(t) = \varphi'(t) + ze^{-\gamma\alpha t} \int_0^t e^{(z+\alpha)s} e^{-(1-\gamma)\alpha s} e^{\gamma\alpha(t-s)} \varphi'(t-s)\, ds.
$$

Using that $|\tau^j| \le k||y_z''||_{L^\infty(I_j)}$ and that $|e^{(z+\alpha)t_n}| \le C$ we get

$$
|\tau^j| \le Cke^{-\gamma\alpha t_{j-1}}||e^{\gamma\alpha t}\varphi'(t)||_{t_n}\big(1 + (\alpha+\varepsilon)/((1-\gamma)\alpha)\big).
$$

Since $|z|$ is bounded we have, *cf.* (2.19), for $\gamma < 1$ and $k$ small,

$$
|r(zk)| \le e^{(2\varepsilon-\gamma\alpha)k}. \tag{2.20}
$$

Combining these bounds with the error representation (2.16) gives the desired estimate.

When $\varphi \equiv 1$ we have $y_z''(t) = ze^{zt}$ and so $|\tau^j| \le Cke^{(\varepsilon-\alpha)t_{j-1}}$. Picking a $\gamma' \in (\gamma, 1)$ and using again (2.20) we find

$$|e_z^n| \le k \sum_{j=1}^{n} e^{(2\varepsilon-\gamma'\alpha)t_{n+1-j}} Cke^{(\varepsilon-\alpha)t_{j-1}} \le Ckt_n e^{-\gamma'\alpha t_n} \le Cke^{-\gamma\alpha t_n}.$$

This completes the proof. □

With these lemmas the general proof of Theorem 2.1 follows with only trivial modifications from the proof in the case $\alpha = 0$.

2.3.2. *The second order rule.* In this section we shall prove the following analogue of Theorem 2.1 for the rule defined in Subsection 2.2.2, in the case $\alpha = 0$.

**Theorem 2.2.** *Let $\beta$ satisfy condition B1 for $\alpha = 0$, and let $I(t_n)$ and $I_k^n$ be given by (2.1) and (2.15) respectively. Then, for any $\gamma < 1$,*

$$|I_k^n - I(t_n)| \le CMk^2\big(t_n^{\mu-2}|\varphi(0)| + t_n^{\mu-1}|\varphi'(0)| + t_n^{\mu}||\varphi''(t)||_{t_n}\big).$$

Using the same techniques as in the proof of Theorem 2.1, the proof of Theorem 2.2 is an immediate consequence of the following lemmas.

**Lemma 2.3.** *Let $y_z(t)$ and $Y_z^n$ be given by (2.5) and (2.13) respectively and let $z \in \mathcal{N}_\theta^0$ for some $\theta \in (0, \pi/2)$. Then for any $\gamma < 1$ there exist constants $c > 0$ and $\rho \in (0, 1)$ such that, with $x = -\mathrm{Re}\, z$,*

$$|Y_z^n - y_z(t_n)| \le Ck^2\big(R^n + ||\varphi''||_{L^\infty(0,t_n)}x^{-1}\big),$$

*where*

$$R^n = \begin{cases} (t_n^{-1}|\varphi(0)| + |\varphi'(0)|)\, e^{-\gamma x t_n}, & \text{when } |zk| \le c, \\ (k^{-1}|\varphi(0)| + |\varphi'(0)|)\,(xk)^{-1}\,\rho^n, & \text{when } |zk| > c. \end{cases}$$

**Lemma 2.4.** *Let $y_z(t)$ and $Y_z^n$ be given by (2.5) and (2.13) respectively. Then, for $|z| \le t_n^{-1}$,*

$$|Y_z^n - y_z(t_n)| \le Ck^2\big(t_n^{-1}|\varphi(0)| + |\varphi'(0)| + t_n||\varphi''||_{L^\infty(0,t_n)}\big).$$

**Proof of Lemma 2.3.** The error, $e_z^n = Y_z^n - y_z(t_n)$, satisfies

$$\bar\partial_2 e_z^n - z\, e_z^n = y_z'(t_n) - \bar\partial_2 y_z^n =: \tau^n,$$

where

$$\tau^n = -k^{-1}\int_{t_{n-1}}^{t_n} (s-t_{n-1})^2 y_z'''(s)\,ds + \tfrac14 k^{-1}\int_{t_{n-2}}^{t_n}(s-t_{n-2})^2 y_z'''(s)\,ds.$$

The error equation is the second order difference equation

$$\big(\tfrac32 - zk\big)e_z^n - 2e_z^{n-1} + \tfrac12 e_z^{n-2} = k\tau^n, \quad n \ge 2, \tag{2.21}$$

$$e_z^0 = 0, \quad e_z^1 = \frac{k\big(\frac23\varphi^1 + \frac13\varphi^0\big)}{1-zk} - \int_0^k e^{zs}\varphi(k-s)\,ds,$$

where we used equation (2.13) with $n = 1$ to find an expression for $Y_z^1$. The characteristic polynomial of the error equation has the zeros,

$$r_\pm = r_\pm(zk) = \frac{1}{\frac{3}{2} - zk}\big(1 \pm \tfrac{1}{2}\sqrt{1+2zk}\big),$$

where the square root sign refers to the branch that has a positive real part. It is now possible to write down the following error representation

$$e_z^n = \frac{r_+^n - r_-^n}{r_+ - r_-} e_z^1 + k \sum_{j=2}^{n} \frac{r_+^{n+1-j} - r_-^{n+1-j}}{r_+ - r_-} \frac{\tau^j}{\frac{3}{2} - zk} =: \Lambda_1 + \Lambda_2. \tag{2.22}$$

Note that $|\tau^j| \leq Ck||y_z'''||_{L^1(I_j \cup I_{j-1})} \leq Ck^2||y_z'''||_{L^\infty(I_j\cup I_{j-1})}$. Differentiating (2.17) we find

$$y_z'''(t) = z^2 e^{zt}\varphi(0) + z e^{zt}\varphi'(0) + \varphi''(t) + z\int_0^t e^{zs}\varphi''(t-s)\,ds. \tag{2.23}$$

Since the scheme (2.13) is second order accurate it follows necessarily that $r_+(zk) = e^{zk} + O((zk)^3)$. Furthermore, it is easily verified that when $\operatorname{Re} zk \leq 0$ we have $|r_-(zk)| \leq 1/2$. Thus we can for any $\gamma' \in (\gamma, 1)$ pick a constant $c > 0$ such that $|r_\pm(zk)| \leq e^{-\gamma' xk}$ when $|zk| \leq c$. We start by proving the bounds in this region. We will frequently use that

$$\left|\frac{r_+^n - r_-^n}{r_+ - r_-}\right| \leq \frac{|r_+|^n + |r_-|^n}{|r_+ - r_-|} \leq Ce^{-\gamma' xkn}, \quad \text{when } |zk| \leq c. \tag{2.24}$$

In order to prove the bound for $\Lambda_1$ we use (2.21) and find

$$\begin{aligned} |e_z^1| \leq & \left|\frac{k\big(\frac{2}{3}\varphi(0) + \frac{1}{3}\varphi(k)\big)}{1 - \frac{2}{3}zk} - \int_0^k e^{zs}\big(\tfrac{2}{3}\varphi(0) + \tfrac{1}{3}\varphi(k)\big)\,ds\right| \\ & + \left|\int_0^k e^{zs}\big(\tfrac{2}{3}\varphi(0) + \tfrac{1}{3}\varphi(k) - \varphi(k-s)\big)\,ds\right| \\ \leq & k\big(|\varphi(0)| + |\varphi(k)|\big)\left|\frac{1}{1-(2/3)zk} - \frac{1}{zk}(e^{zk}-1)\right| + k||\varphi'||_{L^1(I_1)} \\ \leq & Ck^2|z|(|\varphi(0)| + ||\varphi'||_{L^1(I_1)}) + k||\varphi'||_{L^1(I_1)}. \end{aligned} \tag{2.25}$$

Since $||\varphi'||_{L^1(I_1)} \leq k|\varphi'(0)| + k^2||\varphi''||_{L^\infty(I_1)}$ and $|zk| \leq c$ this gives

$$|e_z^1| \leq Ck^2\big(|z|\,|\varphi(0)| + |\varphi'(0)| + k||\varphi''||_{L^\infty(I_1)}\big).$$

By (2.24) we find that for any $\gamma < \gamma'$,

$$|\Lambda_1| \leq Ck^2(t_n^{-1}|\varphi(0)| + |\varphi'(0)|)e^{-\gamma x t_n} + k^3||\varphi''||_{L^\infty(I_1)}.$$

Since $k \leq Cx^{-1}$ this gives the bound for $\Lambda_1$ when $|zk| \leq c$.

When proving the bound for $\Lambda_2$ we first assume that $\varphi(0) = \varphi'(0) = 0$. Then $|\tau^j| \leq k^2||\varphi''||_{t_n}$. Since $e^{-\gamma' xk} \leq (1+\gamma xk)^{-1}$ we immediately get the required bound from (2.24). Next assume that $\varphi'' \equiv 0$. Then $|\tau^j| \le Cxk^2(x|\varphi(0)| + |\varphi'(0)|)e^{-xkj}$ and using (2.24) we find

$$\begin{aligned}|\Lambda_2| &\le Cxk^3(x|\varphi(0)| + |\varphi'(0)|)\sum_{j=1}^{n} e^{-\gamma' xk(n+1-j)}e^{-xkj}\\ &\le Cxk^3(x|\varphi(0)| + |\varphi'(0)|)ne^{-\gamma' xt_n}\\ &\le Ck^2(t_n^{-1}|\varphi(0)| + |\varphi'(0)|)e^{-\gamma xt_n}.\end{aligned}$$

Consider next the region $|zk| \ge c$. It can be verified that $|r_\pm(zk)| \le (1+xk)^{-1/2}$. This implies

$$\Big|\frac{r_+^n - r_-^n}{r_+ - r_-}\Big| = \Big|\sum_{i=0}^{n-1} r_+^i r_-^{n-1-i}\Big| \le nr^{n-1}, \quad \text{where} \quad r = \frac{1}{\sqrt{1+xk}}. \tag{2.26}$$

For the bound of $\Lambda_1$ we this time use, *cf.* (2.25),

$$\begin{aligned}|e_z^1| &\le C\big(|\varphi(0)| + |\varphi(k)|\big)\, x^{-1} + ||\varphi||_{L^\infty(I_1)}x^{-1}\\ &\le Cx^{-1}\big(|\varphi(0)| + k|\varphi'(0)| + k^2||\varphi''||_{L^\infty(I_1)}\big).\end{aligned}$$

Combining this with (2.26) gives, for any $\rho \in ((1+c\cos\theta)^{-1/2}, 1)$,

$$|\Lambda_1| \le Cx^{-1}\big(|\varphi(0)| + k|\varphi'(0)| + k^2||\varphi''||_{L^\infty(I_1)}\big)\rho^n.$$

When proving the bound for $\Lambda_2$ we first assume that $\varphi(0) = \varphi'(0) = 0$. Then $|\tau^j| \le k^2||\varphi''||_{t_n}$ and using (2.26) we get

$$|\Lambda_2| \le Ck^3||\varphi''||_{t_n}\frac{1}{1+xk}\sum_{j=1}^{n-1} j(1+c\cos\theta)^{-j/2}.$$

Since the sum is convergent this is bounded as required. Assume next that $\varphi'' \equiv 0$. Then $|\tau^j| \le Ck(x|\varphi(0)| + |\varphi'(0)|)e^{-xk(j-2)}$ and we get

$$\begin{aligned}|\Lambda_2| &\le Ck^2(x|\varphi(0)| + |\varphi'(0)|)\frac{1}{1+xk}\sum_{j=2}^{n}(n+1-j)r^{n-j}e^{-xk(j-2)}\\ &\le Ck^2(x|\varphi(0)| + |\varphi'(0)|)\frac{1}{1+xk}\, r^{n-2}\sum_{j=2}^{n}\big((1+xk)^{1/2}e^{-xk}\big)^{j-2}\\ &\le Ck^2(x|\varphi(0)| + |\varphi'(0)|)\frac{1}{(1+xk)^2}\left(\frac{1}{1+c\cos\theta}\right)^{n-4} n^2\\ &\le Cx^{-1}(|\varphi(0)| + k|\varphi'(0)|)\rho^n,\end{aligned}$$

with $\rho$ as before. This completes the proof. □

**Proof of Lemma 2.4.** Using the same notation as in the proof of Lemma 2.3 and with $\varepsilon = t_n^{-1}$ we first note that

$$\begin{aligned}|\tau^j| &\le k^2||y_z'''||_{L^\infty(I_j\cup I_{j-1})}\\ &\le k^2\big(\varepsilon^2 e^{\varepsilon t_n}|\varphi(0)| + \varepsilon e^{\varepsilon t_n}|\varphi'(0)| + ||\varphi''||_{t_n} + \varepsilon t_n e^{\varepsilon t_n}||\varphi''||_{t_n}\big).\end{aligned}$$

Since $r_+(zk) = e^{zk} + O((zk)^3)$ and $r_-(0) = 1/3$ we must have $|r_\pm(zk)| \leq e^{2\varepsilon k}$ when $|z| \leq \varepsilon$ and $k$ is small. Thus

$$\left|\frac{r_+^n - r_-^n}{r_+ - r_-}\right| \leq C(|r_+|^n + |r_-|^n) \leq Ce^{2\varepsilon t_n}.$$

Combining this with the error representation (2.22) and using the obvious estimate for $e_z^1$ gives the bound claimed in the lemma. $\square$

Actually, it is possible to prove that if B1 holds for a positive $\alpha$ then

$$|I_k^n - I(t_n)| \leq CMk^2 e^{-\gamma\alpha t_n}\big(t_n^{\mu-2}|\varphi(0)| + t_n^{\mu-1}|\varphi'(0)| + t_n^{\mu}||e^{\gamma\alpha t}\varphi''(t)||_{t_n}\big),$$

for any $\gamma < 1$. To prove this, we can extend the analysis above just like we did for the first order rule in Section 2.3.1. The extension is straightforward but lengthy and we refrain from giving the details.

2.4. **Error bounds for regular kernels.** Throughout this section we assume that the kernel satisfies assumption B2. For the rules defined in Subsections 2.2.1 and 2.2.2 we shall prove that

$$|I_k^n - I(t_n)| \leq Ck^r|\varphi|_{\bar\alpha,r,t_n}, \tag{2.27}$$

where $\bar\alpha$ is any number smaller than $\alpha$, and

$$|\varphi|_{\bar\alpha,r,t_n} := e^{-\bar\alpha t_n}\Big(|\varphi(0)| + \cdots + |\varphi^{(r-1)}(0)| + \int_0^{t_n} e^{\bar\alpha s}|\varphi^{(r)}(s)|\,ds\Big).$$

Note that $\varphi^{(r)}$ only enters the bound in a weighted $L^1$-norm and not in $L^\infty$-norm as in the previous section.

At the end of this section we will demonstrate that since we make such strong assumptions on $\beta$ we can actually use quadrature rules based on slightly less stable numerical schemes for $y_z$, such as the Crank-Nicolson scheme. In this case, however, we only get $\bar\alpha = 0$ in the error estimate.

The bounds for the quadrature error will, as always, be based on (2.9). Assumption B2 implies that $\hat\beta(z)$ is analytic in the region $\operatorname{Re} z > -\alpha$ and we can consequently use the contour $\Gamma_\nu = -\nu + i\mathbb{R}$, oriented upwards, for any $\nu \in (\bar\alpha, \alpha)$. We will derive bounds for $Y_z^n - y_z(t_n)$ of the following form,

$$|Y_z^n - y_z(t_n)| \leq Ck^r|\varphi|_{\bar\alpha,r,t_n}\big(1 + |z|^r\big), \quad \text{for} \quad z \in \Gamma_\nu. \tag{2.28}$$

Since this bound grows as $|z|^r$ and the present assumptions only guarantee $O(|z|^{-1})$ decay for $|\hat\beta(z)|$ we would not get the bound (2.27) by just taking absolute values under the integral in (2.9). What saves us is that the integral against the first terms in the Laurent expansion for $\hat\beta(z)$ vanish. In particular, using assumption B2, we can perform partial integrations in (2.2) and write

$$\hat\beta(z) = \frac{\beta(0)}{z} + \frac{\beta'(0)}{z^2} + \cdots + \frac{\beta^{(r)}(0)}{z^{r+1}} + \hat\beta_{\text{res}}(z), \tag{2.29}$$

where the remainder term satisfies

$$|\hat{\beta}_{\mathrm{res}}(z)| = \Big|\frac{1}{z^{r+1}}\int_0^\infty e^{-zt}\,\beta^{(r+1)}(t)\,dt\Big| \le \frac{C}{|z|^{r+2}}. \tag{2.30}$$

In order to prove that the integral against the first terms in the Laurent expansion vanish we need the following lemma.

**Lemma 2.5.** (i) *If $s > -\nu$ and $j$ is any positive integer, then*

$$\int_{\Gamma_\nu} \frac{1}{z^j} e^{zs}\,dz = 0.$$

(ii) *Let $\psi(z)$ be a function that is analytic and decays as $|z|^{-1}$ when* $\mathrm{Re}\, z \le -\nu$*. Then, for any positive integer $j$,*

$$\int_{\Gamma_\nu} \frac{1}{z^j}\psi(z)\,dz = 0.$$

The first part of the lemma shows that

$$\frac{1}{2\pi i}\int_{\Gamma_\nu} \frac{\beta^{(j-1)}(0)}{z^j} e^{zs}\,dz = 0, \quad \text{when} \quad s > -\nu.$$

Thus, by inserting expansion (2.29) and equation (2.4) into (2.3) and interchanging the order of integration we find

$$I(t_n) = \frac{1}{2\pi i}\int_{\Gamma_\nu} \hat{\beta}_{\mathrm{res}}(z) y_z(t_n)\,dz.$$

Using the second part of the lemma we will prove that this holds also when $y_z(t_n)$ is replaced by $Y_z^n$ for each of the methods we discuss. Consequently

$$I_k^n - I(t_n) = \frac{1}{2\pi i}\int_{\Gamma_\nu} \hat{\beta}_{\mathrm{res}}(z)\big(Y_z^n - y_z(t_n)\big)\,dz. \tag{2.31}$$

Combining (2.28), (2.30) and (2.31) then immediately gives (2.27).

**Proof of Lemma 2.5.**
**Part 1.** Let $R$ be a large number and set

$$\begin{aligned} \Gamma_\nu^R &:= -\nu + i[-R, R], & \Gamma_u^R &:= [-R, -\nu] + iR, \\ \Gamma_l^R &:= -R + i[-R, R], & \Gamma_d^R &:= [-R, -\nu] - iR. \end{aligned}$$

By Cauchy's theorem,

$$\left[\int_{\Gamma_\nu^R} + \int_{\Gamma_u^R} + \int_{\Gamma_d^R} + \int_{\Gamma_l^R}\right] \frac{e^{zs}}{z^j}\,dz = 0.$$

As $R \to \infty$ we have

$$\int_{\Gamma_\nu^R} \frac{e^{zs}}{z^j}\,dz \to \int_{\Gamma_\nu} \frac{e^{zs}}{z^j}\,dz,$$
$$\Big|\int_{\Gamma_{u,d}^R} \frac{e^{zs}}{z^j}\,dz\Big| \leq \frac{1}{R^j}\int_\nu^\infty e^{-sx}\,dx \to 0,$$
$$\Big|\int_{\Gamma_l^R} \frac{e^{zs}}{z^j}\,dz\Big| \leq \frac{e^{-sR}}{R^j}\int_{-R}^{R} dy \to 0,$$

which shows the result.

**Part 2.** This part is very similar. We use the same integration contour as in the first part and simply note that since the integrand decays as $|z|^{-2}$ the integrals over $\Gamma_u^R$, $\Gamma_d^R$ and $\Gamma_l^R$ vanish when $R \to \infty$. □

2.4.1. *The first order rule.* For the first order rule presented in Subsection 2.2.1 we will prove the following error bound.

**Theorem 2.3.** *If $\beta$ satisfies condition B2 and if $I(t_n)$ and $I_k^n$ are given by (2.1) and (2.12), respectively, then for any $\bar\alpha < \alpha$, there exists a constant $C$ such that*

$$|I_k^n - I(t_n)| \leq Ck|\varphi|_{\bar\alpha,1,t_n}.$$

When proving this we will need the following estimate for the error in the Backward Euler approximation.

**Lemma 2.6.** *Let $y_z(t)$ and $Y_z^n$ be defined by (2.4) and (2.10), respectively. Then for any $\bar\alpha < \nu$,*

$$|Y_z^n - y_z(t_n)| \leq Ck|\varphi|_{\bar\alpha,1,t_n}(1+|z|), \quad \textit{for} \quad z \in \Gamma_\nu.$$

**Proof of Lemma 2.6.** We use the same notation as in the proof of Lemma 2.1. Recall in particular that the error representation reads

$$e_z^n = k\sum_{j=1}^n r(zk)^{n+1-j}\,\tau^j, \quad \text{where} \quad r(zk) := (1-zk)^{-1}. \tag{2.32}$$

For any $\bar\alpha < \nu$, pick $\bar\nu \in (\bar\alpha, \nu)$. If $z \in \Gamma_\nu$ and if $k$ is small enough then $|r(zk)| \leq (1+\nu k)^{-1} \leq e^{-\bar\nu k}$. This gives, by (2.32),

$$\begin{aligned}|e_z^n| &\leq k\sum_{j=1}^n e^{-\bar\nu k(n+1-j)}|\tau^j|\\ &\leq k\sum_{j=1}^n e^{-\bar\nu(t_n-t_{j-1})}\int_{t_{j-1}}^{t_j}|y_z''(t)|\,dt \leq k\int_0^{t_n} e^{-\bar\nu(t_n-t)}|y_z''(t)|\,dt.\end{aligned}$$

Recall from (2.17) that

$$y_z''(t) = ze^{zt}\varphi(0) + \varphi'(t) + z\int_0^t e^{z(t-s)}\varphi'(s)\,ds.$$

Thus, using repeatedly that $te^{-\bar{\nu}t} \leq Ce^{-\bar{\alpha}t}$,

$$\begin{aligned}
|e_z^n| \leq & k\Big(|z|\int_0^{t_n} e^{-\bar{\nu}(t_n-t)}e^{-\bar{\nu}t}\,dt\,|\varphi(0)| + \int_0^{t_n} e^{-\bar{\nu}(t_n-t)}|\varphi'(t)|\,dt \\
& + |z|\int_0^{t_n} e^{-\bar{\nu}(t_n-t)}\int_0^{t} e^{-\bar{\nu}(t-s)}|\varphi'(s)|\,ds\,dt\Big) \\
\leq & k(1+|z|)\Big(t_n e^{-\bar{\nu}t_n}|\varphi(0)| + |\varphi|_{\bar{\nu},1,t_n} + \int_0^{t_n}\int_s^{t_n} e^{-\bar{\nu}(t_n-s)}|\varphi'(s)|\,dt\,ds\Big) \\
\leq & Ck(1+|z|)\Big(|\varphi|_{\bar{\alpha},1,t_n} + \int_0^{t_n}(t_n-s)e^{-\bar{\nu}(t_n-s)}|\varphi'(s)|\,ds\Big) \\
\leq & Ck(1+|z|)|\varphi|_{\bar{\alpha},1,t_n}.
\end{aligned}$$

This completes the proof. □

**Proof of Theorem 2.3.** For any $\bar{\alpha} < \alpha$, pick $\nu \in (\bar{\alpha},\, \alpha)$.

We start by verifying that the error representation (2.31) holds. Using Lemma 2.5(i) we have already proved that we can replace $\hat{\beta}(z)$ by $\hat{\beta}_{\text{res}}(z)$ in equation (2.3). Noting that we can write

$$Y_z^n = \sum_{j=1}^{n} r(zk)^{n+1-j}\varphi^j = \frac{1}{1-zk}\sum_{j=1}^{n}(1-zk)^{-(n-j)}\varphi^j$$

we see that $Y_z^n$ satisfies the conditions for $\psi(z)$ in Lemma 2.5(ii) and so

$$\frac{1}{2\pi i}\int_{\Gamma_\nu}\frac{\beta^{(j)}(0)}{z^{j+1}}Y_z^n\,dz = 0, \quad \text{for} \quad j = 0,\, 1,\, 2, \ldots$$

It follows that we can replace $\hat{\beta}(z)$ by $\hat{\beta}_{\text{res}}(z)$ in (2.7). Thus (2.31) is valid for the present method.

Inserting the estimate for $|Y_z^n - y_z(t_n)|$ given in Lemma 2.6 and the estimate $|\hat{\beta}_{\text{res}}(z)| \leq C|z|^{-3}$ into (2.31) we find

$$|I_k^n - I(t_n)| \leq \frac{1}{2\pi}\int_{\Gamma_\nu}\frac{C}{|z|^3}\,Ck|\varphi|_{\bar{\alpha},1,t_n}(1+|z|)\,|dz| \leq Ck|\varphi|_{\bar{\alpha},1,t_n}.$$

This completes the proof. □

Before closing this subsection we prove the following result concerning the decay of $I_k^n$ and $Y_z^n$ for large $n$.

**Theorem 2.4.** *Assume that there exists a constant $\alpha_1$ such that $|\varphi(t)| \leq Ce^{-\alpha_1 t}$ and that $Y_z^n$ is given by (2.10). Then there exists, for any $\bar{\alpha} < \min\{\alpha_1,\, k^{-1}\log|1-zk|\}$, a constant $C$ such that*

$$|Y_z^n| \leq Ce^{-\bar{\alpha}t_n}.$$

*Furthermore, if $\beta$ satisfies B2 and $I_k^n$ is given by (2.12), then there exists, for any $\bar{\alpha} < \min\{\alpha_1,\, k^{-1}\log|1+\alpha k|\}$, a constant $C$ such that*

$$|I_k^n| \leq Ce^{-\bar{\alpha}t_n}.$$

**Proof of Theorem 2.4.** For any complex function $f$ let $\rho(f)$ denote its radius of analyticity. The assumptions in the theorem immediately gives that $\rho(\tilde{\varphi}) \geq e^{\alpha_1 k}$.

Since $\rho(\frac{1}{\delta_k(\xi)-z}) = |1-zk|$ it follows by (2.11) that $\rho(\tilde{Y}_z) \geq \min\{e^{\alpha_1 k},\, |1-zk|\}$. This proves the claim for $Y_z^n$.

Similarly, the claim for the decay in $I_k^n$ follows once we find a lower bound for $\rho(\tilde{I}_k)$. Recall that $\tilde{I}_k(\xi) = \tilde{\beta}(\xi)(\tilde{\varphi}(\xi) - \varphi^0)$ and that $\tilde{\beta}(\xi) := \hat{\beta}(\delta_k(\xi))$. Since $\hat{\beta}(z)$ is analytic whenever $z > -\alpha$ and $\operatorname{Re}\delta_k(\xi) > -\alpha$ whenever $|\xi| < 1+\alpha k$ it follows that $\rho(\tilde{\beta}) \geq 1+\alpha k$ which proves the claim. □

2.4.2. *The second order rule.* In this section we revisit the rule defined in Subsection 2.2.2.

**Theorem 2.5.** *Let $\beta$ satisfy condition B2 and let $I(t_n)$ and $I_k^n$ be given by (2.1) and (2.15) respectively. Then for any $\bar{\alpha} < \alpha$,*

$$|I_k^n - I(t_n)| \leq Ck^2 |\varphi|_{\bar{\alpha},2,t_n}.$$

We begin with the following lemma.

**Lemma 2.7.** *Let $y_z(t)$ and $Y_z^n$ be defined by (2.4) and (2.13) respectively. Then for any $\bar{\alpha} < \nu$, with $k$ small enough,*

$$|Y_z^n - y_z(t_n)| \leq Ck^2 |\varphi|_{\bar{\alpha},2,t_n}(1+|z|^2), \quad \text{for} \quad z \in \Gamma_\nu.$$

**Proof of Lemma 2.7.** We use the same notation as in the proof of Lemma 2.3. Recall in particular equation (2.22),

$$e_z^n = \frac{r_+^n - r_-^n}{r_+ - r_-} e_z^1 + k \sum_{j=2}^{n} \frac{r_+^{n+1-j} - r_-^{n+1-j}}{r_+ - r_-} \frac{\tau^j}{\frac{3}{2} - zk},$$

where

$$r_\pm = r_\pm(zk) = \frac{1}{\frac{3}{2} - zk}\big(1 \pm \tfrac{1}{2}\sqrt{1+2zk}\big).$$

Picking any number $\bar{\nu} \in (\bar{\alpha},\, \nu)$ we will show that

$$\left|\frac{r_+^n - r_-^n}{r_+ - r_-}\right| \leq Ce^{-\bar{\nu}t_n}, \quad \text{when} \quad z \in \Gamma_\nu. \tag{2.33}$$

Moreover

$$|e_z^1| \leq Ck^2\big(|\varphi(0)| + |\varphi'(0)| + ||\varphi''||_{L^1(I_1)}\big)(1+|z|). \tag{2.34}$$

Accepting (2.33) and (2.34) for the moment we find, inserting these into (2.22),

$$\begin{aligned} |e_z^n| \leq & Ce^{-\bar{\nu}t_n}\, Ck^2\big(|\varphi(0)| + |\varphi'(0)| + ||\varphi''||_{L^1(0,k)}\big)(1+|z|) \\ & + k \sum_{j=1}^{n} Ce^{-\bar{\nu}(t_n - t_{j-1})}\, k ||y_z'''||_{L^1(t_{j-2},t_j)} \\ \leq & Ck^2 |\varphi|_{\bar{\nu},2,t_n}(1+|z|) + Ck^2 \int_0^{t_n} e^{-\bar{\nu}(t_n - t)} |y_z'''(t)|\, dt. \end{aligned}$$

Combining this with the expression for $y_z'''(t)$ given in (2.23) as we did in the proof of Lemma 2.6 proves that claim.

We turn to the proof of (2.33). Since $r_+(zk) = e^{zk} + O((zk)^3)$ we can find a constant $R$ such that when $|zk| \le R$ we have $|r_+(zk)| \le e^{-\bar{\nu}k}$. We also have $|r_-(zk)| \le 1/2$ and consequently $|r_-(zk)| \le e^{-\bar{\nu}k}$ follows trivially for $k$ small. If $R \le 1/4$ it is easy to verify that $|r_+ - r_-|^{-1} \le 2$. This completes the proof when $|zk|$ is small. Assume next that $|zk| > R$. Then there exists a constant $\rho < 1$ such that $|r_\pm(zk)| \le \rho$. Consequently

$$\left|\frac{r_+^n - r_-^n}{r_+ - r_-}\right| = \left|\sum_{j=0}^{n-1} r_+^{n-1-j} r_-^j\right| \le n\,\rho^n \le C\left(\tfrac{1}{2}(1+\rho)\right)^n \le Ce^{-\bar{\nu}kn},$$

for $k$ small enough.

Finally we need to prove (2.34). When $|zk| \le 1$ the claim follows directly from (2.25). When $|zk| \ge 1$ we use instead *cf.* (2.25),

$$\begin{aligned} |e_z^1| \le& Ck(|\varphi^0| + |\varphi^1|) + ||\varphi||_{L^1(0,k)} \\ \le& C|z|k^2(|\varphi^0| + |\varphi^1|) + |z|k||\varphi||_{L^1(0,k)}. \end{aligned}$$

This completes the proof. □

**Proof of Theorem 2.5.** We first need to verify that we can replace $\hat{\beta}(z)$ by $\hat{\beta}_{\text{res}}(z)$ in the integral (2.7) so that the error representation (2.31) holds. This amounts to verifying that Lemma 2.5(ii) applies with $\psi(z) = Y_z^n$. This, in turn, follows directly once we note that we can write

$$Y_z^n = \frac{k\left(\frac{2}{3}\varphi^0 + \frac{1}{3}\varphi^1\right)}{1 - \frac{2}{3}zk} \sum_{i=0}^{n-1} r_+^i\, r_-^{n-1-i} + \frac{k}{\frac{3}{2} - zk} \sum_{j=2}^{n} \sum_{i=0}^{n-j} r_+^i r_-^{n-j-i} \varphi^j.$$

and recall that $|r_\pm| \le 1$

Inserting the estimate for $|Y_z^n - y_z(t_n)|$ given by Lemma 2.7 and $|\hat{\beta}_{\text{res}}(z)| \le C|z|^{-4}$ into (2.31) we get

$$|I_k^n - I(t_n)| \le \frac{1}{2\pi} \int_{\Gamma_\nu} \frac{C}{|z|^4}\, Ck^2 |\varphi|_{\bar{\alpha},2,t_n} (1 + |z|^2)\,|dz| \le Ck^2 |\varphi|_{\bar{\alpha},2,t_n}.$$

□

We end this subsection by giving an analogue of Theorem 2.4.

**Theorem 2.6.** *Assume that there exists a constant $\alpha_1$ such that $|\varphi(t)| \le Ce^{-\alpha_1 t}$. Then, if $\operatorname{Re} zk > -\frac{1}{2}$, and $Y_z^n$ is given by (2.13), there exists for any $\bar{\alpha} < \min\{\alpha_1,\, k^{-1}\log|2 - \sqrt{1+2zk}|\}$, a constant $C$ such that*

$$|Y_z^n| \le Ce^{-\bar{\alpha}t_n}.$$

*Furthermore, if $\beta$ satisfies B2 and $I_k^n$ is given by (2.12), then for any $\bar{\alpha} < \min\{\alpha_1,\, (2k)^{-1}\log|1 + 2\alpha k|\}$, there exists a constant $C$ such that,*

$$|I_k^n| \le Ce^{-\bar{\alpha}t_n}.$$

**Proof of Theorem 2.6.** As in the proof of Theorem 2.4 we will prove the claims by finding the radii of analyticity for $\tilde{Y}_z$ and $\tilde{I}_k$. From (2.14) we get

$$\tilde{Y}_z(\xi) = \frac{\tilde{\varphi}(\xi) - \varphi^0 + \frac{1}{2}\varphi^0\xi}{(2k)^{-1}(\xi - \xi_+)(\xi - \xi_-)}, \quad \text{where} \quad \xi_\pm = 2 \pm \sqrt{1 + 2zk}.$$

Since $\operatorname{Re} zk > -\frac{1}{2}$ we have $|\xi_-| \leq |\xi_+|$ and so $\rho(\tilde{Y}_z) \geq \min\{e^{k\alpha_1}, |\xi_-|\}$. This proves the claim for $Y_z^n$.

When determining $\rho(\tilde{\beta})$ we first recall that $\tilde{\beta}(\xi) := \hat{\beta}(\delta_k(\xi))$. For $\xi = re^{i\theta}$ we have

$$\operatorname{Re} \delta_k(\xi) = \frac{1}{k}\big((r\cos\theta - 1)^2 + \tfrac{1}{2} - \tfrac{1}{2}r^2\big) \geq \frac{1}{2k}(1 - r^2). \tag{2.35}$$

Since $\hat{\beta}(z)$ is analytic whenever $\operatorname{Re} z > -\alpha$ it then follows that $\rho(\tilde{\beta}) \geq \sqrt{1 + 2\alpha k}$. Recalling that $\tilde{I}_k(\xi) = \tilde{\beta}(\xi)(\tilde{\varphi}(\xi) - \varphi^0 + \frac{1}{2}\varphi^0\xi)$ the statement in the theorem follows directly. ☐

2.4.3. *Alternative second order rules.* In this section we will discuss some second order accurate methods derived by using less stable numerical schemes for $y_z$ than the ones described in Subsections 2.2.1 and 2.2.2. First we consider the following version of the Crank-Nicolson scheme,

$$\begin{aligned} &\bar{\partial}_1 Y_z^n - z\tfrac{1}{2}(Y_z^n + Y_z^{n-1}) = \tfrac{1}{2}(\varphi^n + \varphi^{n-1}), \quad n \geq 1, \\ &Y_z^0 = 0. \end{aligned} \tag{2.36}$$

It turns out that for this scheme, $\tilde{Y}_z(\xi)$, and hence $I_k^n$, depends on $\varphi^0$ in a way that does not conform to our previous analysis. In order to clarify what happens we introduce the variable $\psi^n := \varphi^n - \varphi^0$ when deriving the quadrature rule. We can now write (2.36) as,

$$k^{-1}(Y_z^n - Y_z^{n-1}) - \tfrac{1}{2}z(Y_z^n + Y_z^{n-1}) = \tfrac{1}{2}(\psi^n + \psi^{n-1}) + \varphi^0.$$

Multiplying this by $\xi^n$ and summing from one to infinity we find

$$k^{-1}(1 - \xi)\tilde{Y}_z(\xi) - \tfrac{1}{2}z(1 + \xi)\tilde{Y}_z(\xi) = \tfrac{1}{2}(1 + \xi)\tilde{\psi}(\xi) + \varphi^0(\xi + \xi^2 + \cdots).$$

Dividing by $\frac{1}{2}(1 + \xi)$ this gives, with $\delta_k(\xi) = 2k^{-1}\frac{1-\xi}{1+\xi}$,

$$(\delta_k(\xi) - z)\tilde{Y}_z(\xi) = \tilde{\psi}(\xi) + 2\varphi^0(\xi + \xi^3 + \xi^5 + \cdots) = \tilde{\varphi}(\xi) - \frac{1}{1 + \xi}\varphi^0.$$

Note that this is not exactly of the form (2.6). However, if $|\xi| < 1$, the residue theorem still applies and (2.8) gives

$$\tilde{I}_k = \hat{\beta}(\delta_k(\xi))\big(\tilde{\varphi}(\xi) - \frac{1}{1 + \xi}\varphi^0\big).$$

Defining the quadrature weights so that $\tilde{\beta}(\xi) = \hat{\beta}(\delta_k(\xi))$ we get the quadrature rule

$$I_k^n = \sum_{j=1}^{n} \beta_{n-j}\,\varphi^j - \varphi^0 \sum_{j=1}^{n} (-1)^j\,\beta_{n-j}. \tag{2.37}$$

We note that for this scheme the radius of analyticity for $\tilde{Y}_z$ and consequently for $\tilde{I}_k$ is only one since we have a pole at $\xi = -1$. This means that we do not get the exponential decay in the error estimate for $Y_z^n$.

**Theorem 2.7.** *If $\beta$ satisfies B2 and $I(t_n)$ and $I_k^n$ are defined by (2.1) and (2.37) respectively then*

$$|I_k^n - I(t_n)| \le Ck^2 |\varphi|_{0,2,t_n}.$$

When proving Theorem 2.7 we use the following estimate for $|Y_z^n - y_z(t_n)|$.

**Lemma 2.8.** *Let $y_z(t)$ and $Y_z^n$ be defined by (2.4) and (2.36) respectively. Then*

$$|Y_z^n - y_z(t_n)| \le Ck^2 |\varphi|_{0,2,t_n}(1+|z|^2), \quad \textit{for} \quad \operatorname{Re} z \le 0.$$

**Proof of Lemma 2.8.** The error equation reads

$$\bar{\partial}_1 e_z^n - z\tfrac{1}{2}(e_z^n + e_z^{n-1}) = \tau^{n-\frac{1}{2}},$$

where

$$\tau^{n-\frac{1}{2}} = \tfrac{1}{2}\big(y_z'(t_n) + y_z'(t_{n-1})\big) - \bar{\partial}_1 y_z^n = \int_{I_n} \Big(\frac{k}{8} - \frac{(s-t_{n-1/2})^2}{2k}\Big) y_z'''(s)\,ds.$$

We have the error representation

$$e_z^n = \frac{k}{1-zk/2} \sum_{j=1}^n r(zk)^{n-j} \tau^{j-1/2}, \quad \text{where} \quad r(zk) = \frac{1+zk/2}{1-zk/2}.$$

Using that $|r(zk)| \le 1$ in the left halfplane we get

$$|e_z^n| \le k \sum_{j=1}^n |\tau^{j-1/2}| \le \tfrac{1}{4}k^2 ||y_z'''||_{L^1(0,t_n)}.$$

Combining this with expression (2.23) for $y_z'''$ completes the proof. ☐

**Proof of Theorem 2.7.** As in the proofs of Theorems 2.3 and 2.5 the claim will follow immediately once we have verified that the error representation (2.31) holds. This, in turn, amounts to verifying that Lemma 2.5(ii) applies with $\psi(z) = Y_z^n$. But this is obvious once we note that

$$Y_z^n = \frac{k}{2-zk} \sum_{j=1}^n r(zk)^{n-j} (\varphi^{j-1} + \varphi^j),$$

and that $|r(zk)| \le 1$. ☐

If we want to find a quadrature rule for $I(t_{n+1/2})$ that depends only on the values of $\varphi$ at the points $t_j$ we can use a modified version of the Crank-Nicolson scheme. Letting $Y_z^{n+1/2}$ denote an approximation of $y_z(t_{n+1/2})$ we use

$$
\begin{aligned}
&k^{-1}(Y_z^{n+1/2} - Y_z^{n-1/2}) - z\tfrac{1}{2}(Y_z^{n+1/2} + Y_z^{n-1/2}) = \varphi^n, \quad n \geq 1, \\
&Y_z^{1/2} = \frac{k}{2 - zk}\varphi^1.
\end{aligned} \tag{2.38}
$$

This time we have to define an adjusted generating function. Setting

$$\check{Y}_z(\xi) := \sum_{n=0}^{\infty} \xi^n Y_z^{n+1/2},$$

we can multiply (2.38) by $\xi^n$, sum from one to infinity and find

$$k^{-1}\big(\check{Y}_z(\xi) - Y_z^{1/2} - \xi\check{Y}_z(\xi)\big) - z\tfrac{1}{2}\big(\check{Y}_z(\xi) - Y_z^{1/2} + \xi\check{Y}_z(\xi)\big) = \tilde{\varphi}(\xi) - \varphi^0.$$

Inserting the expression for $Y_z^{1/2}$ this simplifies to

$$\check{Y}_z(\xi) = \frac{\frac{2}{1+\xi}(\tilde{\varphi}(\xi) - \varphi^0 + \frac{1}{2}\varphi^1)}{\delta_k(\xi) - z}, \quad \text{where} \quad \delta_k(\xi) = \frac{2}{k}\,\frac{1-\xi}{1+\xi}.$$

Using the adjusted generating function we get

$$
\begin{aligned}
\check{I}_k(\xi) &= \frac{1}{2\pi i}\int_{\Gamma_\nu} \hat{\beta}(z)\check{Y}_z(\xi)\,dz \\
&= \hat{\beta}(\delta_k(\xi))\frac{2}{1+\xi}\big(\tilde{\varphi}(\xi) - \varphi^0 + \tfrac{1}{2}\varphi^1\big).
\end{aligned}
$$

If we define the quadrature weights this time so that (this is *not* as before)

$$\tilde{\beta}(\xi) = \hat{\beta}(\delta_k(\xi))\frac{2}{1+\xi},$$

we get the quadrature rule

$$I_k^{n+1/2} = \sum_{j=1}^{n} \beta_{n-j}\varphi^j + \tfrac{1}{2}\beta_n\varphi^1. \tag{2.39}$$

Note that $I_k^{n+1/2}$ depends on neither $\varphi^{n+1}$ nor $\varphi^0$ and that again the radii of analyticity for $\tilde{\beta}$ and $\tilde{I}_k$ may be only one.

**Theorem 2.8.** *If $\beta$ satisfies B2 and $I(t_n)$ and $I_k^n$ are given by (2.1) and (2.39) respectively then*

$$|I_k^{n+1/2} - I(t_{n+1/2})| \leq Ck^2|\varphi|_{0,2,t_{n+1/2}}.$$

As usual, we need a bound for $|Y_z^n - y_z(t_n)|$.

**Lemma 2.9.** *Let $y_z(t)$ and $Y_z^{n+1/2}$ be defined by (2.4) and (2.38) respectively. Then*

$$|Y_z^{n+1/2} - y_z(t_{n+1/2})| \leq Ck^2|\varphi|_{0,2,t_{n+1/2}}(1 + |z|^2), \quad \textit{for} \quad \operatorname{Re} z \leq 0.$$

**Proof of Lemma 2.9.** The error equation is

$$k^{-1}(e_z^{n+1/2} - e_z^{n-1/2}) - z\tfrac{1}{2}(e_z^{n+1/2} + e_z^{n-1/2}) = \tau^n,$$

where

$$\begin{aligned}\tau^n =& \big[y_z'(t_n) - k^{-1}(y_z^{n+1/2} - y_z^{n-1/2})\big] - z\big[y_z^n - \tfrac{1}{2}(y_z^{n+1/2} + y_z^{n-1/2})\big]\\ =& -\frac{1}{2k}\int_{t_{n-1/2}}^{t_{n+1/2}} \big(\tfrac{1}{2}k - |s-t_n|\big)^2 y_z'''(s)\,ds\\ &+ \frac{z}{2k}\int_{t_{n-1/2}}^{t_{n+1/2}} \big[\big(|s-t_n| - \tfrac{1}{2}k\big)^2 + \big(\tfrac{1}{4}k^2 - (s-t_n)^2\big)\big] y_z''(s)\,ds.\end{aligned}$$

This gives the error representation

$$e_z^{n+1/2} = r^n e_z^{1/2} + \frac{k}{1-zk/2}\sum_{j=1}^{n} r^{n-j}\tau^j, \quad \text{where} \quad r = \frac{1+zk/2}{1-zk/2},$$

and since $|r| \le 1$

$$|e_z^{n+1/2}| \le |e_z^{1/2}| + Ck^2\big(||y_z'''||_{L^1(k/2,\,t_{n+1/2})} + |z|\,||y_z''||_{L^1(k/2,\,t_{n+1/2})}\big).$$

Inserting the expressions for $y_z''$ and $y_z'''$ given by (2.17) and (2.23), we obtain the desired bound. ☐

The proof of Theorem 2.8 now follows easily using the same techniques as in the previous sections.

Note that since $\rho(\tilde{I}_k) = \rho(\tilde{\beta}) = 1$ we cannot prove stability theorems corresponding to Theorems 2.4 and 2.6 for the Crank-Nicolson rules.

2.5. **Positivity.** In this section we will demonstrate that when $\beta$ satisfies the positivity condition (1.2) then the quadrature rules derived in Subsections 2.2.1 and 2.2.2 are positive and weakly positive respectively. Recall that *positivity* for a quadrature rule means that is satisfies (1.8), the discrete analogue of (1.2). *Weak positivity* means that (1.8) is satisfied whenever $\varphi^0 = 0$.

First we demonstrate that the assumptions on $\beta$ enforce certain conditions for $\hat{\beta}(z)$.

**Lemma 2.10.** *Assume that $\beta$ satisfies (1.2) and either B1 or B2. Then*

$$\operatorname{Re}\hat{\beta}(z) \ge 0, \quad \text{when } \operatorname{Re} z \ge 0.$$

**Proof of Lemma 2.10.** We extend $\varphi$ in (1.2) to the whole real axis by setting it to zero for $t > T$. Then, by Plancherel's formula, the function $y \mapsto \hat{\varphi}(iy)$ is contained in $L^2(\mathbb{R})$ and so, with $\Gamma_0 = i\mathbb{R}$,

$$\begin{aligned}I :=& \int_0^T \varphi(t)\int_0^t \beta(t-s)\varphi(s)\,ds\,dt = \int_0^T \varphi(t)\frac{1}{2\pi i}\int_{\Gamma_0} e^{zt}\hat{\beta}(z)\hat{\varphi}(z)\,dz\,dt\\ =& \frac{1}{2\pi i}\int_{\Gamma_0}\hat{\varphi}(-z)\hat{\beta}(z)\hat{\varphi}(z)\,dz = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\beta}(iy)|\hat{\varphi}(iy)|^2\,dy,\end{aligned}$$

where we used that $\hat{\varphi}(-z) = \overline{\hat{\varphi}(z)}$ for $z \in \Gamma_0$. Since $I \geq 0$ we must have $\operatorname{Re}\hat{\beta}(z) \geq 0$ for $z \in \Gamma_0$. We will extend this result to the right halfplane by the maximum principle.

Assume that there is a point $z_0$ with positive real part such that $\operatorname{Re}\hat{\beta}(z_0) = -\varepsilon < 0$. Since $|\hat{\beta}(z)| \leq C|z|^{-\mu}$ for some $\mu > 0$ when B1 holds and the same bound is valid for $\mu = 1$ when B2 holds we can pick an $R > |z_0|$ such that $|\hat{\beta}(z)| \leq \varepsilon/2$ for $|z| = R$. The maximum principle then gives a contradiction since $\operatorname{Re}\hat{\beta}(z)$ is harmonic on the half disc $\{z\colon |z| \leq R, \operatorname{Re} z \geq 0\}$ and $\operatorname{Re}\hat{\beta}(z) \geq -\varepsilon/2$ on the boundary. □

**Theorem 2.9.** *If $\beta$ is positive and satisfies either B1 or B2, then the quadrature rule defined in Subsection 2.2.1 is positive and the rule defined in Subsection 2.2.2 is weakly positive.*

**Proof of Theorem 2.9.** We start by proving the claim for the first order rule. We extend the sequence $\{\varphi^n\}_{n=0}^N$ in (1.8) to an infinite sequence by setting $\varphi^n = 0$ for $n > N$. Since $\varphi^0$ does not enter the formula for the first order rule (2.12) we can also set $\varphi^0 = 0$. Then the sequence $\{\sum_{j=0}^n \beta_{n-j}\varphi^j\}_n$ has the generating function $\tilde{\beta}(\xi)\tilde{\varphi}(\xi)$. Here $\rho(\tilde{\beta}) \geq 1$ since $\tilde{\beta}(\xi) = \hat{\beta}(k^{-1}(1-\xi))$ and $\hat{\beta}$ is analytic in the right half plane. Hence, for any $r < 1$ we can write

$$Q_N(\varphi) = \sum_{n=1}^{\infty} \varphi^n \frac{1}{2\pi i}\int_{|\xi|=r} \frac{1}{\xi^{n+1}}\tilde{\beta}(\xi)\tilde{\varphi}(\xi)\,d\xi = \frac{1}{2\pi i}\int_{|\xi|=r} \tilde{\varphi}(\xi^{-1})\tilde{\beta}(\xi)\tilde{\varphi}(\xi)\frac{d\xi}{\xi}.$$

If B1 holds for a positive $\alpha$ or if B2 holds, then the integrand is analytic on the disc $|\xi| \leq 1$ and we can let $r \to 1$. Observing that when $|\xi| = 1$ we have $\tilde{\varphi}(\xi^{-1}) = \tilde{\varphi}(\bar{\xi}) = \overline{\tilde{\varphi}(\xi)}$, we find

$$Q_N(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{\beta}(e^{i\theta})|\tilde{\varphi}(e^{i\theta})|^2\,d\theta. \tag{2.40}$$

By Lemma 2.10 and $\tilde{\beta}(e^{i\theta}) = \hat{\beta}(k^{-1}(1-e^{i\theta}))$ it follows that $Q_N(\varphi) \geq 0$.

When B1 holds for $\alpha = 0$ we find that $\tilde{\beta}(\xi)$ may have a singularity at $\xi = 1$. If $\mu < 1$ then the singularity is integrable and we can still let $r \to 1$ without complications. If $\mu = 1$, then (2.40) still holds but we must interpret it as the limit obtained by adding an infinitesimal semicircle to the right of the singularity. Thus $Q_N(\varphi) \geq 0$ even when $\alpha = 0$.

By equation (2.15) it is clear that the second order rule takes exactly the same form as the first order rule when $\varphi^0 = 0$ (but of course with different coefficients). Thus the same argument as we used for the first order rule will hold once we establish that $\operatorname{Re}\tilde{\beta}(\xi) \geq 0$ when $|\xi| = 1$. But this is an immediate consequence of Lemma 2.10 and (2.35). □

## 3. The Evolution Equation

3.1. **Introduction.** In this section we will use the convolution quadrature rules derived in Chapter 2 to construct first and second order accurate time discrete schemes for the initial value problem stated in Section 1.1,

$$\begin{aligned} u_t(t) + \int_0^t \beta(t-s)Au(s)\,ds &= 0, \quad t>0, \\ u(0) &= v, \end{aligned} \tag{1.1}$$

set in a Hilbert space $H$. We assume that $A$ is self-adjoint, compactly invertible and positive, *i.e.*, there exists a positive constant $\kappa$, such that, with $A(\cdot,\,\cdot)$ the bilinear form induced by $A$,

$$A(\varphi,\,\varphi) \geq \kappa||\varphi||^2, \quad \forall\, \varphi \in \mathrm{dom}(A^{1/2}). \tag{3.1}$$

Since $A$ is self-adjoint and compactly invertible it has a complete set of orthonormal eigenvectors and we can choose $\kappa$ as the smallest eigenvalue. We further assume that $\beta$ satisfies

**B2:** $\quad |\beta^{(j)}(t)| \leq Ce^{-\alpha t}, \quad \text{for} \quad j = 0, 1, \ldots, r+2,$

and that,

$$\mathrm{Re}\,\hat{\beta}(z) > 0, \quad \text{when} \quad \mathrm{Re}\,z > -\alpha. \tag{3.2}$$

In view of Lemma 2.10 this condition can be interpreted as a stronger version of the positivity condition (1.2). In fact, (3.2) is equivalent to assuming that $e^{\bar{\alpha}t}\beta(t)$ is positive definite for any $\bar{\alpha} \in [0,\,\alpha)$. Without loss of generality we further assume that $A$ and $\beta$ are scaled so that $\beta(0) = 1$.

The first order method takes the form,

$$\bar{\partial}U^n + q_n(A_hU) = 0, \tag{1.7}$$

with $\bar{\partial} = \bar{\partial}_1$ and $q_n$ the corresponding first order accurate convolution quadrature derived in Subsection 2.2.1. For the second order method we analogously use $\bar{\partial} = \bar{\partial}_2$ and take for $q_n$ the corresponding second order accurate convolution quadrature rule derived in Subsection 2.2.2. The schemes thus obtained are implicit, requiring us to solve a Poisson equation of the form $(I + k^2\beta_0 A)U^n = f$, for some right hand side $f$ and some quadrature weight $\beta_0$, at each time step. Since $A$ is an elliptic operator it is, theoretically speaking, possible to solve this equation. However, when $A$ is a differential operator, the equation cannot, in general, be solved with a finite number of arithmetic operations. But since our analysis is valid for a general operator $A$, we can apply the time discretisation methods to the spatially discrete equation, (1.6), in which case we only need to solve the finite dimensional problem $(I + k^2\beta_0 A_h)u_h = f_h$. This gives a fully discrete scheme for approximating (1.1).

Note that the assumption that $A$ is compactly invertible implies that it has a complete set of orthonormal eigenvectors. This in turn means that we can construct the solution to (1.1) by forming a suitably weighted sum of solutions to the scalar analogou of equation (1.1) obtained by replacing

$A$ by an eigenvalue, $\lambda$. Consequently, the results for the full equation will follow easily once we understand how the following equations,

$$\begin{aligned} u_t(t) + \lambda \int_0^t \beta(t-s) u(s)\, ds &= 0, \quad t > 1, \\ u(0) &= 1, \end{aligned} \tag{3.3}$$

and their discrete counterparts behave when $\lambda$ runs through the spectrum of $A$.

We will begin this chapter by studying a scalar model problem, namely equation (3.3) with $\beta(t) = e^{-\alpha t}$ and the first order discretisation of that equation. These equations can be solved analytically and we will demonstrate that both the continuous and the discrete equations have exponentially decaying solutions.

After proving some technical lemmas we will then proceed to show that many features of the model problem are also present for the whole class of equations specified by kernels satisfying (3.2) and B2. In Subsection 3.4 we prove that there exist positive constants $C$ and $\nu$ such that,

$$||u^{(j)}(t)|| \le Ce^{-\nu t}||A^{j/2}v||, \qquad \text{for } j = 0, 1, 2,$$

where $u$ is the solution to (1.1). Then, in Subsections 3.5 and 3.6 we will prove that the solutions to the first and second order discrete equations satisfy

$$||U^n|| \le Ce^{-\nu_k t}||v||,$$

where $\nu_k = \nu + O(k)$. Note however that we do not claim that the constant $\nu$ is sharp. For the first order method we will further prove that for any $\gamma < \nu$ there exists a constant $C$ such that

$$||U^n - u(t_n)|| \le Ck\, e^{-\gamma t_n}||Av||,$$

Similarly for the second order method

$$||U^n - u(t_n)|| \le Ck^2 e^{-\gamma t_n}(\log k^{-1}||Av|| + ||A^{3/2}v||).$$

The constants $C$ and $\nu$ depend on the ellipticity constant $a$ in (3.1) but do not depend on $A$ or $k$ in any other way. Since (3.1) also holds for $A_h$, with the same ellipticity constant $a$, the bounds derived for $A$ also hold for $A_h$. In Section 3.7 we will use this fact to prove error estimates for the fully discrete schemes obtained by applying our time discretisation techniques to the spatially semidiscrete equation, (1.6).

Most of the material regarding the continuous problem and the first order discretisation is adapted from McLean and Thomée [9].

3.2. **The model problem.** In this section we shall study a model problem obtained by setting $\beta(t) = e^{-\alpha t}$ in equation (3.3),

$$u_t(t) + \lambda \int_0^t e^{-\alpha(t-s)} u(s)\, ds = 0, \quad \text{for } t > 0, \quad \text{and } u(0) = 1. \tag{3.4}$$

We will derive the exact solution to this equation, and the solution to the corresponding first order accurate discrete equation, and demonstrate that both these solutions decay exponentially.

The easiest way to find the solution to (3.4) is simply to differentiate the equation and note that the resulting equation can be written

$$u_{tt} + \alpha u_t + \lambda u = 0, \quad \text{when } t > 0, \quad \text{and } u(0) = 1, \quad u_t(0) = 0. \tag{3.5}$$

The characteristic polynomial of this ordinary differential equation is now directly solvable and the solutions may be written down immediately. However, this technique generally cannot be applied when $\beta$ is not a pure exponential. We will therefore proceed to solve the equation using transform techniques which may be generalised to other $\beta$. Taking the Laplace transform of (3.4) we find

$$z\hat{u}(z) - 1 + \lambda\hat{\beta}(z)\hat{u}(z) = 0,$$

where $\hat{\beta}(z) = (z+\alpha)^{-1}$. With $\alpha_1 := \frac{1}{2}\alpha$ we then get

$$\hat{u}(z) = \frac{1}{z + \lambda\hat{\beta}(z)} = \frac{z + 2\alpha_1}{z^2 + 2\alpha_1 z + \lambda} = \frac{z + 2\alpha_1}{(z - z_+)(z - z_-)}, \tag{3.6}$$

where

$$z_\pm := -\alpha_1 \pm \sqrt{\alpha_1^2 - \lambda}.$$

Note that $z_\pm$ are the zeros of the characteristic polynomial to equation (3.5). Setting

$$\nu_0 := -\max\{\operatorname{Re} z_+, \operatorname{Re} z_-\} = \begin{cases} \alpha_1 - \sqrt{\alpha_1^2 - \lambda}, & \text{when } \lambda < \alpha_1^2, \\ \alpha_1, & \text{when } \lambda \geq \alpha_1^2, \end{cases} \tag{3.7}$$

we find that $\hat{u}(z)$ is analytic when $\operatorname{Re} z > -\nu_0$. For any $\nu < \nu_0$ we can thus invoke the Laplace inversion formula and find

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma_\nu} e^{zt}\frac{z + 2\alpha_1}{(z - z_+)(z - z_-)}\,dz, \quad \text{where } \Gamma_\nu = -\nu + i\mathbb{R}. \tag{3.8}$$

Using the residue theorem we obtain

$$u(t) = e^{z_+ t}\frac{z_+ + 2\alpha_1}{z_+ - z_-} + e^{z_- t}\frac{z_- + 2\alpha_1}{z_- - z_+}.$$

With $\omega := \sqrt{|\alpha_1^2 - \lambda|}$ this can be written as

$$u(t) = \begin{cases} e^{-\alpha_1 t}\big(\cosh(\omega t) + \frac{\alpha_1}{\omega}\sinh(\omega t)\big), & \text{when } \lambda < \alpha_1^2, \\ e^{-\alpha_1 t}\big(\cos(\omega t) + \frac{\alpha_1}{\omega}\sin(\omega t)\big), & \text{when } \lambda \geq \alpha_1^2, \end{cases} \tag{3.9}$$

where, for $\lambda = \alpha_1^2$ we should interpret the right hand side as the limit when $\omega \to 0$, *i.e.*, $u(t) = e^{-\alpha_1 t}(1 + \alpha_1 t)$.

We see that the solution decays exponentially, $|u(t)| \leq Ce^{-\nu_0 t}$ (when $\lambda = \alpha_1^2$, we get $|u(t)| \leq C(1+t)e^{-\nu_0 t}$). It is important to note that if $\lambda > \alpha_1^2$, then $\nu_0$ does not depend on $\lambda$. Viewing (3.4) as the result of a spectral decomposition of the full problem (1.1), with $\lambda$ an eigenvalue of $A$, this means that all high eigenmodes have the same rate of decay, *i.e.*, we have no smoothing. This was expected since we remarked in the introduction that when $\beta(t) = e^{-\alpha t}$, equation (1.1) is equivalent to the damped wave equation (1.3).

If $\beta$ is not a pure exponential, then $\hat{u}(z)$ need not be a rational function and in that case we cannot, in general, use the residue theorem to find the solution. We can, however, use the integral representation directly to show that the solution decays exponentially. This will be done for general kernels in Subsection 3.4. As an introduction we indicate what happens if we apply this technique to the model problem. We first recall from Lemma 2.5 that $\int_{\Gamma_\nu} e^{zt} z^{-1}\, dz = 0$. We can therefore obtain an absolutely convergent integral representation, *cf.* (3.6),

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma_\nu} e^{zt}\left[\frac{1}{z+\lambda\hat{\beta}(z)} - \frac{1}{z}\right] dz = \frac{-\lambda}{2\pi i}\int_{\Gamma_\nu} e^{zt}\frac{1}{z(z-z_+)(z-z_-)}\, dz.$$

Taking absolute values under the integral we trivially find $|u(t)| \leq C\lambda e^{-\nu t}$. Using the techniques in the proof of Lemma 3.5, one can actually obtain a bound that does not depend on $\lambda$; $|u(t)| \leq Ce^{-\nu t}$. In either case the constant $C$ depends on both $\nu$ and $\nu_0$. When $\nu \nearrow \nu_0$ then $C$ tends to infinity. We conclude that this method underestimates the decay rate slightly.

For the model problem, it is also possible to give an explicit solution to the first order discrete scheme under consideration. In Appendix B.2 we show that if the derivative is approximated with a one step backward difference operator and the integral by the associated first order accurate convolution quadrature, then the solution is given by

$$(3.10) \qquad U^n = \begin{cases} e^{-\alpha_{1,k}t_n}\big(\cosh(\omega_k^r t_n) + \frac{\alpha_1}{\omega}\sinh(\omega_k^r t_n)\big), & \text{when } \lambda < \alpha_1^2, \\ e^{-\alpha_{1,k}t_n}\big(\cos(\omega_k^i t_n) + \frac{\alpha_1}{\omega}\sin(\omega_k^i t_n)\big), & \text{when } \lambda \geq \alpha_1^2, \end{cases}$$

where

$$\begin{aligned} \alpha_{1,k} &= \frac{1}{2k}\log(1+2\alpha_1 k + \lambda k^2) = \alpha_1 + (\lambda - 2\alpha_1^2)k + O(k^2), \\ \omega_k^r &= \frac{1}{k}\log\frac{1+\alpha_1 k + \omega k}{1+\alpha_1 k - \omega k} = \omega - \alpha_1\omega k + O(k^2), \\ \omega_k^i &= \frac{1}{k}\arctan\frac{k\omega}{1+\alpha_1 k} = \omega - \alpha_1\omega k + O(k^2). \end{aligned}$$

Comparing (3.9) and (3.10) we see that the behaviour of the approximate solution mimics that of the exact solution. We again find that the size of $\lambda$ is crucial; when $\lambda$ is smaller than a threshold, then $U^n$ is a sum of two exponentially decaying components whose exponential decay rates can be arbitrarily small. When $\lambda \geq \alpha_1^2$ the exponential decay rate of the approximate solution is given by $(2k)^{-1}\log(1+2\alpha_1 k + \lambda k^2)$. This means that initially, as $\lambda$ grows larger than $\alpha_1^2$, the exponential decay rate stays fairly constant at around $\alpha_1$. But when $\lambda \geq ck^{-1}$ then the solution decays faster again. This can be interpreted as a smoothing effect in the discrete solution.

Finally we note that when $\lambda \leq 2\alpha_1^2$ the approximate solution actually decays *slower* than the exact one. Consequently we cannot hope to prove an asymptotic error estimate that is sharp in the sense that the bound decays with the same rate as the exact solution.

3.3. **Technical lemmas.** When we use the Laplace transform to find the rate of decay for the solution to the model problem, the central question is to find the domain of analyticity for $\hat{u}(z)$, since this determines how far to the left the integration contour in (3.8) could be chosen. We address this question by using that for the general, scalar problem

$$\hat{u}(z) = \frac{1}{z + \lambda\hat{\beta}(z)} = \frac{1/\hat{\beta}(z)}{z/\hat{\beta}(z) + \lambda}.$$

Assumption (3.2) guarantees that $1/\hat{\beta}(z)$ and $z/\hat{\beta}(z)$ both are analytic when $\operatorname{Re} z > -\alpha$, and it remains to determine when $z/\hat{\beta}(z) + \lambda \neq 0$. For the model problem, $z/\hat{\beta}(z) + \lambda = z^2 + 2\alpha_1 z + \lambda$ and the question was then resolved by locating the zeros to this polynomial. In the general case, the denominator will not be exactly a polynomial but the lemmas in this section say that it is, as the following sections will prove, sufficiently close to one to make the analysis go through.

Set

$$\psi(z) := \frac{z}{\hat{\beta}(z)}, \quad \alpha_1 := -\tfrac{1}{2}\beta'(0), \quad q(z) := z^2 + 2\alpha_1 z. \tag{3.11}$$

For the model problem we had $\alpha_1 = \alpha/2$ and $\psi(z) = q(z)$. For a general kernel we have instead the following three lemmas.

**Lemma 3.1.** *Let $\beta$ satisfy (3.2) and B2. Then $\beta'(0) \leq -\alpha$.*

**Lemma 3.2.** *Let $\beta$ satisfy (3.2) and B2. Then given any $\epsilon > 0$ there are constants $C$ and $M$ such that*

$$|\psi(z)| + |z\psi'(z)| \leq C(1 + |z|^2), \quad \text{and} \quad |\psi(z) - q(z)| \leq M,$$

*when* $\operatorname{Re} z \geq -\alpha + \epsilon$.

**Lemma 3.3.** *Let $\beta$ satisfy (3.2) and B2, let $0 < \alpha_2 < \min(\alpha, \alpha_1)$, let $M$ be as in Lemma 3.2, and assume that $\Lambda$ is such that*

$$\sqrt{\Lambda - \alpha_1^2} \geq \frac{2M}{\alpha_1 - \alpha_2} + 1. \tag{3.12}$$

(i) *For $\lambda \geq \Lambda$, we have, with $z_\pm = -\alpha_1 \pm i\sqrt{\lambda - \alpha_1^2}$,*

$$|\psi(z) + \lambda| \geq \tfrac{1}{2}|q(z) + \lambda| = \tfrac{1}{2}|(z - z_+)(z - z_-)|, \quad \text{when } \operatorname{Re} z \geq -\alpha_2.$$

(ii) *For any $\kappa \in (0, \Lambda)$, there are $\alpha_3 \in (0, \alpha_2)$ and $c > 0$ such that, for $\lambda \in [\kappa, \Lambda]$,*

$$|\psi(z) + \lambda| \geq c(1 + |z|^2), \quad \text{when } \operatorname{Re} z \geq -\alpha_3.$$

For the proofs of these lemmas we refer to [9].

3.4. **The general continuous problem.** The purpose of this section is to prove that the solution to equation (1.1) decays exponentially. Using the norm $|v|_r := ||A^{r/2}v||$ we will prove

**Theorem 3.1.** *Let $u$ be defined by (1.1) and assume that $\beta$ satisfies (3.2) and B2. Then there exist positive constants $\nu$ and $C$ such that*

$$||u^{(j)}(t)|| \leq Ce^{-\nu t}|v|_j, \quad \text{for } t \geq 0, \quad \text{and } j = 0,\, 1,\, 2.$$

*With $\kappa$ the smallest eigenvalue of $A$, and with the notation of Lemma 3.3 we can choose $\nu = \alpha_2$ if $\kappa \geq \Lambda$ and $\nu = \alpha_3$ otherwise.*

When proving the theorem we start by using the energy method to show that the solution and its first two derivatives are bounded. Then for large $t$ we demonstrate the exponential decay by estimating an integral representation of the solution, derived by using Laplace transform methods.

We begin by providing the integral representation for the solution to the scalar problem and then a lemma stating that this solution decays exponentially for large $t$.

**Lemma 3.4.** *Let $\psi$ be defined by (3.11), and let $\alpha_2$ and $\alpha_3$ be as in Lemma 3.3. Then, with $\Gamma_\nu = -\nu + i\mathbb{R}$, the solution to (3.3) is given by*

(3.13)
$$u(t) = \frac{-\lambda}{2\pi i}\int_{\Gamma_\nu} e^{zt}\frac{1}{z(\psi(z)+\lambda)}\,dz, \quad \text{where } \nu = \begin{cases} \alpha_3, & \text{if } \kappa \leq \lambda < \Lambda, \\ \alpha_2, & \text{if } \lambda \geq \Lambda. \end{cases}$$

**Lemma 3.5.** *Let $u$ be defined by (3.3) and let $\nu$ be as in Lemma 3.4. Then, for $j = 0,\, 1$,*

$$|u^{(j)}(t)| \leq C\lambda^{j/2}e^{-\nu t} \quad \text{when} \quad t \geq 1.$$

Before proving these lemmas we give the proof of Theorem 3.1.

**Proof of Theorem 3.1.** We start by using the energy method to show that $||u^{(j)}(t)||$ is bounded for $j = 0,\, 1,\, 2$.

For $j = 0$ we take the scalar product of (1.1) by $u(t)$,

$$\tfrac{1}{2}\frac{d}{dt}||u(t)||^2 + \int_0^t \beta(t-s)(Au(s),\, u(t))\,dt = 0.$$

Fixing any $T$, integrating over $[0,\, T]$ and using (3.2), we find that $||u(T)|| \leq ||v||$. Furthermore, if $v \in \mathrm{dom}(A^{r/2})$ then the function $A^{r/2}u(t)$ satisfies (1.1) with $v$ replaced by $A^{r/2}v$. Consequently, for any $r \geq 0$,

$$|u(T)|_r \leq |v|_r, \quad \forall\, T \in \mathbb{R}_+. \tag{3.14}$$

For $j = 1$ we differentiate (1.1) and integrate by parts,

$$u_{tt}(t) + \beta(t)Au(0) + \int_0^t \beta(t-s)Au_t(s)\,ds = 0.$$

Taking inner products with $u_t(t)$, this gives,

$$\tfrac{1}{2}\frac{d}{dt}||u_t(t)||^2 + \beta(t)(Av,\, u_t(t)) + \int_0^t \beta(t-s)(Au_t(s),\, u_t(t))\, ds = 0.$$

Integrating over $[0,\, T]$ and using (3.2) we find, since $u_t(0) = 0$,

$$\tfrac{1}{2}||u_t(T)||^2 + \int_0^T \beta(t)(Av,\, u_t(t))\, dt \le 0.$$

Integrating by parts and using (3.14) this yields,

$$\begin{aligned}\tfrac{1}{2}||u_t(T)||^2 &\le \left| \big[\beta(t)(Av,\, u(t))\big]_0^T - \int_0^T \beta'(t)(Av,\, u(t))\, dt \right| \\ &\le |\beta(T)|\,|v|_1\,|u(T)|_1 + |v|_1^2 + \int_0^T |\beta'(t)|\,|v|_1\,|u(t)|_1\, dt \le C|v|_1^2,\end{aligned}$$

where, in the last inequality, we used (3.14) with $r = 1$ and the exponential decay of $|\beta'(t)|$.

For $j = 2$ we differentiate (1.1),

$$u_{tt}(t) + Au(t) + \int_0^t \beta'(t-s)Au(s)\, ds = 0.$$

Using again (3.14) we find

$$||u_{tt}(t)|| \le |u(t)|_2 + \int_0^t |\beta'(t-s)|\,|u(s)|_2\, ds \le C|v|_2. \tag{3.15}$$

Next we use Lemma 3.5 and a standard spectral argument to show the exponential decay for $||u||$ and $||u_t||$ when $t \ge 1$. Since $A$ is compactly invertible it follows that it has an orthonormal set of eigenvectors $\{\psi_i\}_{i=1}^N$, where $N$ may be infinity, that span $H$. Calling the associated eigenvalues $\lambda_i$ we let $u_i$ denote the solution to the scalar problem (3.3), with $\lambda = \lambda_i$. With $a_i := (v,\, \psi_i)$ it then follows that $u(t) = \sum_{i=1}^N a_i u_i(t)\psi_i$. By Lemma 3.5 and the orthogonality of the eigenvectors we find, for $t \ge 1$,

$$||u^{(j)}(t)||^2 \le \sum_{i=1}^N a_i^2 C\lambda_i^j e^{-2\nu t} = Ce^{-2\nu t}\sum_{i=1}^N a_i^2\lambda_i^j = Ce^{-2\nu t}|v|_j^2, \quad j = 0, 1.$$

As a direct consequence we also get $|u(t)|_2 \le Ce^{-\nu t}|v|_2$. In order to prove the statement for $j = 2$ we combine this with (3.15),

$$||u_{tt}(t)|| \le Ce^{-\nu t}|v|_2 + \int_0^t Ce^{-\alpha(t-s)}\, Ce^{-\nu s}|v|_2\, ds.$$

Since $\nu < \alpha$ the exponential decay of $||u_{tt}||$ follows. $\square$

**Proof of Lemma 3.4.** Taking the Laplace transform of (3.3) we find

$$\hat{u}(z) = \frac{1}{z + \lambda\hat{\beta}(z)} = \frac{1}{\hat{\beta}(z)\big(\psi(z) + \lambda\big)}.$$

Lemma 3.3 and assumption (3.2) give that $|\psi(z)+\lambda|>0$ and $\operatorname{Re}\hat{\beta}(z)>0$, when $\operatorname{Re} z \geq -\nu$. Consequently $\hat{u}(z)$ is analytic in this region and so

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma_\nu} e^{zt}\frac{1}{z+\lambda\hat{\beta}(z)}\,dz, \quad \text{with} \quad \Gamma_\nu = -\nu + i\mathbb{R}. \tag{3.16}$$

Recall from Lemma 2.5 that for $t$ non-negative $\int_{\Gamma_\nu} e^{zt}z^{-1}\,dz = 0$. The statement in the theorem then follows since

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma_\nu} e^{zt}\left[\frac{1}{z+\lambda\hat{\beta}(z)} - \frac{1}{z}\right]dz = \frac{1}{2\pi i}\int_{\Gamma_\nu} e^{zt}\frac{-\lambda}{z(\psi(z)+\lambda)}\,dz.$$

□

**Proof of Lemma 3.5.** We will bound the integral given in (3.13) by using the lower bounds for $|\psi(z)+\lambda|$ provided by Lemma 3.3.
**Case 1, $\kappa \leq \lambda < \Lambda$.** Invoking Lemma 3.3(ii) we get

$$|u(t)| \leq Ce^{-\nu t}\int_{\Gamma_\nu}\frac{\Lambda}{|z|(1+|z|^2)}\,|dz| \leq Ce^{-\nu t}.$$

For the derivative, simply differentiate under the integral sign in (3.13),

$$u'(t) = \frac{-\lambda}{2\pi i}\int_{\Gamma_\nu} e^{zt}\frac{1}{\psi(z)+\lambda}\,dz. \tag{3.17}$$

Invoking Lemma 3.3(ii) once more we find,

$$|u'(t)| \leq Ce^{-\nu t}\int_{\Gamma_\nu}\frac{\Lambda}{1+|z|^2}\,|dz| \leq Ce^{-\nu t}.$$

**Case 2, $\lambda \geq \Lambda$.** Partial integration in (3.13) gives

$$u(t) = \frac{-\lambda}{2\pi i}\int_{\Gamma_\nu}\frac{e^{zt}}{t}\frac{\lambda+\psi(z)+z\psi'(z)}{z^2(\psi(z)+\lambda)^2}\,dz. \tag{3.18}$$

By Lemma 3.2 and Lemma 3.3(i) we find, for $t \geq 1$,

$$|u(t)| \leq Ce^{-\nu t}\int_{\Gamma_\nu}\frac{\lambda^2+\lambda|z|^2}{|z|^2|z-z_+|^2|z-z_-|^2}\,|dz| \tag{3.19}$$

Setting $\mu = \sqrt{\lambda-\alpha_1^2}$ and $\theta = \alpha_1 - \nu$ we have $|z-z_\pm|^2 = \theta^2 + (y \mp \mu)^2$. Hence, exploiting the symmetry,

$$\begin{aligned}|u(t)| \leq Ce^{-\nu t}\Big[&\int_0^{\frac{1}{2}\mu}\frac{\lambda^2}{[\nu^2+y^2][\frac{1}{4}\mu^2][\mu^2]}\,dy\\ &+\int_{\frac{1}{2}\mu}^{2\mu}\frac{\lambda^2}{[\frac{1}{4}\mu^2][\theta^2+(y-\mu)^2][\mu^2]}\,dy\\ &+\int_{2\mu}^{\infty}\frac{\lambda(\nu^2+y^2)}{[\nu^2+y^2][\theta^2+\frac{1}{4}y^2][\mu^2]}\,dy\Big] \leq Ce^{-\nu t},\end{aligned}$$

where we used that $\lambda/\mu \le \Lambda/(\Lambda-\alpha_1^2) \le C$. Partial integration in (3.17), in conjunction with Lemma 3.3(i), similarly gives

$$\begin{aligned}|u'(t)| &\le Ce^{-\nu t}\int_0^\infty \frac{\lambda|z|}{|z-z_+|^2|z-z_-|^2}\,|dz|\\ &\le Ce^{-\nu t}\Big[\int_0^{2\mu}\frac{\lambda\mu}{[\theta^2+(y-\mu)^2]\,[\mu^2]}\,dy+\int_{2\mu}^\infty\frac{\lambda y}{[\theta^2+\frac14 y^2][y\mu]}\,dy\Big]\\ &\le C\sqrt{\lambda}e^{-\nu t}.\end{aligned}$$

□

3.5. **A first order discretisation in time.** To obtain a first order time discretisation we approximate the derivative in (1.1) by the one step backward difference operator $\bar\partial_1$ defined in Subsection 2.2.1 and the integral term by the associated first order accurate convolution quadrature (2.12), $q_n(U)=\sum_{j=1}^n \beta_{n-j}U^j$, where the quadrature weights are defined by $\tilde\beta(\xi)\equiv\hat\beta(\delta_k(\xi))$ and $\delta_k(\xi)=k^{-1}(1-\xi)$. This gives the approximation

$$\begin{aligned}&\bar\partial_1 U^n + q_n(AU)=0,\quad n\ge 1,\\ &U^0=v.\end{aligned}\tag{3.20}$$

The following two theorems state that this scheme is exponentially stable and first order accurate.

**Theorem 3.2.** *The solution, $U^n$, of the scheme (3.20) satisfies*

$$||U^n||\le Ce^{-\nu t_n}||v||,$$

*where, with the notation from Lemma 3.3 and $\kappa$ the smallest eigenvalue of $A$,*

$$\nu=\begin{cases}k^{-1}\log(1+\alpha_2 k), & \text{when } \kappa\ge\Lambda,\\ k^{-1}\log(1+\alpha_3 k), & \text{when } \kappa<\Lambda.\end{cases}$$

**Theorem 3.3.** *Let $u$ and $U^n$ be defined by (1.1) and (3.20), respectively. Then there exists a $\gamma>0$ such that, for $k$ small,*

$$||U^n-u(t_n)||\le Ck\,e^{-\gamma t_n}|v|_2.$$

*With $\kappa$ the smallest eigenvalue of $A$ and using the notation from Lemma 3.3 we can pick $\gamma$ as any number smaller than $\alpha_2$ if $\kappa\ge\Lambda$ and any number smaller than $\alpha_3$ if $\kappa<\Lambda$.*

When proving these theorems we will first prove the corresponding results for the scalar case. The general case will then follow by a spectral decomposition of the operator $A$.

The following lemma gives a useful integral representation for the solution to the discrete, scalar equation.

**Lemma 3.6.** *Let $U^n$ be defined by the scalar version of equation (3.20), with $A = \lambda$ and $v = 1$. Then, with $z_k = \delta_k(e^{-zk})$,*

$$U^n = \frac{-\lambda}{2\pi i}\int_{\Gamma_\nu^k} e^{zt_n}\frac{e^{-zk}}{z_k(\psi(z_k)+\lambda)}\,dz, \quad \text{where } \Gamma_\nu^k = -\nu + i[-\pi/k,\,\pi/k],$$

*and where, with the notation from Lemma 3.3,*

$$\nu = \begin{cases} k^{-1}\log(1+\alpha_2 k), & \text{when } \lambda \geq \Lambda, \\ k^{-1}\log(1+\alpha_3 k), & \text{when } \lambda \in [\kappa,\,\Lambda). \end{cases}$$

By setting

$$U_\star^n := \frac{-\lambda}{2\pi i}\int_{\Gamma_\nu^k} e^{zt_n}\frac{1}{z_k(\psi(z_k)+\lambda)}\,dz \tag{3.21}$$

we can write the representation given in the lemma as

$$U^n = U_\star^{n-1} = U_\star^n - k\bar{\partial}U_\star^n. \tag{3.22}$$

Theorem 3.2 is then an immediate consequence of the following lemma.

**Lemma 3.7.** *With $U_\star^n$ given by (3.21), and with $\nu$ as in Theorem 3.2 we have*

$$|U_\star^n| \leq Ce^{-\nu t_n}.$$

The scalar analogue of Theorem 3.3 is a simple consequence of (3.22) and the following lemma.

**Lemma 3.8.** *Let $U_\star^n$ be defined by (3.21) and let $u$ be the solution of (3.3). Then, with $\nu$ as in Theorem 3.2,*

$$|U_\star^n - u(t_n)| \leq Ck\,\lambda e^{-\nu t_n}.$$

This lemma is proved by estimating the difference between the integrals in (3.21) and (3.13). But before giving the details on this we give the proof of Theorem 3.3.

**Proof of Theorem 3.3.** We start by combining Lemmas 3.6 and 3.8 to prove an estimate for the error in the scalar equation. Let $u$, $U^n$ and $U_\star^n$ be as above. Then, using (3.22),

$$U^n - u(t_n) = \big(U_\star^{n-1} - u(t_{n-1})\big) - \int_{t_{n-1}}^{t_n} u'(s)\,ds.$$

Lemma 3.8 immediately provides a bound for the first term. When estimating the integral recall from Theorem 3.1 that $|u'(t)| \leq C\sqrt{\lambda}e^{-\nu t} \leq C\sqrt{\lambda}e^{-\nu t}$, since $\lambda \geq \kappa$. Taken together, these estimates show, for the scalar equation,

$$|U^n - u(t_n)| \leq Ck\,\lambda e^{-\nu t_n}. \tag{3.23}$$

We prove the corresponding bound for the non-scalar equation by a spectral decomposition. Using the same notation as in the proof of Theorem 3.1

and with $U_i^n$ denoting the time discrete solution corresponding to $u_i(t_n)$ we get, by (3.23),

$$\begin{aligned}||U^n - u(t_n)||^2 &= \sum_{i=1}^{\infty} a_i^2 |U_i^n - u_i(t_n)|^2 \le \sum_{i=1}^{\infty} a_i^2 C k^2 \lambda_i^2 e^{-2\nu t_n} \\ &\le C k^2 e^{-2\nu t_n} \sum_{i=1}^{\infty} a_i^2 \lambda_i^2 = C k^2 e^{-2\nu t_n} |v|_2^2,\end{aligned}$$

and since $\nu \to \alpha_j$ as $k \to 0$ the theorem follows at once. $\square$

**Proof of Lemma 3.6.** Setting $A = \lambda$ in equation (3.20), multiplying by $\xi^n$ and summing from one to infinity we find

$$k^{-1}\big(\tilde{U}(\xi) - U^0 - \xi \tilde{U}(\xi)\big) + \lambda\big(\tilde{U}(\xi) - U^0\big)\tilde{\beta}(\xi) = 0.$$

Since $U^0 = 1$ and $\delta_k(\xi) = k^{-1}(1-\xi)$, this simplifies to

$$\big(\delta_k(\xi) + \lambda\tilde{\beta}(\xi)\big)\tilde{U}(\xi) = k^{-1} + \lambda\tilde{\beta}(\xi).$$

We claim that with $\nu$ as given in the lemma, the radius of analyticity for $\tilde{U}(\xi)$ is at least $r = e^{\nu k}$. To see this, note that when $|\xi| \le r$ we have $\operatorname{Re}\delta_k(\xi) \ge k^{-1}(1-r) = -\alpha_j$, for $j = 2, 3$, respectively. Lemma 3.3 and assumption (3.2) then give $|\psi(\delta_k(\xi)) + \lambda| > 0$ and $\operatorname{Re}\hat{\beta}(\delta_k(\xi)) > 0$. Therefore $|\delta_k(\xi) + \lambda\tilde{\beta}(\xi)| = |\hat{\beta}(\delta_k(\xi))(\psi(\delta_k(\xi)) + \lambda)| > 0$, and so

$$U^n = \frac{1}{2\pi i}\int_{|\xi|=r} \xi^{-(n+1)} \frac{k^{-1} + \lambda\hat{\beta}(\delta_k(\xi))}{\delta_k(\xi) + \lambda\hat{\beta}(\delta_k(\xi))}\, d\xi.$$

Changing variables, $\xi = e^{-zk}$, and setting $z_k := \delta_k(e^{-zk})$ we find

$$U^n = \frac{1}{2\pi i}\int_{\Gamma_\nu^k} e^{zt_n} \frac{1 + k\lambda\hat{\beta}(z_k)}{z_k + \lambda\hat{\beta}(z_k)}\, dz.$$

Recall that $r = e^{\nu k} > 1$, which gives

$$\frac{1}{2\pi i}\int_{\Gamma_\nu^k} e^{zt_n}\frac{1}{z_k}\, dz = \frac{1}{2\pi i}\int_{|\xi|=r} \xi^{-(n+1)}\frac{1}{1-\xi}\, d\xi = 0,$$

and consequently

$$\begin{aligned}U^n =& \frac{1}{2\pi i}\int_{\Gamma_\nu^k} e^{zt_n}\left[\frac{1 + k\lambda\hat{\beta}(z_k)}{z_k + \lambda\hat{\beta}(z_k)} - \frac{1}{z_k}\right] dz \\ =& \frac{1}{2\pi i}\int_{\Gamma_\nu^k} e^{zt_n}\frac{-\lambda(1 - kz_k)}{z_k(\psi(z_k) + \lambda)}\, dz.\end{aligned}$$

Since $kz_k = 1 - e^{-zk}$ the representation in the lemma then follows. $\square$

**Proof of Lemma 3.7.** The positivity of $q_n$ implies that $U^n$ is bounded (see [8]), and since $U_\star^n = U^{n+1}$ the same must hold for $U_\star^n$. Consequently, we only need to prove the exponential decay for $t_n \ge 1$. We will do this by estimating the integral in (3.21) much like we estimated the integral (3.13) in the proof of Lemma 3.5. Comparing the two integrals, we see that the main difference is that in (3.21) the quantity $z_k$ enters rather that $z$. We therefore start by deriving some properties of $z_k$, these will also be used in the proof of Lemma 3.8.

First we prove that there exist positive constants $m$, $M$, and $C$ such that

$$m|z| \leq |z_k| \leq M|z|, \tag{3.24}$$

$$|z - z_k| \leq C|z|^2 k, \tag{3.25}$$

when $z \in \Gamma_\nu^k$ and $\nu k \leq 1$. We form the quotient

$$q(zk) := \left|\frac{z_k}{z}\right| = \frac{|1 - e^{-zk}|}{|zk|},$$

whence the proof of (3.24) amounts to showing that $q(zk)$ is bounded away from zero and infinity. If $z \in \Gamma_\nu^k$ and $\nu k \leq 1$, then $|zk|$ is bounded and hence both the numerator and the denominator are bounded as well. We have $\lim_{|zk|\to 0} q(zk) = 1$ and thus there exists a $\sigma > 0$ such that $\frac{1}{2} \leq q(zk) \leq \frac{3}{2}$ when $|zk| \leq \sigma$. When $|zk| \geq \sigma$, then clearly both the numerator and the denominator are bounded away from zero which completes the proof of (3.24). In order to prove (3.25) we simply note that since $|zk|$ is bounded in the region under consideration there exists a constant $C$ such that

$$k|z - z_k| = |zk - (1 - e^{-zk})| \leq C|zk|^2.$$

Next we analyze $\operatorname{Re} z_k$ and $\operatorname{Im} z_k$ separately. Set $y = \operatorname{Im} z$ so that $z = -\nu + iy$. Then set

$$x_k := \operatorname{Re} z_k = k^{-1}(1 - e^{\nu k}\cos yk),$$
$$y_k := \operatorname{Im} z_k = k^{-1}(e^{\nu k}\sin yk).$$

Note that $x_k \geq k^{-1}(1 - e^{\nu k}) = -\nu + O(k)$, so, with $\theta = \alpha_1 - \nu$ and with $k$ small enough,

$$x_k + \alpha_1 = -\theta + O(k) \geq \tfrac{1}{2}\theta, \quad \forall y. \tag{3.26}$$

Since $\frac{d}{dy}y_k = e^{\nu k}\cos ky$ there exists an $\eta > 0$ such that,

$$\frac{3}{4} \leq \frac{dy_k}{dy} \leq \frac{3}{2}, \quad \text{for } |y| \leq \eta/k, \tag{3.27}$$

provided $\nu k$ is small. Moreover,

$$x_k + \alpha_1 \geq k^{-1}(1 - e^{\nu k}\cos ky) \geq \tfrac{1}{2}k^{-1}(1 - \cos\eta), \quad \text{for } |y| \geq \eta/k. \tag{3.28}$$

We are now prepared to start estimating the integral in (3.21).

**Case 1,** $\lambda \in [\kappa, \Lambda)$. By Lemma 3.3(ii), and $|z_k| \geq m|z|$, we find

$$|U_\star^n| \leq Ce^{-\nu t_n}\int_{\Gamma_\nu^k} \frac{\Lambda}{|z|(1 + |z|^2)}\,|dz| \leq Ce^{-\nu t_n}\int_{\Gamma_\nu^k} \frac{1}{1 + y^3}\,dy \leq Ce^{-\nu t_n}.$$

**Case 2, $\lambda \geq \Lambda$.** We integrate by parts in (3.21). Note that $z_k|_{\partial\Gamma_\nu^k} = k^{-1}(1+e^{\nu k})$. Hence we get for the end-point terms, with $\bar{x} = 1 + e^{\nu k}$,

$$\left|\frac{-\lambda}{2\pi i}\left[\frac{e^{zt_n}}{t_n}\frac{1}{z_k\big(\psi(z_k)+\lambda\big)}\right]_{\partial\Gamma_\nu^k}\right| \leq \frac{C\lambda e^{-\nu t_n}}{k^{-1}|\psi(k^{-1}\bar{x})+\lambda|}, \quad \text{for } t_n \geq 0.$$

By Lemma 3.3(i) we get, $|\psi(k^{-1}\bar{x})+\lambda| \geq c|k^{-1}\bar{x}-z_+|\,|k^{-1}\bar{x}-z_-| \geq c(k^{-2}+\lambda)$. It thus remains to bound the following integral, *cf.* (3.18),

$$I_1 = \frac{-\lambda}{2\pi i}\int_{\Gamma_\nu^k}\frac{e^{zt_n}}{t_n}\frac{\psi(z_k)+\lambda+z_k\psi'(z_k)}{z_k^2\big(\psi(z_k)+\lambda\big)^2}\frac{dz_k}{dz}\,dz.$$

By Lemma 3.3(i) and (3.24) we get $|\psi(z_k)+z_k\psi'(z_k)| \geq C|z_k|^2 \geq C|z|^2$. Since $|dz_k/dz| = |e^{zk}| = e^{-\nu k}$, and $|z_k| \geq m|z|$, we find, for $t_n \geq 1$,

$$\begin{aligned} I_1 \leq Ce^{-\nu t_n}\Bigg[&\int_0^{\eta/k}\frac{\lambda^2+\lambda|z|^2}{|z|^2|z_k-z_+|^2|z_k-z_-|^2}\,dy \\ &+\int_{\eta/k}^{\pi/k}\frac{\lambda^2+\lambda|z|^2}{|z_k|^2|z_k-z_+|^2|z_k-z_-|^2}\,dy\Bigg]. \end{aligned}$$

In the first term we change variables so that $s = y_k(y)$. By virtue of (3.26) and (3.27) the resulting integral is bounded by the integral in (3.19). For the second term we note that (3.28) gives $|z_k|^2|z_k-z_+|^2|z_k-z_-|^2 \geq k^{-2}(k^{-2}+\lambda)^2$ and consequently this term is also bounded as desired. ☐

**Proof of Lemma 3.8.** By (3.21) and (3.13) we find

$$\begin{aligned} |U_\star^n - u(t_n)| \leq C\lambda e^{-\nu t_n}\Bigg[&\int_{\Gamma_\nu^k}\left|\frac{1}{z_k(\psi(z_k)+\lambda)}-\frac{1}{z(\psi(z)+\lambda)}\right||dz| \\ &+\int_{\Gamma_\nu\setminus\Gamma_\nu^k}\frac{1}{|z(\psi(z)+\lambda)|}\,|dz|\Bigg] =: C\lambda e^{-\nu t_n}(I_1+I_2). \end{aligned}$$

We can write

$$I_1 = \int_{\Gamma_\nu^k}\left|\frac{\big(z\psi(z)-z_k\psi(z_k)\big)+\lambda(z-z_k)}{zz_k\big(\psi(z_k)+\lambda\big)\big(\psi(z)+\lambda\big)}\right||dz|. \tag{3.29}$$

Using Lemma 3.2 in conjunction with (3.24) and (3.25) we find

$$|z\psi(z)-z_k\psi(z_k)| \leq C|z|^2|z-z_k| \leq C|z|^4k. \tag{3.30}$$

Consequently,

$$I_1 \leq Ck\int_{\Gamma_\nu^k}\frac{|z|^2+\lambda}{|(\psi(z_k)+\lambda)(\psi(z)+\lambda)|}\,|dz|.$$

**Case 1, $\kappa \leq \lambda < \Lambda$.** Using Lemma 3.3(ii) and $|z_k| \geq m|z|$ we find

$$I_1 \leq Ck\int_{\Gamma_\nu^k}\frac{|z|^2+\Lambda}{(1+|z|^2)(1+|z_k|^2)}\,|dz| \leq Ck\int_0^\infty\frac{1+y^2+\Lambda}{1+y^4}\,dy \leq Ck,$$

and

$$I_2 \le C \int_{\Gamma_\nu \setminus \Gamma_\nu^k} \frac{1}{|z|(1+|z|^2)}\,|dz| \le C \int_{\pi/k}^{\infty} \frac{1}{y^3}\,dy \le Ck^2.$$

**Case 2,** $\lambda \ge \Lambda$**.** Cauchy's inequality gives

$$I_1 \le Ck\Big[\int_{\Gamma_\nu^k} \frac{|z|^2+\lambda}{|\psi(z)+\lambda|^2}\,|dz| \int_{\Gamma_\nu^k} \frac{|z|^2+\lambda}{|\psi(z_k)+\lambda|^2}\,|dz|\Big]^{1/2} =: Ck[J_1 J_2]^{1/2}.$$

We start by estimating $J_1$. Set $y = \operatorname{Im} z$ so that $z = -\nu + iy$. Then, by Lemma 3.3(i), we get, with $\mu = \sqrt{\lambda - \alpha_1^2}$ and $\theta = \alpha_1 - \nu$,

(3.31)
$$|\psi(z)+\lambda|^2 \ge c|z-z_+|^2|z-z_-|^2 = c\big[\theta^2 + (y-\mu)^2\big]\big[\theta^2+(y+\mu)^2\big].$$

Symmetry then implies that

$$J_1 \le C\int_0^{\pi/k} \frac{\theta^2+y^2+\lambda}{\big[\theta^2+(y-\mu)^2\big]\big[\theta^2+(y+\mu)^2\big]}\,dy.$$

The integral is clearly bounded, but we need to prove that the bound does not depend on $\lambda$, or, equivalently, $\mu$. We find

$$J_1 \le C\int_0^{2\mu} \frac{\theta^2+\mu^2+\alpha_1^2}{\big[\theta^2+(y-\mu)^2\big]\big[\theta^2+\mu^2\big]}\,dy + C\int_{2\mu}^{\infty} \frac{\theta^2+y^2}{\big[\theta^2+\frac{1}{4}y^2\big]^2}\,dy \le C.$$

Next we need to estimate $J_2$. By Lemma 3.3(i) we get, with $\eta$ as in (3.28),

$$J_2 \le C\int_0^{\eta/k} \frac{|z|^2+\lambda}{|z_k-z_+|^2|z_k-z_-|^2}\,dy + C\int_{\eta/k}^{\pi/k} \frac{|z|^2+\lambda}{|z_k-z_+|^2|z_k-z_-|^2}\,dy.$$

In the first term we have, by (3.26), $|z_k - z_\pm|^2 \ge \frac{1}{4}\theta^2 + (y_k \mp \mu)^2$. By a change of variables, $s = y_k(y)$, we can, by (3.27), bound the resulting integral by $J_1$. In the second term we use (3.28) and find $|z_k - z_\pm|^2 \ge ck^{-2} + \mu^2$. This immediately provides the required bound.

Finally we complete the estimate for $I_2$. By Lemma 3.3(i) we obtain

$$|z(\psi(z)+\lambda)| \ge c\,y\,|z-z_+|\,|z-z_-|.$$

If $\mu \le \frac{1}{2}\pi/k$, then $|z(\psi(z)+\lambda)| \ge cy^3$ and it follows trivially that $I_2 \le Ck^2$. If $\mu > \frac{1}{2}\pi/k$ we get instead

$$\begin{aligned} I_2 &\le C\int_{\pi/k}^{2\mu} \frac{1}{k^{-1}\big[\theta^2+(y-\mu)^2\big]^{1/2}\mu}\,dy + \int_{2\mu}^{\infty} \frac{1}{cy^3}\,dy \\ &\le C\frac{k}{\mu}\int_{-\mu}^{\mu} \frac{1}{\big[\theta^2+y^2\big]^{1/2}}\,dy + C\mu^{-2} \le Ck\frac{1}{\mu}\log\mu + Ck^2. \end{aligned}$$

Since the function $x \mapsto x^{-1}\log x$ is monotonically decreasing for $x > e$ and $\mu \ge ck^{-1}$ we find $I_2 \le Ck^2 \log k^{-1}$. $\square$

**Remark.** If $\beta(t) = e^{-\alpha t}$, then Theorem 3.3 holds for any $\gamma < \nu_0$, with $\nu_0$ defined by (3.7). To prove this, we use the same techniques as in the proofs given for Lemmas 3.6 and 3.8, but instead of invoking Lemma 3.3 we use the exact expression $\psi(z) + \lambda = (z - z_+)(z - z_-)$. The lemmas then hold for any $\nu < \nu_0$ and the claim follows.

3.6. **A second order discretisation in time.** To obtain a second order discretisation we approximate the derivative by the two step backward difference operator $\bar\partial_2$ from Subsection 2.2.2. For the integral term we use the associated second order accurate convolution quadrature given by (2.15),

$$q_n(\varphi) := \sum_{j=1}^{n} \beta_{n-j}\varphi^j + \tfrac{1}{2}\beta_{n-1}\varphi^0,$$

where the quadrature weights are defined by

$$\tilde\beta(\xi) \equiv \hat\beta(\delta_k(\xi)), \quad \text{and} \quad \delta_k(\xi) = k^{-1}(\tfrac{3}{2} - 2\xi + \tfrac{1}{2}\xi^2).$$

When choosing the initial conditions we must be careful so that the generating function $\tilde U(\xi)$ takes a form which makes it amenable to our analysis. We will demonstrate that the choice in the following scheme is suitable,

$$\begin{aligned} &\bar\partial_2 U^n + q_n(AU) = 0, \quad \text{when } n \geq 2, \\ &\bar\partial_1 U^1 + \beta_0\big(\tfrac{2}{3}AU^1 + \tfrac{1}{3}AU^0\big) = 0, \quad U^0 = v. \end{aligned} \tag{3.32}$$

We will prove two theorems stating that this scheme is exponentially stable and second order accurate, save for a $\log k^{-1}$-term.

**Theorem 3.4.** *The solution $U^n$ of the scheme (3.32) satisfies*

$$||U^n|| \leq Ce^{-\nu t_n}||v||,$$

*where, with the notation from Lemma 3.3, and $\kappa$ the smallest eigenvalue of $A$,*

$$\nu = \begin{cases} (2k)^{-1}\log(1 + 2\alpha_2 k), & \text{when } \kappa \geq \Lambda, \\ (2k)^{-1}\log(1 + 2\alpha_3 k), & \text{when } \kappa < \Lambda. \end{cases}$$

**Theorem 3.5.** *Let $u$ and $U^n$ be defined by (1.1) and (3.32), respectively. Then, for $n \geq 3$, and $k$ small,*

$$||U^n - u(t_n)|| \leq Ck^2 e^{-\gamma t_n}(\log k^{-1}|v|_2 + |v|_3),$$

*where $\gamma$ can be chosen as in Theorem 3.3.*

As in Section 3.5 the theorems will be easily proved once we have established the corresponding results for the scalar equations. These, in turn, will be proved by using the following representation result.

**Lemma 3.9.** *Let $U^n$ be defined by the scalar version of equation (3.32), with $A = \lambda$ and $v = 1$. Then, with $z_k = \delta_k(e^{-zk})$,*

$$U^n = \frac{-\lambda}{2\pi i}\int_{\Gamma_\nu^k} e^{zt_n}\frac{e^{-zk}(3 - e^{-zk})^2}{4z_k(\psi(z_k) + \lambda)}\,dz, \quad n \geq 2,$$

*where, with the notation from Lemma 3.3,*

$$\nu = \begin{cases} (2k)^{-1}\log(1+2\alpha_2 k), & \text{when } \lambda \geq \Lambda, \\ (2k)^{-1}\log(1+2\alpha_3 k), & \text{when } \lambda \in [\kappa,\, \Lambda). \end{cases}$$

By setting

$$U_\star^n := \frac{-\lambda}{2\pi i}\int_{\Gamma_\nu^k} e^{zt_n}\frac{1}{z_k(\psi(z_k)+\lambda)}\,dz. \tag{3.33}$$

we can decompose the representation given in the lemma as

$$\begin{aligned} U^n &= U_\star^n - (U_\star^n - 2U_\star^{n-1} + U_\star^{n-2}) + \tfrac{1}{4}(U_\star^{n-1} - 2U_\star^{n-2} + U_\star^{n-3}) \\ &= U_\star^n - k^2\bar\partial^2 U^n + \tfrac{1}{4}k^2\bar\partial^2 U^{n-1}. \end{aligned} \tag{3.34}$$

Theorem 3.4 is then an immediate consequence of the following lemma, which is proved using the representation above.

**Lemma 3.10.** *With $U^n$ the solution to the scalar version of equation (3.32), with $A=\lambda$ and $v=1$, and with $\nu$ as in Lemma 3.9,*

$$|U^n| \leq Ce^{-\nu t_n}.$$

Next we provide an analogue of Lemma 3.8.

**Lemma 3.11.** *Let $U_\star^n$ be defined by (3.33) and let $u$ be the solution of (3.3). Then, with $\nu$ as in Lemma 3.9,*

$$|U_\star^n - u(t_n)| \leq Ck^2\lambda e^{-\nu t_n}(\log k^{-1} + \sqrt{\lambda}).$$

Before proving the lemmas we show how they can be used to prove Theorem 3.5.

**Proof of Theorem 3.5.** We start the proof by combining Lemmas 3.9 and 3.11 to show an estimate for the error in the scalar equation. Let $u$, $U^n$, and $U_\star^n$ be as above. Setting $e_\star^n := U_\star^n - u(t_n)$ we can, by (3.34), write

$$\begin{aligned} U^n - u(t_n) =& \big(U_\star^n - u(t_n)\big) - k^2\bar\partial^2 e_\star^n + \tfrac{1}{4}k^2\bar\partial^2 e_\star^{n-1} - k^2\bar\partial^2 u^n + \tfrac{1}{4}k^2\bar\partial^2 u^{n-1} \\ =& e_\star^n - k^2\bar\partial^2 e_\star^n + \tfrac{1}{4}k^2\bar\partial^2 e_\star^{n-1} - \int_{t_{n-2}}^{t_n}(k - |s-t_{n-1}|)u''(s)\,ds \\ &+ \tfrac{1}{4}\int_{t_{n-3}}^{t_{n-1}}(k - |s-t_{n-2}|)u''(s)\,ds. \end{aligned}$$

By Lemma 3.11 we can immediately bound the first three terms as desired. When estimating the integrals we recall from Theorem 3.1 that $|u''(t)| \leq C\lambda e^{-\nu t}$. Taken together these estimates show, for the scalar equation,

$$|U^n - u(t_n)| \leq Ck^2\lambda e^{-\nu t_n}(\log k^{-1} + \sqrt{\lambda}).$$

For the non-scalar equation we use spectral representation. With $\{\lambda_i\}$ the eigenvalues of $A$ we get, just as in the proof of Theorem 3.3,

$$\begin{aligned}||U^n - u(t_n)||^2 &\leq Ck^4 e^{-2\nu t_n} \sum_{i=1}^{\infty} (v,\, \psi_i)^2 \big(\log^2 k^{-1} \lambda_i^2 + \lambda_i^3\big) \\ &= Ck^4 e^{-2\nu t_n} \big(\log^2 k^{-1} ||Av||^2 + ||A^{3/2} v||^2\big),\end{aligned}$$

and since $\nu \to \alpha_j$ as $k \to 0$ the theorem follows at once. □

**Proof of Lemma 3.9.** Setting $A = \lambda$ in equation (3.32), multiplying by $\xi^n$ and summing from 2 to $\infty$ we obtain

$$\begin{aligned}&k^{-1}\big[\tfrac{3}{2}(\tilde{U}(\xi) - \xi U^1 - U^0) - 2\xi(\tilde{U}(\xi) - U^0) + \tfrac{1}{2}\xi^2 \tilde{U}(\xi)\big] \\ &\qquad + \lambda\big[(\tilde{U}(\xi) - U^0)\tilde{\beta}(\xi) - U^1 \xi \beta_0 + \tfrac{1}{2} U^0 \xi (\tilde{\beta}(\xi) - \beta_0)\big] = 0.\end{aligned}$$

With $\delta_k(\xi) = k^{-1}(\frac{3}{2} - 2\xi + \frac{1}{2}\xi^2)$ this simplifies to

$$\begin{aligned}&\big(\delta_k(\xi) + \lambda\tilde{\beta}(\xi)\big)\tilde{U}(\xi) = \\ &\qquad \big(\tfrac{3}{2}k^{-1}\xi + \lambda\beta_0\xi\big)U^1 + \big(\tfrac{3}{2}k^{-1} - 2k^{-1}\xi + \lambda(1 - \tfrac{1}{2}\xi)\tilde{\beta}(\xi) + \tfrac{1}{2}\lambda\beta_0\xi\big)U^0.\end{aligned}$$

The initial conditions state that $U^0 = 1$ and hence

$$\big(\tfrac{3}{2}k^{-1} + \lambda\beta_0\big)U^1 = \big(\tfrac{3}{2}k^{-1} - \tfrac{1}{2}\lambda\beta_0\big).$$

Consequently

$$\big(\delta_k(\xi) + \lambda\tilde{\beta}(\xi)\big)\tilde{U}(\xi) = k^{-1}\big(\tfrac{3}{2} - \tfrac{1}{2}\xi\big) + (1 - \tfrac{1}{2}\xi)\lambda\tilde{\beta}(\xi). \tag{3.35}$$

Note that $\beta_0$ disappeared and we ended up with a rather neat expression. Had we used another relation between $AU^0$ and $AU^1$ in the equation for $U^1$ this would not have happened.

Next we prove that the radius of analyticity for $\tilde{U}$ is at least $r = e^{\nu k}$, with $\nu$ as in the lemma. For $\xi = re^{i\varphi}$ we have

$$\begin{aligned}\operatorname{Re} \delta_k(\xi) &= k^{-1}\big(\tfrac{3}{2} - 2r\cos\varphi + \tfrac{1}{2}r^2 \cos 2\varphi\big) \\ &= k^{-1}\big((1 - r\cos\varphi)^2 + \tfrac{1}{2} - \tfrac{1}{2}r^2\big) \geq \tfrac{1}{2}k^{-1}(1 - r^2).\end{aligned}$$

Thus, for $|\xi| \leq r$, we find that $\operatorname{Re}\, \delta_k(\xi) \geq \frac{1}{2}k^{-1}(1 - e^{2\nu k}) = -\alpha_j$, for $j = 2, 3$, respectively. Lemma 3.3 and assumption (3.2) then give, just as in the proof of Lemma 3.6, $|\delta_k(\xi) + \lambda\hat{\beta}(\delta_k(\xi))| > 0$. Therefore, by (3.35),

$$U^n = \frac{1}{2\pi i} \int_{|\xi| = r} \xi^{-(n+1)} \frac{k^{-1}(3 - \xi) + \lambda\hat{\beta}(\delta_k(\xi))(2 - \xi)}{2\big(\delta_k(\xi) + \lambda\hat{\beta}(\delta_k(\xi))\big)}\, d\xi.$$

Changing variables so that $\xi = e^{-zk}$ and setting $z_k := \delta_k(e^{-zk})$ we find

$$U^n = \frac{1}{2\pi i} \int_{\Gamma_\nu^k} e^{zt_n} \frac{(3 - e^{-zk}) + k\lambda\hat{\beta}(z_k)(2 - e^{-zk})}{2\big(z_k + \lambda\hat{\beta}(z_k)\big)}\, dz. \tag{3.36}$$

Since $r > 1$ and $kz_k = \frac{1}{2}(3 - e^{-zk})(1 - e^{-zk})$ it follows that

$$\frac{1}{2\pi i}\int_{\Gamma_\nu^k} e^{zt_n}\frac{3 - e^{-zk}}{2z_k}\,dz = \frac{1}{2\pi i}\int_{|\xi|=r} \xi^{-(n+1)}\frac{1}{1-\xi}\,d\xi = 0.$$

Subtracting this expression from the integral in (3.36) we get

$$U^n = \frac{1}{2\pi i}\int_{\Gamma_\nu^k} e^{zt_n}\frac{-\lambda\hat{\beta}(z_k)(3 - e^{-zk}) + \lambda\hat{\beta}(z_k)kz_k(2 - e^{-zk})}{2z_k(z_k + \lambda\hat{\beta}(z_k))}\,dz.$$

Dividing through by $\hat{\beta}(z_k)$ and noting again that $kz_k = \frac{1}{2}(3-e^{-zk})(1-e^{-zk})$ we find the representation claimed in the lemma. □

**Proof of Lemma 3.10.** The weak positivity of $q_n$ implies that $U^n$ is bounded, [8]. Using the decomposition (3.34) our claim will then follow once we show that $U_\star^n$ decays exponentially for $t_n \geq 1$. This can be proved using the techniques in the proof of Lemma 3.7, we just need to verify that the estimates (3.24), ... , (3.28) hold with the present definition of $z_k$.

First we prove that there exist positive constants $m$, $M$, and $C$ such that

$$m|z| \leq |z_k| \leq M|z|, \tag{3.37}$$

$$|z - z_k| \leq C|z|^3k^2, \tag{3.38}$$

when $z \in \Gamma_\nu^k$ and $\nu k \leq 1$. As in the proof of Lemma 3.7 we prove (3.37) by forming the quotient

$$q(zk) = \Big|\frac{z_k}{z}\Big| = \Big|\frac{(3 - e^{-zk})(1 - e^{-zk})}{2zk}\Big|.$$

Again $\lim_{|zk|\to 0} q(zk) = 1$ and when $zk$ is bounded away from the origin clearly both numerator and denominator are bounded and non-zero. For (3.38) we this time get

$$k|z - z_k| = |zk - \big(\tfrac{3}{2} - 2e^{-zk} + \tfrac{1}{2}e^{-2zk}\big)| \leq C|zk|^3.$$

We proceed to prove the analogues of (3.26), (3.27) and (3.28). Now

$$x_k := \operatorname{Re} z_k = k^{-1}\big(\tfrac{3}{2} - 2e^{\nu k}\cos yk + \tfrac{1}{2}e^{2\nu k}\cos 2yk\big),$$

$$y_k := \operatorname{Im} z_k = k^{-1}\big(2e^{\nu k}\sin yk - \tfrac{1}{2}e^{2\nu k}\sin 2yk\big),$$

where $y = \operatorname{Im} z$. First we note that

$$x_k = k^{-1}(1 - e^{\nu k}\cos y)^2 + k^{-1}\tfrac{1}{2}(1 - e^{2\nu k}) \geq k^{-1}\tfrac{1}{2}(1 - e^{2\nu k}) = -\nu + O(k),$$

so for $k$ small enough we have, with $\theta = \alpha_1 - \nu$,

$$x_k + \alpha_1 = \theta + O(k) \geq \tfrac{1}{2}\theta, \quad \forall y. \tag{3.39}$$

Since $\frac{d}{dy}y_k = 2e^{\nu k}\cos yk - e^{2\nu k}\cos 2yk$ there exists an $\eta > 0$ such that

$$\frac{3}{4} \leq \frac{dy_k}{dy} \leq \frac{3}{2}, \quad \text{for } |y| \leq \eta/k, \tag{3.40}$$

provided $\nu k$ is small enough. Furthermore,

$$x_k + \alpha_1 \geq k^{-1}\tfrac{1}{2}(1-\cos\eta)^2, \quad \text{for } |y| \geq \eta/k. \tag{3.41}$$

Using (3.37), (3.39), (3.40) and (3.41) we can estimate $|U_\star^n|$ with exactly the same techniques as in the proof of Lemma 3.10. □

**Proof of Lemma 3.11.** Combining (3.13) with (3.33) we find that $|U_\star^n - u(t_n)| \leq C\lambda e^{-\nu t_n}(I_1 + I_2)$, where $I_2$ is exactly as in the proof of Lemma 3.8 and $I_1$ is of the form (3.29) but with the second order accurate $z_k$. Recall that we have already proved that $I_2 \leq Ck^2 \log k^{-1}$. When estimating $I_1$ we note that now, by (3.38), $|z_k\psi(z_k) - z\psi(z)| \leq C|z|^5k^2$, *cf.* (3.30). Accordingly

$$I_1 \leq Ck^2 \int_{\Gamma_\nu^k} \frac{|z|^3 + \lambda|z|}{|(\psi(z_k)+\lambda)(\psi(z)+\lambda)|}\,|dz|. \tag{3.42}$$

**Case 1, $\kappa \leq \lambda < \Lambda$.** Lemma 3.3(ii) together with $|z_k| \geq m|z|$ yield

$$\begin{aligned} I_1 &\leq Ck^2 \int_{\Gamma_\nu^k} \frac{|z|^3 + \Lambda|z|}{(1+|z|^2)(1+|z_k|^2)}\,|dz| \\ &\leq Ck^2 \int_0^{\pi/k} \frac{(\nu^2+y^2)^{3/2}}{1+y^4}\,dy \leq Ck^2 \log k^{-1}. \end{aligned}$$

**Case 2, $\lambda \geq \Lambda$.** Using Cauchy's inequality we find

$$I_1 \leq Ck^2 \left[ \int_{\Gamma_\nu^k} \frac{|z|^3+\lambda|z|}{|\psi(z)+\lambda|^2}\,|dz| \int_{\Gamma_\nu^k} \frac{|z|^3+\lambda|z|}{|\psi(z_k)+\lambda|^2}\,|dz| \right]^{1/2} =: Ck^2[J_1J_2]^{1/2}.$$

With $\mu$ and $\theta$ as in the proof of Lemma 3.8 we use (3.31) and find, when $2\mu \leq \pi/k$,

$$\begin{aligned} J_1 &\leq C\int_0^{2\mu} \frac{\nu^3+\mu^3}{\left[\theta^2+(y-\mu)^2\right]\left[\theta^2+\mu^2\right]}\,dy + \int_{2\mu}^{\pi/k} \frac{\nu^3+y^3}{\left[\theta^2+\frac{1}{4}y^2\right]\left[\theta^2+y^2\right]}\,dy \\ &\leq C\mu + C(1+\log k^{-1}) \leq C(\sqrt{\lambda} + \log k^{-1}). \end{aligned}$$

If $\mu > \frac{1}{2}\pi/k$ use the same analysis but skip the second term.

Next we need to estimate $J_2$. By Lemma 3.3 we get, with $\eta$ as in (3.40),

$$J_2 \leq C\int_0^{\eta/k} \frac{|z|^3+\lambda|z|}{|z_k-z_+|^2|z_k-z_-|^2}\,dy + C\int_{\eta/k}^{\pi/k} \frac{|z|^3+\lambda|z|}{|z_k-z_+|^2|z_k-z_-|^2}\,dy.$$

In the first term we have, by (3.39), $|z_k - z_\pm|^2 \geq \frac{1}{4}\theta^2 + (y_k \mp \mu)^2$. By a change of variables, $s = y_k(y)$, we can, by (3.40), bound the resulting integral by $J_1$. In the second term we use (3.41) and find $|z_k - z_\pm|^2 \geq ck^{-2} + \mu^2$. This immediately provides the required bound. □

3.7. **The fully discrete equation.** In this section we will demonstrate how the results in Sections 3.5 and 3.6 can be used to obtain results for fully discrete schemes for the initial-boundary value problem

$$
\begin{aligned}
(3.43)\qquad & u_t(t) - \int_0^t \beta(t-s)\Delta u(s)\,ds = 0, \quad \text{in } \Omega, \quad t>0,\\
& u = 0, \quad \text{on } \partial\Omega, \quad t>0,\\
& u = v, \quad \text{in } \Omega, \quad t = 0,
\end{aligned}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. We will seek an approximate solution in a finite element space $S_h \subset H_0^1(\Omega)$ of piecewise linear functions on a triangulation of $\Omega$. With $h$ the maximum length of an edge in the triangulation we assume that

$$\inf_{\chi\in S_h} \big(||v-\chi|| + h||\nabla(v-\chi)||\big) \le Ch^2||v||_2, \quad \text{for any} \quad v \in H_0^1(\Omega)\cap H^2(\Omega),$$

where $||\cdot||$ and $||\cdot||_r$ denote the norms of $L^2(\Omega)$ and $H^r(\Omega)$, respectively.

Following the plan outlined in Section 1.2 we first discretise in space. The spatially discrete equation takes the form, *cf.* (1.6),

$$(3.44)\qquad u_{h,t}(t) - \int_0^t \beta(t-s)\Delta_h u_h(s)\,ds = 0, \quad u_h(0) = v_h,$$

where $\Delta_h\colon S_h \to S_h$ is defined by $(-\Delta_h\psi, \chi) = (\nabla\psi, \nabla\chi)$ for all $\chi \in S_h$. In order to prove that $u_h$ decays exponentially we apply Theorem 3.1 with $A = -\Delta_h$, $H = S_h$ and with $\kappa$ the smallest eigenvalue of $-\Delta$. This is allowed since this $\kappa$ is a lower bound for the smallest eigenvalue of $-\Delta_h$. As a result we find that

$$(3.45)\qquad ||u_h(t)|| \le Ce^{-\nu t}||v_h||,$$

where the constant $\nu$ is the same constant as that obtained when applying Theorem 3.1 with $A = -\Delta$. Using the exponential stability of $u$ and $u_h$, the following two theorems are proved in [9].

**Theorem 3.6.** *Assume that $\beta$ satisfies B2 and (1.2), that $||v_h - v|| \le Ch^2|v|_2$, and that $u$ and $u_h$ are the solutions to (3.43) and (3.44), respectively. Then, with $\nu$ as in Theorem 3.1 applied with $A = -\Delta$,*

$$||u_h(t) - u(t)|| \le Ch^2 e^{-\nu t}|v|_3, \quad \textit{for} \quad t \ge 0.$$

Applying the first order accurate time discretisation described in Subsection 3.6 to (3.44) we get the following fully discrete scheme,

$$(3.46)\qquad \bar\partial_1 U^n - \sum_{j=1}^{n} \beta_{n-j}\Delta_h U^j = 0, \quad \text{for } n \ge 1, \quad U^0 = v_h,$$

where the quadrature weights $\beta_n$ are defined by $\tilde\beta(\xi) \equiv \hat\beta(k^{-1}(1-\xi))$.

**Theorem 3.7.** *Assume that $\beta$ satisfies B2 and (1.2), that $||v_h - v|| \le Ch^2|v|_2$, and that $u$ and $U$ are the solutions of (3.43) and (3.46) respectively. Then, with $\gamma$ as in Theorem 3.3 applied with $A = -\Delta$, and $k$ small,*

$$||U^n - u(t_n)|| \le Ce^{-\gamma t_n}\big(h^2|v|_3 + k|v|_2\big).$$

**Proof of Theorem 3.7.** With $R_h$ the Ritz projection onto $S_h$ we find $||v_h - R_h v|| \le ||v_h - v|| + ||v - R_h v|| \le Ch^2|v|_2$. Thus, by (3.45), it is sufficient to consider $v_h = R_h v$. Applying Theorem 3.3 with $A = -\Delta_h$ to (3.46) we get, since $\Delta_h R_h v = P_h \Delta v$,

$$||U^n - u_h(t_n)|| \le Cke^{-\gamma t_n}||\Delta_h R_h v|| \le Cke^{-\gamma t_n}|v|_2.$$

Note again that $\gamma$ can be chosen as if Theorem 3.3 was invoked for $A = -\Delta$ since $\inf \sigma(-\Delta) \le \inf \sigma(-\Delta_h)$.

To complete the proof we finally observe that Theorem 3.6 provides the required bound for $|u_h(t_n) - u(t_n)|$. □

Next we apply the second order accurate scheme described in Section 3.6,

$$\bar\partial_2 U^n - \sum_{j=1}^{n} \beta_{n-j}\Delta_h U^j - \tfrac{1}{2}\beta_{n-1}\Delta_h U^0 = 0, \quad \text{for } n \ge 2, \tag{3.47}$$
$$\bar\partial_1 U^1 - \beta_0\big(\tfrac{2}{3}\Delta_h U^1 + \tfrac{1}{3}\Delta_h U^0\big) = 0, \quad U^0 = v_h,$$

where the quadrature weights $\beta_n$ are defined by $\tilde\beta(\xi) \equiv \hat\beta(k^{-1}(\frac{3}{2} - 2\xi + \frac{1}{2}\xi^2))$. When proving an error estimate for this method we will need to assume the following inverse property for $S_h$,

$$||\chi||_1 \le Ch^{-1}||\chi||, \quad \forall \chi \in S_h. \tag{3.48}$$

**Theorem 3.8.** *Assume that $\beta$ satisfies B2 and (1.2), that $||v_h - v|| \le Ch^2|v|_2$, that $u$ and $U$ are the solutions of (3.43) and (3.47) respectively, and that (3.48) holds. Then, with $\gamma$ as in Theorem 3.5 applied with $A = -\Delta$, and $k$ small,*

$$||U^n - u(t_n)|| \le C(h^2 + k^2 \log k^{-1})e^{-\gamma t_n}|v|_3.$$

**Proof of Theorem 3.8.** As in the proof of Theorem 3.7 we only need to consider the case $v_h = R_h v$. Theorem 3.5 applied with $A = -\Delta_h$ then gives

$$\begin{aligned}||U^n - u_h(t_n)|| &\le Ck^2e^{-\gamma t_n}\big(\log k^{-1}||\Delta_h R_h v|| + ||(-\Delta_h)^{3/2}R_h v||\big)\\ &\le Ck^2e^{-\gamma t_n}\big(\log k^{-1}|v|_2 + ||(-\Delta_h)^{1/2}P_h\Delta v||\big).\end{aligned}$$

Using that $||(-\Delta_h)^{1/2}\chi||^2 = (\nabla\chi, \nabla\chi) = |\chi|_1^2$ for any $\chi \in S_h$, we find

$$||(-\Delta_h)^{1/2}P_h\Delta v|| = |P_h\Delta v|_1 \le |R_h\Delta v|_1 + |(P_h - R_h)\Delta v|_1.$$

Since $|R_h\varphi|_1^2 = (\nabla R_h\varphi, \nabla\varphi) \le |R_h\varphi|_1|\varphi|_1$ for any $\varphi \in H_0^1$, the first term is bounded by $|v|_3$. For the second we use the inverse estimate (3.48),

$$|(P_h - R_h)\Delta v|_1 \le Ch^{-1}||(P_h - R_h)\Delta v|| \le Ch^{-1}\,Ch|\Delta v|_1 \le C|v|_3.$$

Combining this with Theorem 3.6 completes the proof. □

In all the theorems in this section we assume that $v \in \mathrm{dom}((-\Delta)^{3/2})$. Note that this is a very strong condition. In particular, we assume that $\Delta v = 0$ on $\partial\Omega$.

## Appendix A. Some practical aspects of convolution quadrature

A.1. **Introduction.** In this appendix we will briefly discuss some aspects of convolution quadrature rules not covered in the main text. First we will show how, in some cases, the quadrature weights $\beta_n$, defined indirectly by

$$\tilde{\beta}(\xi) \equiv \sum_{n=0}^{\infty} \beta_n \xi^n = \hat{\beta}(\delta_k(\xi)), \tag{A.1}$$

can actually be computed. Then we will present some numerical experiments that corroborate that the rules defined in subsections 2.2.1 and 2.2.2 do exhibit $O(k)$ and $O(k^2)$ convergence, respectively. But the experiments also show something that the analysis did not, namely that the constant involved is usually larger for these quadrature rules than the corresponding constant for traditional methods such as the rectangle and the trapezoidal rule.

A.2. **Computing the quadrature weights.** In general, it is not an easy task to numerically compute the quadrature weights from the formula (A.1). In principle, one would first have to numerically compute the Laplace transform from some pointwise values of $\beta(t)$ and then find the coefficients in a power series expansion. Both of these operations are difficult to do with a high degree of accuracy. The reason this is not always such a serious problem is that sometimes we know, or at least assume, that the function $\beta(t)$ must have some specific form, that we have for instance $\beta(t) = c\sin(\omega t)$ and that we only have to measure certain parameters, in the example $c$ and $\omega$. Then we can analytically find an expression for $\beta_n$ that depends on these constants. One case where this is possible is when it is assumed that $\beta(t) = c\,t^{\mu-1}$ for some constants $c$ and $\mu \in (0, 1)$. Then $\hat{\beta}(z) = c\,\Gamma(\mu) z^{-\mu}$ and by inserting the appropriate expression for $\delta_k(\xi)$ one can then determine the quadrature weights, as functions of $c$, $k$, and $\omega$, see [4].

The situation we will focus on in this section is when $\beta(t)$ is a sum or product of simple exponential and trigonometric functions, (exp, sin, cos, sinh, cosh). Any such function can be written as a sum $\beta(t) = \sum_{j=1}^{J} c_j e^{-\omega_j t}$, where the $\omega_j$ may be complex, and so, if we can find the quadrature weights in the case $\beta(t) = e^{-\omega t}$ then by linearity we can find the weights for any function in the above class.

We thus suppose that $\beta(t) = e^{-\omega t}$, whence $\hat{\beta}(z) = (z+\omega)^{-1}$, where $\omega$ may be complex. In order to find the quadrature weights for the first order rule we insert $\delta_k = k^{-1}(1-\xi)$ into (A.1) above. This gives

$$\tilde{\beta}(\xi) = \hat{\beta}(\delta_k(\xi)) = \frac{1}{\delta_k(\xi)+\omega} = \frac{k}{1-\xi+\omega k} = \sum_{n=0}^{\infty} \frac{k}{(1+\omega k)^{n+1}} \xi^n,$$

and we find that $\beta_n = k\,(1+\omega k)^{-(n+1)}$.

For the second order method we similarly insert $\delta_k(\xi) = k^{-1}(\frac{3}{2} - 2\xi + \frac{1}{2}\xi^2)$ and find, with $\xi_\pm = 2 \pm \sqrt{1 - 2\omega k}$,

$$\tilde{\beta}(\xi) = \hat{\beta}(\delta_k(\xi)) = \frac{1}{\delta_k(\xi) + \omega} = \frac{2k}{\xi^2 - 4\xi + 3 + 2\omega k} = \frac{2k}{(\xi - \xi_+)(\xi - \xi_-)}$$
$$= \frac{2k}{\xi_+ - \xi_-}\Big(\frac{1}{\xi_- - \xi} - \frac{1}{\xi_+ - \xi}\Big) = \frac{2k}{\xi_+ - \xi_-}\sum_{n=0}^{\infty}\Big(\frac{1}{\xi_-^{n+1}} - \frac{1}{\xi_+^{n+1}}\Big)\xi^n,$$

and consequently

$$\beta_n = \frac{2k}{\xi_+ - \xi_-}\Big(\frac{1}{\xi_-^{n+1}} - \frac{1}{\xi_+^{n+1}}\Big), \quad \xi_\pm = 2 \pm \sqrt{1 - 2\omega k}. \tag{A.2}$$

Note that we can extend the class of functions considered above to include all functions that are obtained by multiplying an exponential type function by a polynomial. The effect on the transform side is simple to create multiple roots in the denominator of $\hat{\beta}(z)$, which forces us to adjust the formulas but this does, conceptually speaking, not present any problem.

### A.3. **Some numerical experiments.**

A.3.1. *Regular kernels.* In the first experiment we investigated how well the quadrature rules estimate convolutions with a regular kernel, in our experiment we set $\beta(t) = e^{-\alpha t}\cos(\omega t)$. We evaluated the convolution of $\beta(t)$ with the function $\varphi(t) = \cos(\nu^2 t^2) - t$ using several different methods. We used the parameter values $\alpha = 0.6$, $\omega = 7$ and $\nu^2 = 5$ and integrated over $[0, 1]$. In Table A.1 we present the errors with the different methods, $I$ and $I_k$ denote the exact value of the integral and the estimates, respectively, and $N$ is the number of quadrature points. (We obtained the "exact" value by estimating the integral with Simpson's rule with $N = 10^4$.)

We see that all the methods tested display the expected rate of convergence, although the convolution quadrature rules take longer to actually reach the asymptotic range. What is more significant however, is that even though the new rules converge with the correct accuracy order, the quadrature error is significantly larger than that obtained by using the very simplest methods.

Recalling that the bounds we derived for the quadrature error in Section 2 depend in an explicit way on $\varphi(0)$ and $\varphi'(0)$ we next adjusted $\varphi$ slightly so that these values vanish. We set $\varphi(t) = \cos(\nu^2 t^2) - 1$ and repeated the experiment, the result is presented in Table A.2.

The convolution quadrature rules, and the first order accurate in particular, clearly handle these integrals better than the ones in the previous example but we see that the simple methods still perform better.

We finally remark that the numerical values presented come from one particular set of parameter values. In fact, the data are quite sensitive to the choice of these parameters but the specific experiment reported here is representative in the respect that it is almost always the case that the quadrature errors for the convolution quadrature rules are larger, sometimes the

| | First order Conv. Quad. | | RHS Rect. Rule | |
|---|---|---|---|---|
| $N$ | $\lvert I_k - I\rvert$ | conv rate | $\lvert I_k - I\rvert$ | conv rate |
| 20 | 0.404E-1 | N/A | 0.270E-1 | N/A |
| 40 | 0.244E-1 | 1.66 | 0.138E-1 | 1.96 |
| 80 | 0.135E-1 | 1.81 | 0.699E-2 | 1.98 |
| 160 | 0.707E-2 | 1.90 | 0.351E-2 | 1.99 |
| 320 | 0.362E-2 | 1.95 | 0.176E-2 | 1.99 |
| 640 | 0.183E-2 | 1.98 | 0.882E-3 | 2.00 |
| | Second order Conv. Quad. | | Trapezoidal Rule | |
| $N$ | $\lvert I_k - I\rvert$ | conv rate | $\lvert I_k - I\rvert$ | conv rate |
| 20 | 0.107E-1 | N/A | 0.193E-2 | N/A |
| 40 | 0.357E-2 | 3.00 | 0.478E-3 | 4.03 |
| 80 | 0.101E-2 | 3.52 | 0.119E-3 | 4.01 |
| 160 | 0.268E-3 | 3.77 | 0.298E-4 | 4.00 |
| 320 | 0.688E-4 | 3.89 | 0.754E-5 | 4.00 |
| 640 | 0.174E-4 | 3.94 | 0.186E-5 | 4.00 |

TABLE A.1. Regular kernel, $\varphi(0) \neq 0$, $I = 0.666\text{E-}1$.

| | First order Conv. Quad. | | RHS Rect. Rule | |
|---|---|---|---|---|
| $N$ | $\lvert I_k - I\rvert$ | conv rate | $\lvert I_k - I\rvert$ | conv rate |
| 20 | 0.667E-2 | N/A | 0.160E-1 | N/A |
| 40 | 0.603E-2 | 1.10 | 0.848E-2 | 1.89 |
| 80 | 0.394E-2 | 1.53 | 0.436E-2 | 1.95 |
| 160 | 0.224E-2 | 1.76 | 0.221E-2 | 1.97 |
| 320 | 0.119E-2 | 1.88 | 0.111E-2 | 1.99 |
| 640 | 0.615E-3 | 1.94 | 0.558E-3 | 2.00 |
| | Second order Conv. Quad. | | Trapezoidal Rule | |
| $N$ | $\lvert I_k - I\rvert$ | conv rate | $\lvert I_k - I\rvert$ | conv rate |
| 20 | 0.274E-1 | N/A | 0.122E-2 | N/A |
| 40 | 0.757E-2 | 3.62 | 0.303E-3 | 4.05 |
| 80 | 0.195E-2 | 3.87 | 0.756E-4 | 4.01 |
| 160 | 0.495E-4 | 3.95 | 0.188E-4 | 4.00 |
| 320 | 0.124E-4 | 3.98 | 0.472E-5 | 4.00 |
| 640 | 0.312E-5 | 3.99 | 0.118E-5 | 4.00 |

TABLE A.2. Regular kernel, $\varphi(0) = \varphi'(0) = 0$, $I = 0.310\text{E-}1$.

trapezoidal rule outperforms the second order accurate convolution quadrature rule by a factor of 10 to 20.

A.3.2. *Oscillatory kernels.* One claim we made in the introduction was that convolution quadratures are particularly good for oscillatory kernels. We

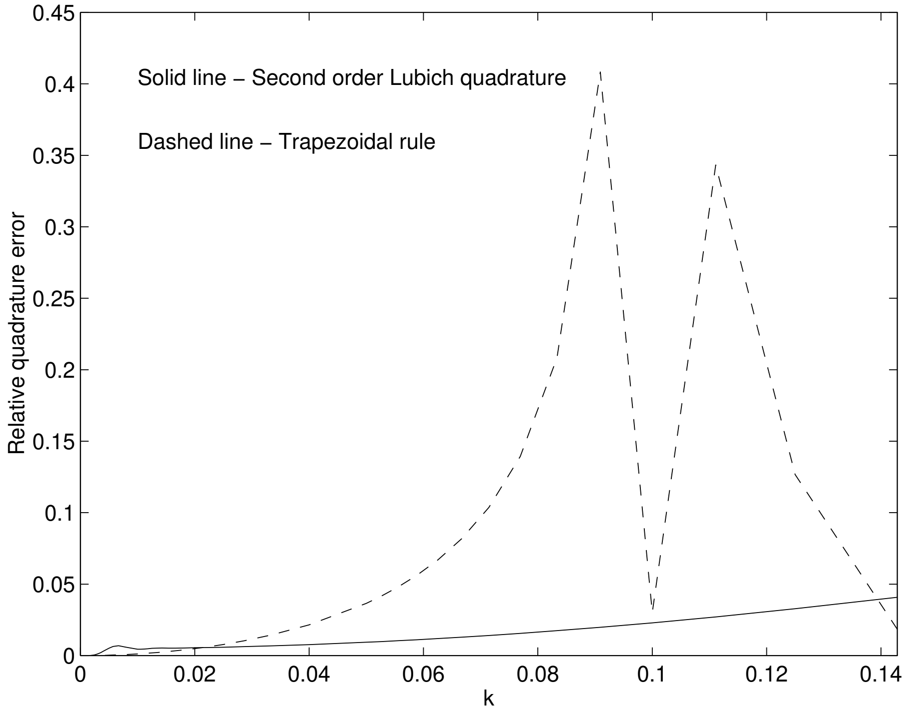



FIGURE A.1. Plot of relative quadrature error, $|(I_k^n - I)/I|$, for the second order accurate convolution quadrature rule and for the Trapezoidal rule.

investigate numerically an example where the kernel consists of a sum of a low and a high frequency component. In particular,

$$\beta(t) = c_1 \cos(\omega_1 t) + c_2 \sin(\omega_2 t), \quad \varphi(t) = (t - t_0)^2,$$

for some constants $\omega_1$, $\omega_2$, $c_1$, $c_2$ and $t_0$. Evaluating the convolution integral over $[0, 1]$ symbolically we get

$$\begin{aligned} I =& c_1 \omega_1^{-3} \big(\omega_1^2 t_0^2 \sin\omega_1 + 2\omega_1(1 + t_0(\cos\omega_1 - 1)) - 2\sin\omega_1\big) \\ &+ c_2 \omega_2^{-3} \big(2(\cos\omega_2 - 1) + 2\omega_2 t_0 \sin\omega_2 + \omega_2^2((1 - t_0)^2 - t_0^2 \cos\omega_2)\big). \end{aligned}$$

In the example we use the values $c_1 = 0.6$, $c_2 = 0.4$, $\omega_1 = \pi/2$, $\omega_2 = 20\pi$ and $t_0 = 1/4$.

In Figure A.1 we plot the relative error for the second order accurate convolution quadrature rule and the trapezoidal rule. We note that for large $k$ the trapezoidal rule gives highly inaccurate results, as was expected since the rule cannot possibly find the correct answer when $k$ is larger than the wavelength for the high frequency oscillation. The convolution quadrature rule on the other hand gives reasonable estimates even when $k$ is large but does not seem to converge well when $k$ tends to zero. In order to clarify what happens we separate the errors in the constant phase and the errors

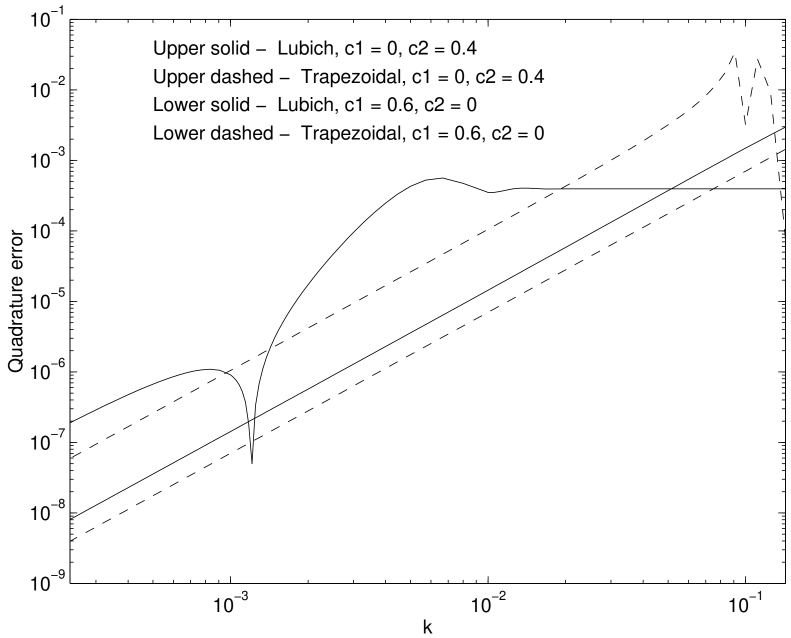



FIGURE A.2. Relative quadrature errors presented for each of the two modes separately.

in the oscillatory phase. In Figure A.2 we give a plot of the relative error when $\beta(t) = c_1 \cos(\omega_1 t)$ and $\beta(t) = c_2 \sin(\omega_2 t)$ respectively. We see that both methods do a good job with the low frequency term, but that the convolution quadrature rule has a larger constant than the trapezoidal rule. For the high frequency component we find that the latter rule does a good job once $k$ is sufficiently smaller than the period of the kernel. The new rule shows a more complicated behaviour, it does a reasonable job even for very large $k$, but for small $k$ it performs significantly worse than the trapezoidal rule. Not until $k$ is extremely small does it reach the asymptotic range, and even then the constant is larger than for the trapezoidal rule. The explanation is given by Figure A.3 where we plot the quadrature weights $\beta_n$ against $kn$ for different values of $k$. When $\omega_2 k$ is large, the convolution quadrature weights approximate a smoothed kernel and simply ignore the high frequency component. Then for intermediate values of $\omega_2 k$ the high frequency component starts to show but it is not an accurate approximation to the kernel this is the region where the new method performs particularly bad. Then when $k$ is extremely small, the method finally manages to resolve the kernel fully and we reach the asymptotic range.

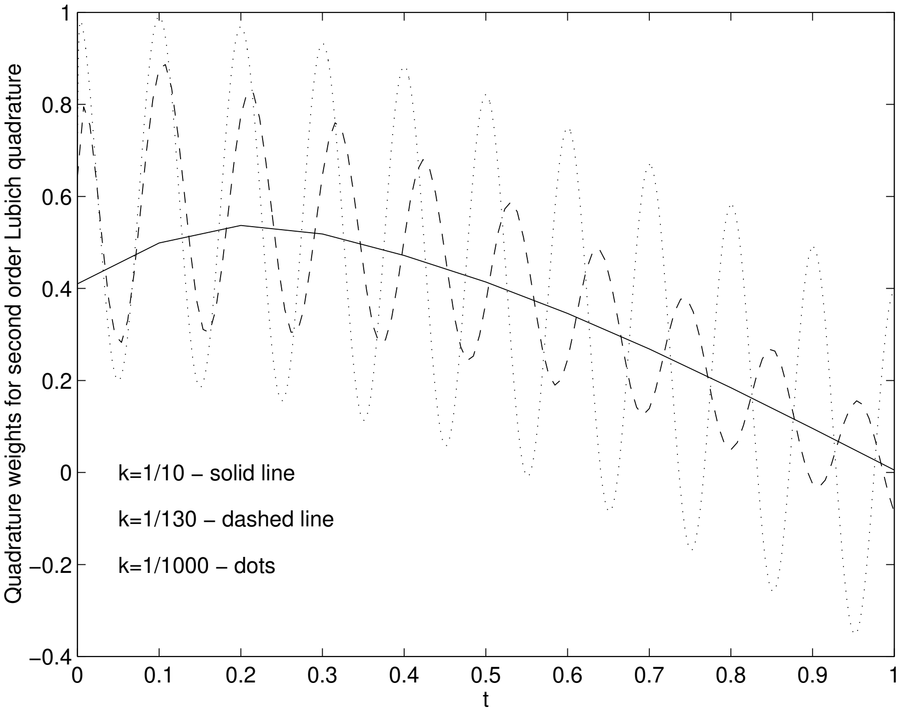



FIGURE A.3. Convolution quadrature weights.

## Appendix B. The model problem revisited

B.1. **Introduction.** The main purpose of this section is to show that for the model problem, most of the statements we proved in Section 3 for discretisations obtained using convolution quadrature also hold for discretisations obtained using the right hand side rectangle rule and the trapezoidal rule. Furthermore, numerical examples indicate that the numerical solutions for all the schemes tested actually decay at almost exactly the same rate as the continuous problem, a fact that we were not able to prove theoretically. Assuming that the results obtained for the model problem are representative for all equations with kernels as in Section 3 this indicates that our analysis can be both improved and extended.

In Subsection B.3 we also address the question of what happens when the initial data is non-smooth. Since we do not have any theoretical results for this case, we will just make some comments based on numerical experiments.

We consider the following schemes.

1. The first order scheme considered in Subsection 3.5,

$$\bar{\partial}_1 U^n + k \sum_{j=1}^{n} \beta_{n-j} A\, U^j = 0, \quad n \geq 1, \quad U^0 = v.$$

Recall from Appendix A that when $\beta(t) = e^{-\alpha t}$, the convolution quadrature weights take the form $\beta_n = k\,(1+\alpha k)^{-(n+1)}$.

2. The first order scheme obtained by combining a one step backward difference operator with the right hand side rectangle rule,

$$\bar{\partial}_1 U^n + k \sum_{j=1}^{n} \beta(t_{n-j}) A\, U^j = 0, \quad n \geq 1, \quad U^0 = v.$$

3. The following Crank-Nicolson-type scheme,

$$\bar{\partial}_1 U^n + q_{n-1/2}(A\, U) = 0, \quad n \geq 1, \quad U^0 = v,$$

where $q_{n-1/2}(U)$ is an approximation of $\int_0^{t_{n-1/2}} \beta(t_{n-1/2} - s) u(s)\, ds$, defined by

$$q_{n-1/2}(\varphi) = k \sum_{j=1}^{n-1} \beta(t_{n-j}) \frac{\varphi^j + \varphi^{j-1}}{2} + \tfrac{1}{2} k \beta(0) \frac{\varphi^n + \varphi^{n-1}}{2}.$$

4. The second order scheme considered in Subsection 3.6,

$$\bar{\partial}_2 U^n + \sum_{j=1}^{n} \beta_{n-j} A\, U^j + \tfrac{1}{2}\beta_{n-1} A\, U^0 = 0, \quad \text{when } n \geq 2,$$
$$\bar{\partial}_1 U^1 + \beta_0 \big(\tfrac{2}{3} A\, U^1 + \tfrac{1}{3} A\, U^0\big) = 0, \quad U^0 = v,$$

where the quadrature weights are given by (A.2).

5. The second order scheme obtained by using a two step backward difference operator and the trapezoidal rule,

$$\begin{aligned}
&\bar{\partial}_2 U^n + \sum_{j=1}^{n} \tfrac{1}{2}\big(\beta(t_{n-j})A\,U^j + \beta(t_{n-j+1})A\,U^{j-1}\big) = 0, \quad n \geq 2,\\
&\bar{\partial}_1 U^1 + \tfrac{1}{2}\big(\beta(t_1)A\,U^0 + \beta(0)A\,U^1\big) = 0, \quad U^0 = v.
\end{aligned}$$

B.2. **The scalar problem.** We start by showing that for the scalar problem we can quite easily find exact expressions for the solutions to methods 1, 2 and 3. Note that if we set $A = \lambda$, $v = 1$ and $\eta = (1+\alpha k)^{-1}$ we can write the equation for Method 1 as

$$\eta^{-1}(U^n - U^{n-1}) + \lambda k^2 \big(U^n + \sum_{j=1}^{n} \eta^{n-j} U^j\big) = 0. \tag{B.1}$$

Using also the equation for the previous timestep

$$U^{n-1} - U^{n-2} + \lambda k^2 \sum_{j=1}^{n-1} \eta^{n-j} U^j = 0,$$

we can rid ourselves of the sum and rewrite the equation as a second order difference equation

$$\begin{aligned}
&(1 + \alpha k + \lambda k^2)U^n - (2+\alpha k)U^{n-1} + U^{n-2} = 0, \quad n \geq 2,\\
&U^0 = 1, \quad U^1 = \frac{1+\alpha k}{1 + \alpha k + \lambda k^2},
\end{aligned}$$

where we used (B.1) with $n = 1$ to find the expression for $U^1$. Curiously enough, this equation can also be written as, *cf.* (3.5),

$$\bar{\partial}_1^2 U^2 + \alpha \bar{\partial}_1 U^n + \lambda U^n = 0.$$

Using standard techniques for difference equations we find, *cf.* (3.9),

$$U^n = \begin{cases} e^{-\frac{1}{2}\alpha_k t_n}\big(\cosh(\omega_k^r t_n) + \frac{\alpha}{2\omega}\sinh(\omega_k^r t_n)\big), & \text{when } \lambda < \frac{1}{4}\alpha^2,\\ e^{-\frac{1}{2}\alpha_k t_n}\big(\cos(\omega_k^i t_n) + \frac{\alpha}{2\omega}\sin(\omega_k^i t_n)\big), & \text{when } \lambda \geq \frac{1}{4}\alpha^2, \end{cases}$$

where

$$\begin{aligned}
\alpha_k &= \frac{1}{k}\log(1 + \alpha k + \lambda k^2) = \alpha + (\lambda - \tfrac{1}{2}\alpha^2)k + O(k^2),\\
\omega_k^r &= \frac{1}{k}\log\frac{1 + \frac{1}{2}\alpha k + \omega k}{1 + \frac{1}{2}\alpha k - \omega k} = \omega - \frac{\alpha\omega}{2}k + O(k^2),\\
\omega_k^i &= \frac{1}{k}\arctan\frac{k\omega}{1 + \frac{1}{2}\alpha k} = \omega - \frac{\alpha\omega}{2}k + O(k^2).
\end{aligned}$$

Note that the threshold value is exactly the same as for the continuous case and also that the coefficient in front of the sine-term is exact.

The same techniques can be applied for Method 2 and give the solution

$$U^n = \begin{cases} e^{-\frac{1}{2}\alpha_k t_n}\big(\cosh(\omega_k^r t_n) + B^r \sinh(\omega_k^r t_n)\big), & \text{when } \frac{1}{4}(1-e^{-\alpha k})^2 > e^{-\alpha k}\lambda k^2, \\ e^{-\frac{1}{2}\alpha_k t_n}\big(\cos(\omega_k^i t_n) + B^i \sin(\omega_k^i t_n)\big), & \text{when } \frac{1}{4}(1-e^{-\alpha k})^2 \le e^{-\alpha k}\lambda k^2, \end{cases}$$

where

$$\begin{aligned} \alpha_k &= \alpha + \frac{1}{k}\log(1+\lambda k^2) = \alpha + \lambda k + O(k^2), \\ \omega_k^r &= \frac{1}{2k}\log\frac{1+e^{-\alpha k} + \sqrt{(1-e^{-\alpha k})^2 - e^{-\alpha k}\lambda k^2}}{1+e^{-\alpha k} - \sqrt{(1-e^{-\alpha k})^2 - e^{-\alpha k}\lambda k^2}} \\ &= \frac{1}{k}\operatorname{arctanh}\frac{\sqrt{4e^{-\alpha k}(1+\lambda k^2)}}{1+e^{-\alpha k}} = \omega + \frac{1}{24}\frac{\lambda(8\lambda-\alpha^2)}{\omega}k^2 + O(k^3), \\ \omega_k^i &= \frac{1}{k}\arctan\frac{\sqrt{4e^{-\alpha k}(1+\lambda k^2)}}{1+e^{-\alpha k}} = \omega - \frac{1}{24}\frac{\lambda(8\lambda-\alpha^2)}{\omega}k^2 + O(k^3). \end{aligned}$$

Note that the phase is second order accurate.

It is possible to perform the same analysis and find a closed expression for Method 3 as well. It actually turns out that $\alpha_k \equiv \alpha$ ! We will later see that this is not such good news as it may at first seem. The expressions for the other coefficients are rather complicated and we do not give them explicitly.

We have seen that all the three one step methods decay exponentially with almost exactly the same rate as the continuous equation. Numerical experiments show that the same holds for Methods 4 and 5. One representative example is presented in Figure B.1. In this example we used $\lambda = 5$, $\alpha = 1$ and $k = 0.1$. The top diagram shows the solutions $U^n$ for the five methods. Since they all decay exponentially it is hard to gather any information from this graph. In the middle diagram we multiplied $U^n$ by $e^{\alpha k n/2}$, the inverse of the rate of decay for the continuous problem. It now becomes clear that the first order methods decay much faster than the continuous problem. Method 3 and 5 have almost exactly the correct decay rate whereas Method 4 decays somewhat too slowly. In order to show how well they capture the phase the bottom diagram shows $(U^n - u(t_n))e^{\alpha k n/2}$. We see that Methods 3 and 5 perform remarkably well, note that at $T = 20$ what is shown is the error times $e^{10} \approx 22\,000$.

One important question we need to pose when investigating the scalar model problem is what happens to the solutions when $\lambda$ increases ? We can then not hope to get an accurate answer but if we want to get a stable method when applying the methods in an infinite dimensional space with an unbounded operator $A$ the solution to the scalar problem must tend to zero as $\lambda$ tends to infinity. For Methods 1 and 2 the explicit expressions for $\alpha_k$ given above show that they do posses this property. Method 3 however, does not, since $\alpha_k \equiv \alpha$. For Methods 4 and 5 numerical experiments unequivocally show that they do satisfy this strong stability property. In Figure B.2 we show how the middle graph in Figure B.1 changes when $\lambda = 30$, the solid line that does not decay represents Method 3.

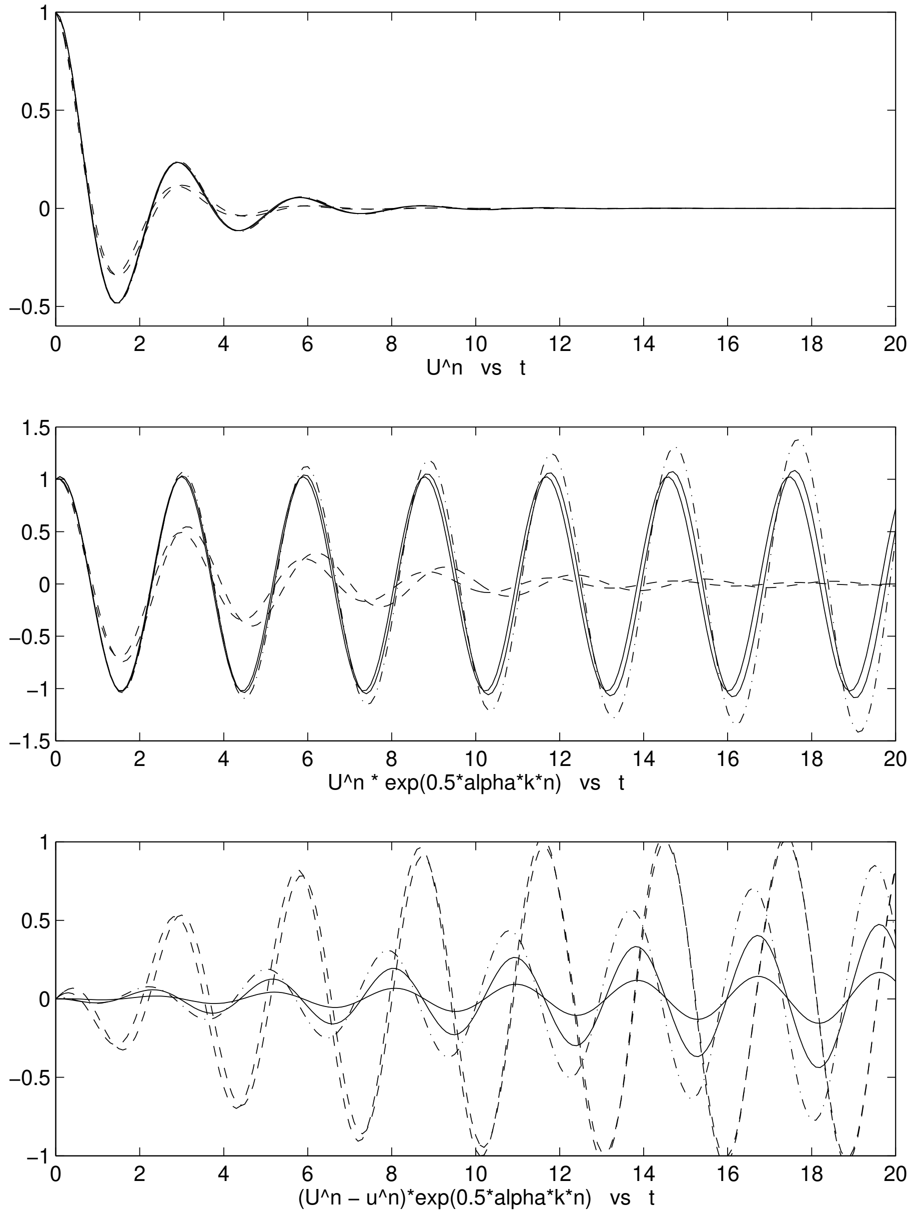



FIGURE B.1. Plots of $U^n$, $U^n\,e^{\alpha t_n/2}$ and $(U^n - u^n)\,e^{\alpha t_n/2}$. The dashed lines represent Methods 1 and 2, the solid lines represent Methods 3 and 5, and dash/dot line represents Method 4.

B.3. **The fully discrete case.** In this subsection we shall with two simple examples illustrate what can happen when the methods presented above are applied to discretise a partial differential equation. In this case the equation

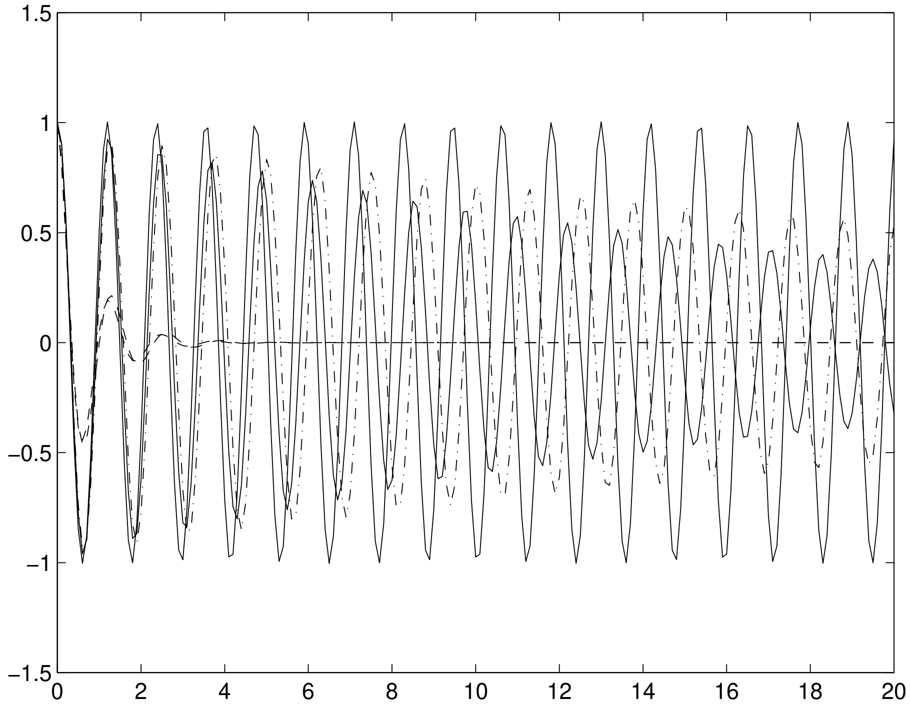



FIGURE B.2. Plot of $U^n\, e^{\alpha t_n/2}$ for a large $\lambda$. The dashed lines represent Methods 1 and 2, the solid represent Methods 3 and 5, and dash/dot represents Method 4.

under consideration is

$$\begin{aligned} &u_t - \int_0^t e^{-\alpha(t-s)} u_{xx}\, ds = 0, \quad \text{when } t > 0, \quad 0 < x < \pi, \\ &u(x=0) = u(x=\pi) = 0, \quad \text{when } t > 0, \\ &u(t=0) = v. \end{aligned}$$

The space discretisation is done with a standard Galerkin finite element method with piecewise linear testfunctions, giving $O(h^2)$ accuracy in space.

We first applied the methods to a problem with smooth initial data,

$$v(x) = \sin(x) + \tfrac{1}{2}\sin(8x) + \tfrac{1}{5}\sin(16x).$$

and with $\alpha = 0.3$, $h = \pi/50$ and $k = 0.05$. The result of the computation at time $t = 2$ is shown in Figure B.3. We see that Method 1 performs significantly worse than the other two methods, as was expected since it is only first order accurate in time. We also see that Method 3 gives almost exactly the correct values.

Next we ran an experiment with exactly the same parameter values but with non-smooth initial data, $v = \chi_{[0,\pi/2]}$, the result is given in the top diagram in Figure B.4. It is clear that none of the methods performed well

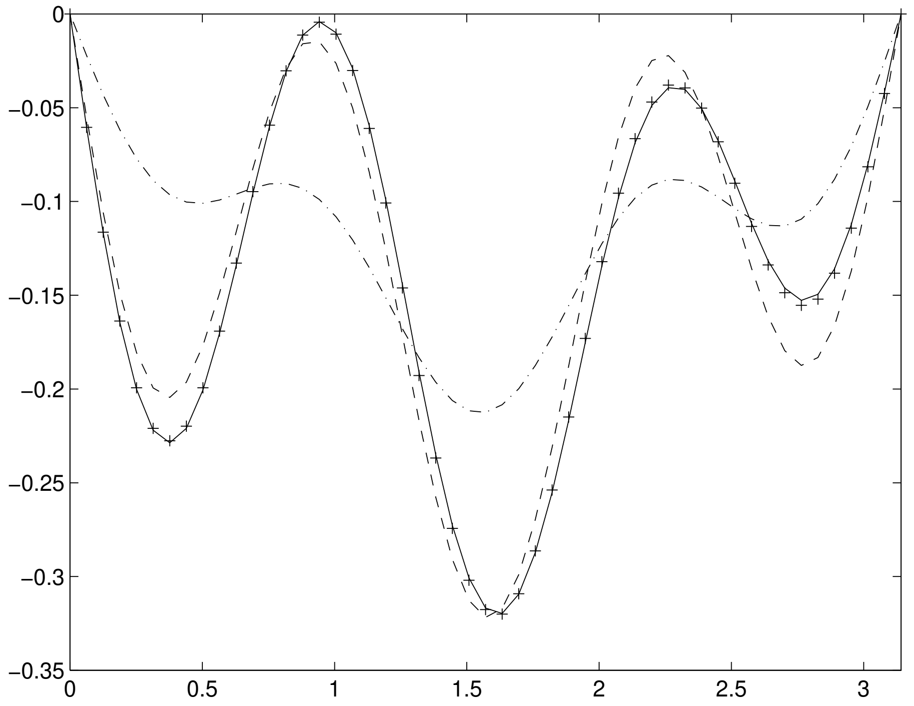
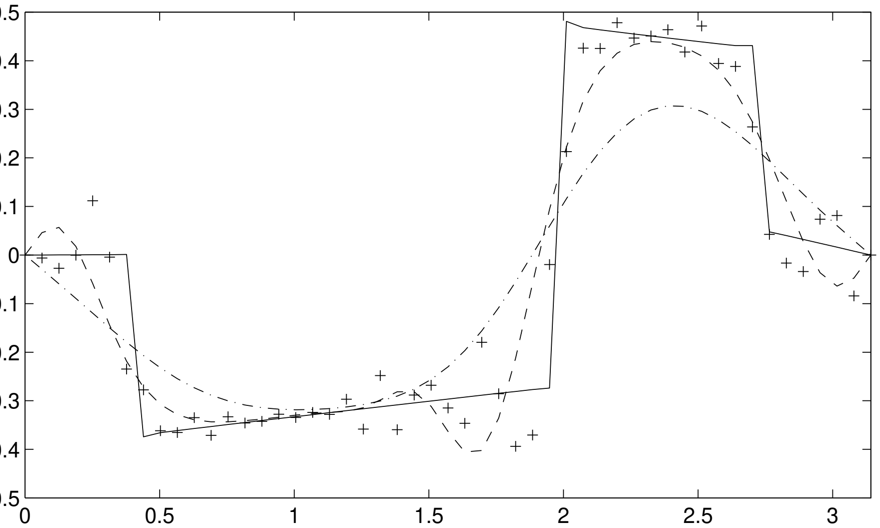



FIGURE B.3. Plots of $u$ (solid line) and $U^n$ for Methods 1, 3 and 4, displayed with dash/dot line, plus signs and dashed line, respectively.

on this test. But what is most striking is that Method 3, that performed so very well on the smooth initial data problem, has started oscillating. The problem gets even worse if we refine in space, in the lower diagram we show the result when using $h = \pi/150$. These problems arise because of the problem discussed in the previous section, namely that $\alpha_k \equiv \alpha$.

We do not present the results for Methods 2 and 5 simply because it turned out that these methods behave similarly as Methods 1 and 4 respectively.



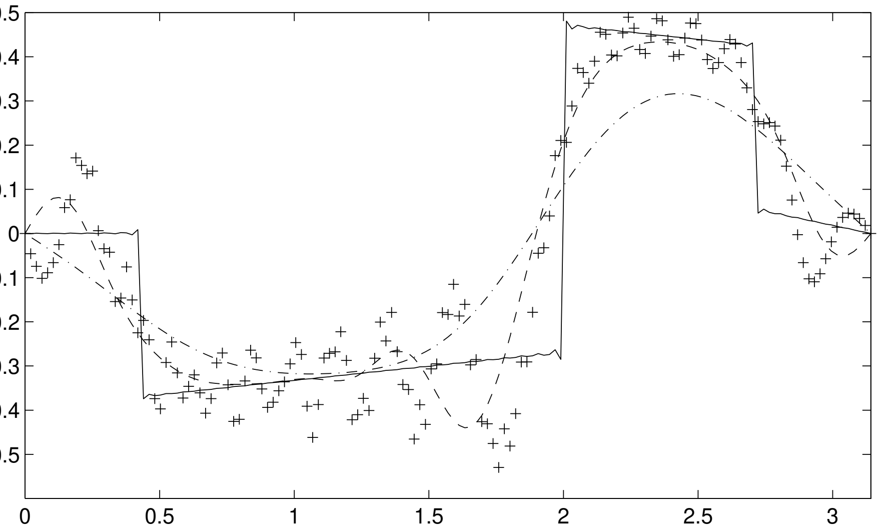




FIGURE B.4. Plots of $u$ (solid line) and $U^n$ for Methods 1, 3 and 4, displayed with dash/dot line, plus signs and dashed line, respectively.

## References

[1] C. M. Dafermos and J. A. Nohel. Energy methods for nonlinear hyperbolic Volterra integrodifferential equations. *Comm. Partial Differential Equations*, 4(3):219–278, 1979.

[2] Stig-Olof Londen. On an integrodifferential Volterra equation with a maximal monotone mapping. *J. Differential Equations*, 27(3):405–420, 1978.

[3] C. Lubich. Convolution quadrature and discretized operational calculus. I. *Numer. Math.*, 52(2):129–145, 1988.

[4] Ch. Lubich, I. H. Sloan, and V. Thomée. Nonsmooth data error estimates for approximations of an evolution equation with a positive-type memory term. *Math. Comp.*, 65(213):1–17, 1996.

[5] R. C. MacCamy. An integro-differential equation with application in heat flow. *Quart. Appl. Math.*, 35(1):1–19, 1977/78.

[6] R. C. MacCamy. A model for one-dimensional, nonlinear viscoelasticity. *Quart. Appl. Math.*, 35(1):21–33, 1977/78.

[7] R. C. MacCamy and J. S. W. Wong. Stability theorems for some functional equations. *Trans. Amer. Math. Soc.*, 164:1–37, 1972.

[8] W. McLean and V. Thomée. Numerical solution of an evolution equation with a positive-type memory term. *J. Austral. Math. Soc. Ser. B*, 35(1):23–70, 1993.

[9] W. McLean and V. Thomée. Asymptotic behaviour of numerical solutions of an evolution equation with memory. *Asymptot. Anal.*, 14(3):257–276, 1997.

[10] W. McLean, V. Thomée, and L. B. Wahlbin. Discretization with variable time steps of an evolution equation with a positive-type memory term. *J. Comput. Appl. Math.*, 69(1):49–69, 1996.

[11] Olof J. Staffans. On a nonlinear hyperbolic Volterra equation. *SIAM J. Math. Anal.*, 11(5):793–812, 1980.

[12] Vidar Thomée. *Galerkin finite element methods for parabolic problems*, volume 25 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1997.

Per-Gunnar Martinsson, Department of Mathematics, Chalmers University of Technology, S-412 96 Göteborg, Sweden

*E-mail address*: pgm@math.chalmers.se