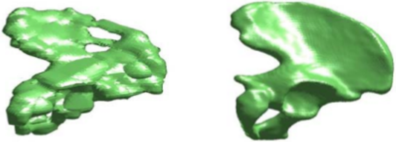
CSE 397, GEO 391, ME 397, ORI 397

Computational and Variational Methods for Inverse Problems

Instructor: Omar Ghattas
Co-Instructor: Georg Stadler

Spring 2014, Mon/Wed 10:00–12:00, RLM 7.112



January 13, 2014

Organizational issues

- Meeting time: each session is 2 hours to compensate for canceled classes due to travel (we may not always need 2 hrs)
- Office hours: after class or by appointment
- Prerequisites:
  - Graduate standing or consent of instructor
  - Background in numerical linear algebra, partial differential equations, and nonlinear optimization is desirable
  - The required mathematical background will be covered when needed (albeit quickly; but a mathematically mature student will be able to acquire the necessary mathematical and computational background from the lectures)
- Required work: about 6 assignments consisting of theoretical problems (paper & pencil) and numerical/coding exercises based on Matlab with Comsol or FEniCS (high-level finite element toolkits)

Course outline

- introduction and examples of inverse problems with PDEs
- ill-posed problems and regularization
  - theoretical issues
  - different regularization methods
  - choice of regularization parameter
- variational methods, weak forms
- computing derivatives via adjoints
  - steady and unsteady problems
  - discrete vs. continuous
  - linear and nonlinear PDEs
  - distributed, boundary, and finite-dimensional parameters and measurements
- numerical optimization methods
  - line search globalization
  - steepest descent
  - Newton method
  - Gauss-Newton method
  - inexact Newton-conjugate gradient method
- inequality constraints on parameters
- Bayesian approach to inverse problems (time permitting)

Anatomy of an inverse problem

General form of an inverse problem:

$$F(m) \qquad = d$$

- F: *parameter-to-observable map*; given *m*, map is defined by solution of *forward problem* to obtain *observables*
- m: *model parameters* or *parameter field* (also called *model* or *image*)
- d: *data* (also called *observations* or *measurements*)

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Variational methods: Our focus is on problems where F is defined by solution of PDEs; will use weak/variational forms of PDEs

Computational methods: We will focus on numerical methods for solution of large-scale inverse problems

Computational & Variational Methods for Inverse Problems

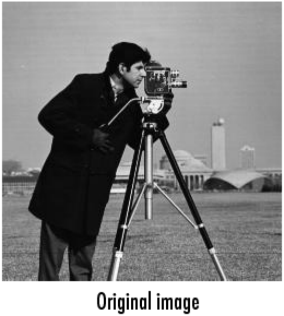
Why inverse problems?

- maturing state of *forward problem* (models, discretizations, solvers, hardware)
- availability of powerful algorithms for *large-scale optimization*
- growing interest in *decision-making*, which cannot be undertaken until models are calibrated to data

Where do inverse problems arise?

- geosciences
- engineering
- biosciences
- medical imaging
- image processing
- . . . wherever the goal is to learn about a system that cannot be directly observed

Example I: Image deblurring & denoising



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Original image

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Blurred image

- F: blurring operator
- parameters: left image (original image we seek to recover)
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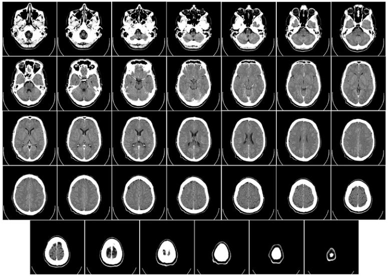
(we will study the blurring problem since it illustrates several of the difficulties of inverse problems)

Example II: Computed tomography



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- parameters: brain tissue radiodensity field
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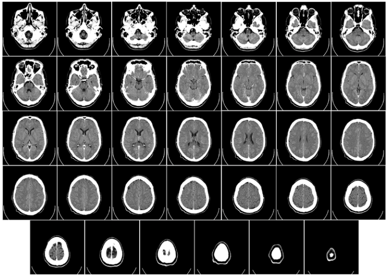

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- image shows 3D reconstruction of brain tissue radiodensity
- many medical imaging procedures involve inverse problems (MRI, PET, SPECT, ...).

Example III: Infer permeability in groundwater flow model

$$-\nabla \cdot (a \nabla u) = f \quad \text{in } \Omega \subset \mathbb{R}^d \quad + \text{boundary conditions}$$

f: source term

a: permeability

u: pressure

Example III: Infer permeability in groundwater flow model

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u: pressure

- Given a, F solves the PDE for u and extracts the pressures at the observation wells
- data: pressure u at points in Ω
- parameter field: $a = a(x)$
- inverse problem: given observations of u, find a

Example IV: Inverse scattering

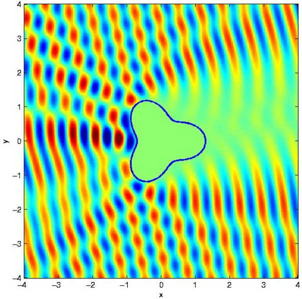
Use (acoustic/elastic/electromagnetic) waves scattered by object to infer its shape

For instance, the acoustic wave equation is given by:

$$u_{tt} - \frac{1}{s(x)^2}\Delta u = 0 \quad \text{in } \Omega \subset \mathbb{R}^d \quad \text{with bdry. \& initial cond.}$$

$s(x)$: spatially varying wave speed

$u(x,t)$: wave field



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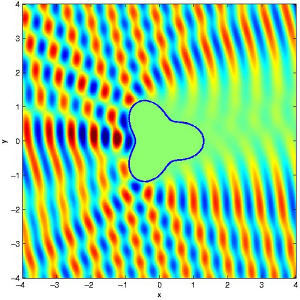

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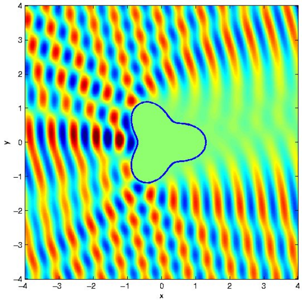

Applications: detection of airborne or submerged vehicles, ocean bathymetry, medical ultrasound, TSA body scans...

Example V: Seismic (or electromagnetic) inversion

Infer medium wavespeed from propagating acoustic/elastic/electromagnetic waves

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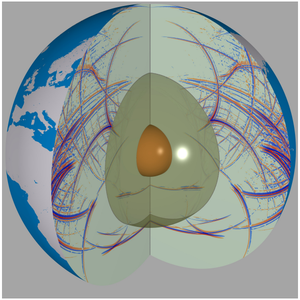

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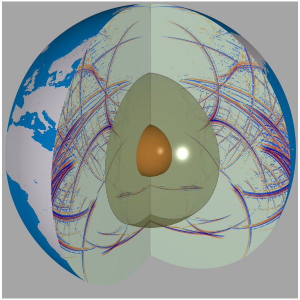

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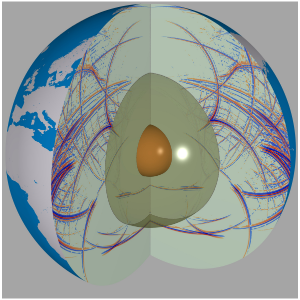
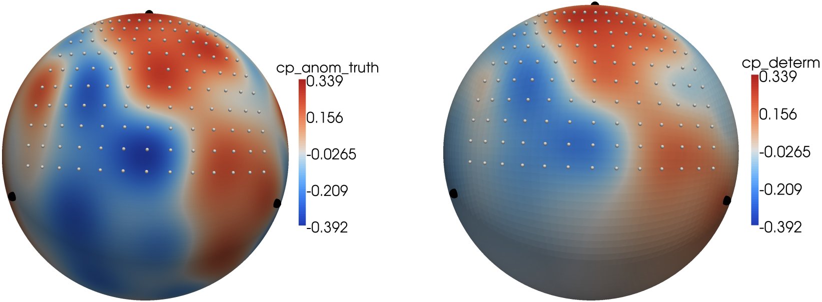


Applications: study of earth's interior, geophysical exploration,...

Example V: Wave-based material inversion

Propagate acoustic/elastic/electromagnetic waves through unknown medium

Example: "Truth" wave speed anomaly on the left and solution of inverse problem on right. Black dots correspond to earthquake sources, white dots are receiver measurement points.



Wave speed anomaly at a depth of 70km

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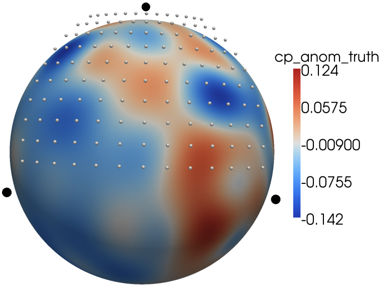

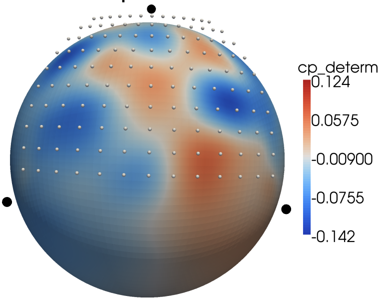

Wave speed anomaly at a depth of 700km

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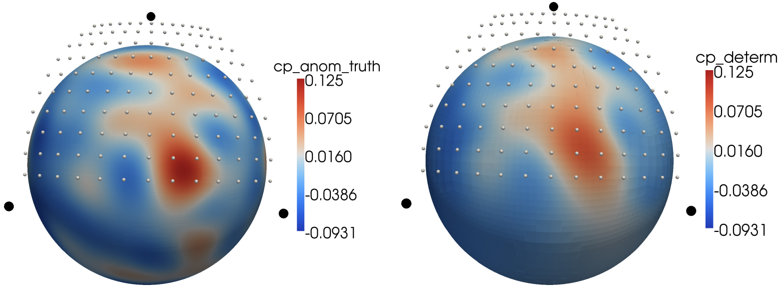
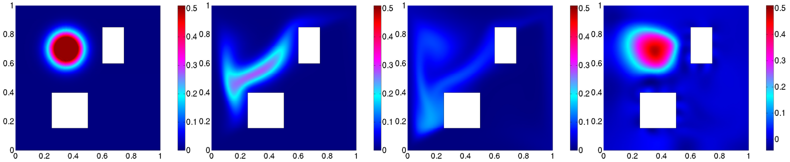


Wave speed anomaly at a depth of 1400km

Example VI: Inference of initial contaminant concentration in an atmospheric transport model

Transport of a concentration field $u(x,t)$ by diffusion and advection. Use measurements at boundaries of white squares to infer the initial concentration $u_0(x)$.

$$
\begin{aligned}
u_t - \kappa \Delta u + \boldsymbol{v} \cdot \nabla u &= 0 \quad \text{in } \Omega \times [0,T] \\
\kappa \nabla u \cdot \boldsymbol{n} &= 0 \quad \text{on } \partial\Omega \times [0,T] \\
u(x,0) &= u_0 \quad \text{in } \Omega
\end{aligned}
$$



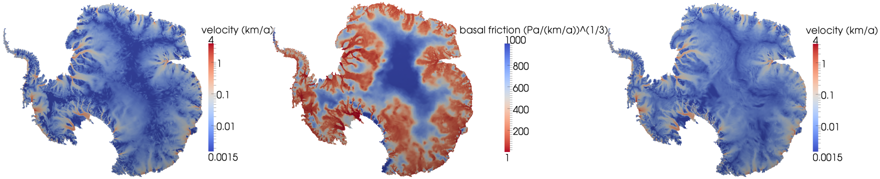
Applications: Detecting contaminant in ground water, ocean, or atmosphere

Example VII: Inference of basal friction in Antarctica

Creeping, viscous, incompressible, non-Newtonian flow:

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0 && \text{in } \Omega \\
-\boldsymbol{\nabla} \cdot [\eta(\boldsymbol{u})(\boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla} \boldsymbol{u}^T) - \boldsymbol{I} p] &= \rho \boldsymbol{g} && \text{in } \Omega \\
\boldsymbol{\sigma n} &= \boldsymbol{0} && \text{on } \Gamma_t \\
\boldsymbol{u} \cdot \boldsymbol{n} = 0, \;\; \boldsymbol{T \sigma n} + \exp(\beta) \boldsymbol{T u} &= \boldsymbol{0} && \text{on } \Gamma_b
\end{aligned}
$$

Invert for "friction" field β at base of ice sheet given InSAR observations of velocity u on top surface of ice



left: InSAR surface velocities; middle: inferred β; right: reconstructed surface velocities

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This is the approach mainly used in this class

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Reminder: survey

Note: no class on Wed. 1/15!