hp-Adaptive Finite Elements for Coupled Wave Propagation Problems

Leszek Demkowicz
ICES, The University of Texas at Austin

Computational Methods for Coupled problems in Science and Engineering IV
Kos, Greece, June 20-22, 2011

Teams and sponsorship:

Air Force project: P. Gatto, M. Paszyński (Cracow), W. Rachowicz (Cracow),

NSF Petascale project: K. Kim, A. Yilmaz, P. Gatto

Sonic tools modeling: P. Matuszyk, C. Torres-Verdin

Navy sonar project: J. Zitelli

Air Force project on stochastic inversion: O. Ghattas, J. Bramwell

Outline of Presentation

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- Old way of doing business: Asymptotic stability and weak coupling: (visco)elasticity coupled with acoustics.
  - Bone conduction of sound in the human head (subwavelength regime)
  - Sonic logging (0 - 50 wavelengths)
  - Sonars (50 - 3000 wavelengths)

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  - Bone conduction of sound in the human head (subwavelength regime)
  - Sonic logging (0 - 50 wavelengths)
  - Sonars (50 - 3000 wavelengths)
- A new paradigm: DPG method with optimal testing
  - model acoustic, Maxwell and elastodynamic problems
  - Pekeris problem

Asymptotic Stability (Mikhlin)

FE classics:

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FE classics:

- If the bilinear form is symmetric (hermitian) and positive-definite,

$$b(u,v) = \overline{b(v,u)}, \quad b(v,v) > 0$$

$u, v \in$ a Hilbert space V,

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$u, v \in$ a Hilbert space V,

- then

$$\left\{ \begin{array}{l} u \in V \\ J(u) := \frac{1}{2} b(u,u) - l(u) \to \text{min} \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} u \in V \\ b(u,v) = l(v), \ v \in V \end{array} \right.$$

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- and, Bubnov-Galerkin method delivers the *best approximation error* in the energy norm,

$$\left\{ \begin{array}{l} u_h \in V_h \subset V \\ b(u_h, v_h) = l(v_h), \; v_h \in V_h \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} u_h \in V_h \\ \|u - u_h\|_E \to \min \end{array} \right.$$

where $\|v\|_E^2 = b(v,v)$.

Asymptotic Stability (Mikhlin)

You cannot do better*

*In terms of energy norm.

Asymptotic Stability (Mikhlin)

Compact perturbation:

†D, *Computers & Mathematics with Applications*, **27**(12),69–84, 1994
D, J.T. Oden, *Comput. Methods Appl. Mech. Engrg.*,**133** (3-4), 287–318, 1996.

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- If we perturb $b(u,v)$ with a compact contribution,

$$b(u,v) + c(u,v)$$

$(|c(u,v)| \leq C\|u\|_H \|v\|_V,\ V \overset{c}{\hookrightarrow} H)$,

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- then the best approximation error property is achieved asymptotically[†],

$$\frac{\|u - u_{hp}\|_E}{\inf_{w_{hp}} \|u - w_{hp}\|_E} \to 0 \text{ as } \frac{h}{p} \to 0$$

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- Is h/p small enough to observe this in practice ?

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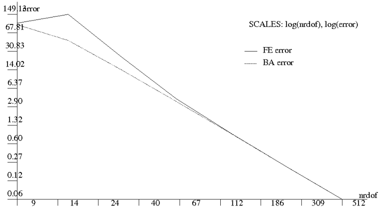
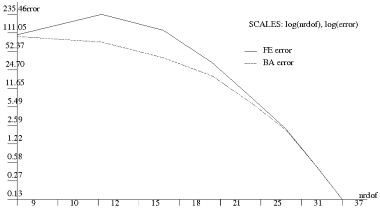
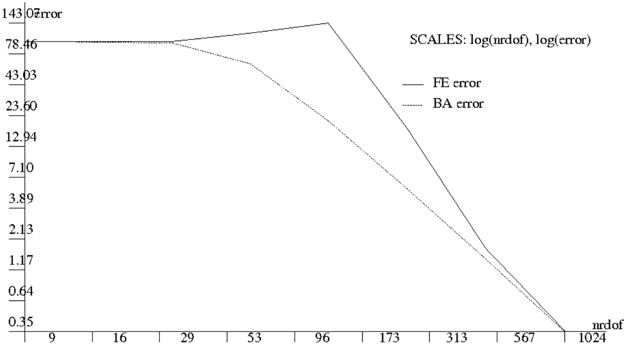
Pollution (Babuška, Ihlenburg)





Vibrations of an elastic bar, $k = 32$ (5 wavelengths). FE and best approximation (BA) errors for uniform h- ($p = 2$) and p-refinements.

Pollution



Vibrations of an elastic bar, $k = 160$ (25 wavelengths). FE and best approximation (BA) errors for uniform h- ($p = 2$) refinements.

Bone Conduction of Sound in Human Head[‡]

L. Demkowicz, P. Gatto, M. Paszyński, W. Rachowicz

[‡] *CMAME*, 2011, in print

Motivation

- Investigate **bone conduction of sound** in the human head in environments with extremely high sound pressure levels (about 120 dB).
- Quantify the **physiological effects** of bone conduction of sound.
- Develop strategies to prevent **hearing impairment**.

Main Challenges

- Acoustics/elasticity coupled problem.
- Modeling of thin-walled structures.
- Geometry modeling: surface reconstruction from scan data.
- Choice of appropriate material data and *qualitative* validation.

Governing Equations

continuity eq. $\frac{d\varrho}{dt} + \varrho \,\mathrm{div}\, \boldsymbol{v} = 0 \xrightarrow{\textit{linearize}} \frac{\partial \varrho}{\partial t} + \varrho_0 \,\mathrm{div}\, \boldsymbol{v} = 0$

momentum eq. $\varrho \frac{d\boldsymbol{v}}{dt} + \nabla p = 0 \xrightarrow{\textit{linearize}} \varrho_0 \frac{\partial \boldsymbol{v}}{\partial t} + \nabla p = 0$

state eq. $p = p(\varrho) \xrightarrow{\textit{linearize}} p = c^2 \varrho$

$$\xrightarrow{\textit{frequency domain}} \quad \frac{i\omega}{c^2} p + \varrho_0 \,\mathrm{div}\, \boldsymbol{v} = 0 \quad ; \quad i\omega \varrho_0 \boldsymbol{v} + \nabla p = 0$$

Variational Formulation

$$0 = \int_{\Omega_a} \left(\frac{i\omega}{c^2} p + \varrho_0 \,\mathrm{div}\, \boldsymbol{v} \right) q = \int_{\Omega_a} \left(\frac{i\omega}{c^2} pq + \frac{1}{i\omega} \nabla p \cdot \nabla q \right) + \varrho_0 \int_{\partial\Omega_a} \boldsymbol{v} \cdot \boldsymbol{n} q$$

Governing Equations

momentum eq.	$\varrho_s \frac{d\boldsymbol{v}}{dt} = \operatorname{div}\boldsymbol{\sigma} + \boldsymbol{f}$
strain def.	$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$
constitutive rel.	$\boldsymbol{\sigma} = E\,\boldsymbol{\varepsilon}$

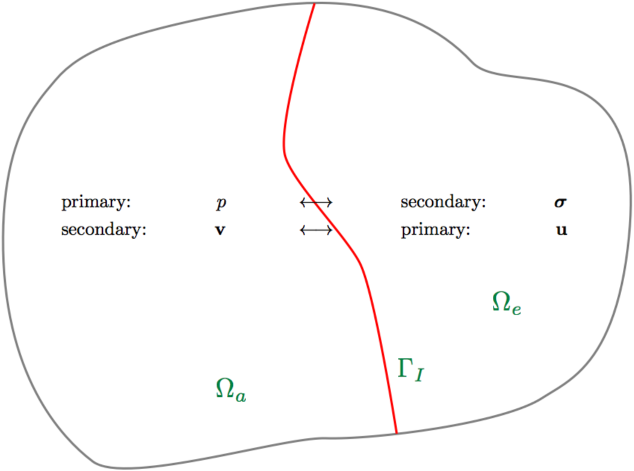
$$\xrightarrow{\textit{frequency domain}} \quad -\omega^2\varrho_s\boldsymbol{u} - \operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f} \quad ; \quad \boldsymbol{\sigma} = E\,\nabla\boldsymbol{u}$$

Variational Formulation

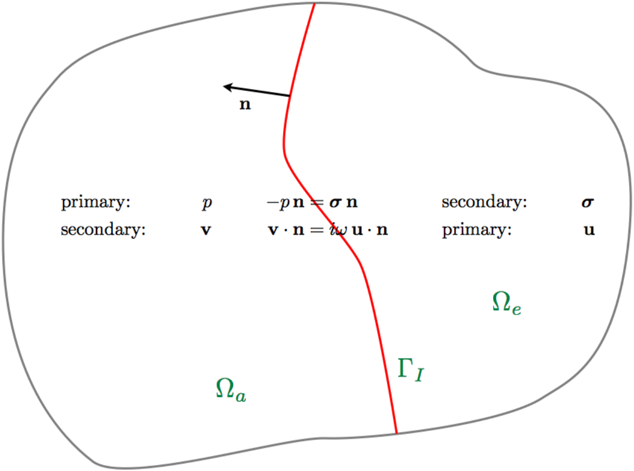
$$\int_{\Omega_e} \boldsymbol{f}\cdot\boldsymbol{v} = \int_{\Omega_e} \Big(-\omega^2\varrho_s\boldsymbol{u} - \operatorname{div}\boldsymbol{\sigma} \Big)\cdot\boldsymbol{v} =$$

$$\int_{\Omega_e} \Big(-\omega^2\varrho_s\boldsymbol{u}\cdot\boldsymbol{v} + (E\,\nabla\boldsymbol{u}) : \nabla\boldsymbol{v} \Big) + \int_{\partial\Omega_e} (\boldsymbol{\sigma}\,\boldsymbol{n})\cdot\boldsymbol{v} \qquad \forall\,\boldsymbol{v}$$

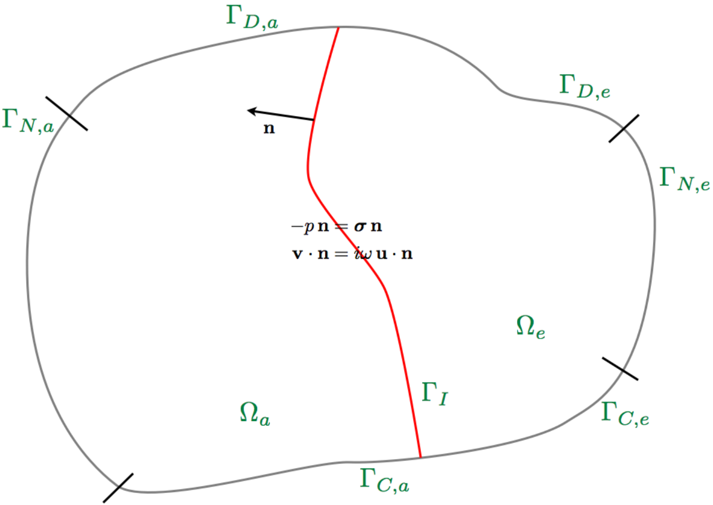
Weak Coupling



Weak Coupling



Boundary Conditions



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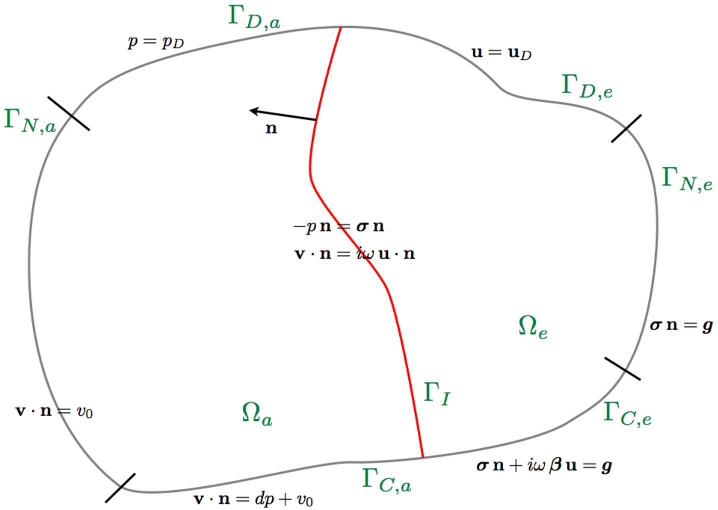


Variational Formulation

Find $\boldsymbol{u} \in \tilde{\boldsymbol{u}}_D + \boldsymbol{V}$, $p \in \tilde{p}_D + V$ such that

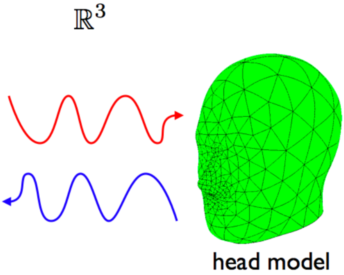
$$\begin{array}{llcccl} \text{elasticity} & b_{ee}(\boldsymbol{u},\boldsymbol{v}) & + & b_{ae}(p,\boldsymbol{v}) & = & l_e(\boldsymbol{v}) & \forall\, \boldsymbol{v} \in \boldsymbol{V} \\ \text{acoustics} & b_{ea}(\boldsymbol{u},q) & + & b_{aa}(p,q) & = & l_a(q) & \forall\, q \in V \end{array}$$

- Test spaces: $\boldsymbol{V} = H^1_{0,\Gamma_{D,e}}(\Omega_e)^3$, $V = H^1_{0,\Gamma_{D,a}}(\Omega_a)$.
- Symmetric forms $b_{ee}(\cdot\,,\cdot)$ and $b_{aa}(\cdot\,,\cdot)$.
- Divide acoustics formulation by $-\omega^2 \varrho_f$, then:

$$b_{ea}(\boldsymbol{v},q) = \int_{\Gamma_I} \boldsymbol{v}\cdot\boldsymbol{n}q = b_{ae}(q,\boldsymbol{v})$$

- Symmetric form $b(\cdot\,,\cdot) = b_{ee}(\cdot\,,\cdot) + b_{ae}(\cdot\,,\cdot) + b_{ea}(\cdot\,,\cdot) + b_{aa}(\cdot\,,\cdot)$.
- Fredholm operator of the second type.

Perfectly Matched Layers (PML)



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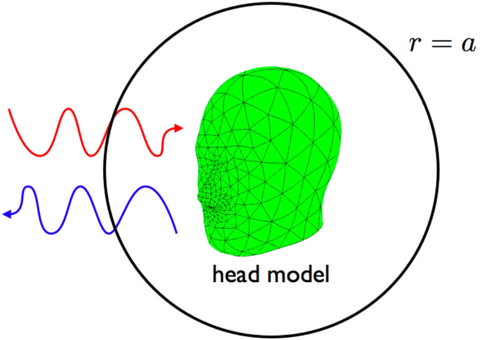

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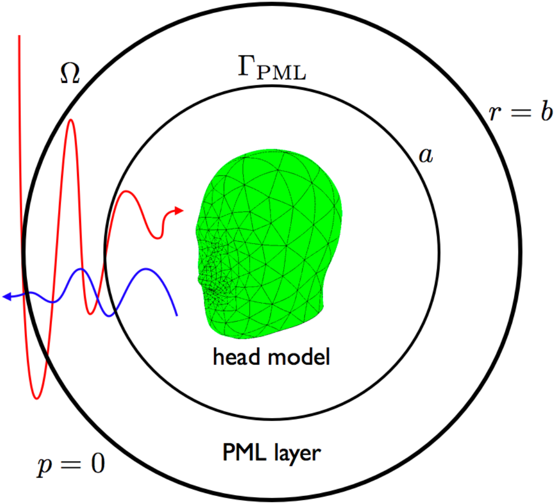

Variational Formulation

$$\begin{array}{lclclcll} \texttt{elem\_E} & \rightarrow & b_{ee}(\boldsymbol{u},\boldsymbol{v}) & + & b_{ae}(p,\boldsymbol{v}) & = & l_e(\boldsymbol{v}) & \forall\, \boldsymbol{v} \\ \texttt{elem\_A} & \rightarrow & b_{ea}(\boldsymbol{u},q) & + & b_{aa}(p,q) & = & l_a(q) & \forall\, q \end{array}$$

Elasticity

$$b_{ee}(\boldsymbol{u},\boldsymbol{v}) = \int_{\Omega_e} \left((E\,\nabla \boldsymbol{u}) : \nabla \boldsymbol{v} - \varrho_s \omega^2 \boldsymbol{u}\cdot\boldsymbol{v} \right) + i\omega \int_{\Gamma_{C,e}} (\boldsymbol{\beta}\,\boldsymbol{u})\cdot\boldsymbol{v}$$

$$b_{ea}(\boldsymbol{u},q) = \int_{\Gamma_I} (\boldsymbol{u}\cdot\boldsymbol{n})q$$

$$l_e(\boldsymbol{v}) = \int_{\Omega_e} \boldsymbol{f}\cdot\boldsymbol{v} + \int_{\Gamma_{N,e}\cup\Gamma_{C,e}} \boldsymbol{g}\cdot\boldsymbol{v}$$

Compact perturbations in red.

Variational Formulation

$$
\begin{array}{llllllll}
\texttt{elem\_E} & \rightarrow & b_{ee}(\boldsymbol{u},\boldsymbol{v}) & + & b_{ae}(p,\boldsymbol{v}) & = & l_e(\boldsymbol{v}) & \forall\, \boldsymbol{v} \\
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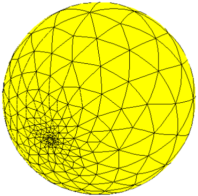
\end{array}
$$

Acoustics

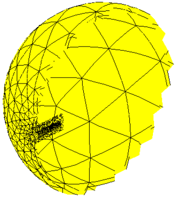
$$
\begin{aligned}
b_{ae}(p,\boldsymbol{v}) &= \int_{\Gamma_I} p(\boldsymbol{v}\cdot\boldsymbol{n}) \\
b_{aa}(p,q) &= \int_{\Omega_a} \left(\frac{1}{c^2\varrho_f} pq - \frac{1}{\omega^2\varrho_f}\nabla p\cdot\nabla q\right) - \frac{i}{\omega}\int_{\Gamma_{C,a}} dpq \\
l_a(q) &= -\frac{i}{\omega}\int_{\Gamma_{N,a}\cup\Gamma_{C,a}} v_0 q
\end{aligned}
$$

Compact perturbations in red.

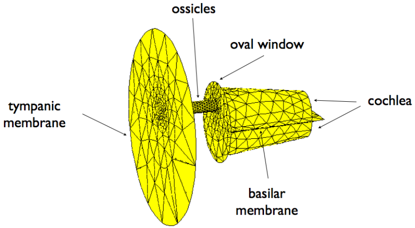
Simplified Model



Head model



Cross-section



Middle ear

Model is generated by NETGEN and Geometry Modeling Package:

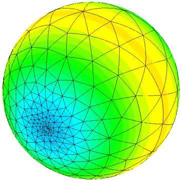
- 16 004 tetrahedra
- 3 228 prisms
- 56 pyramids
- 1 070 190 dof's for $p = 5$

Numerical Results

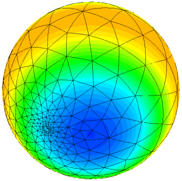
material	E [MPa]	ν	ϱ [kg/m^3]	c_p [m/s]	c_s [m/s]
tissue (brain)	0.67	0.48	1040	75	15
skull (bone)	6500	0.22	1412	2293	1374
cochlea (water)			1000	1500	
air			1.2	344	

Material constants.

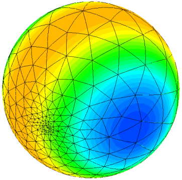
Numerical Results



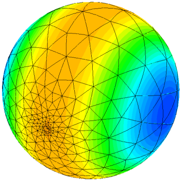
$\theta = 0$



$\theta = \pi/6$



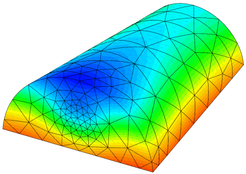
$\theta = \pi/3$



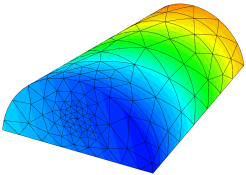
$\theta = \pi/2$

Total acoustic pressure (incident and scattered) on skull, $\omega = 400\pi$.

Numerical Results



$\theta = 0$



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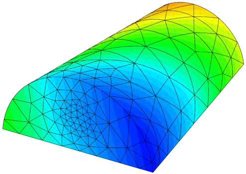

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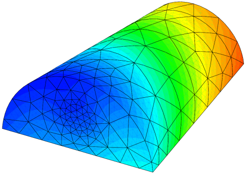

$\theta = \pi/2$

Total pressure (elastic and acoustic) on upper half of cochlea, $\omega = 400\pi$.

Simulation of Sonic Measurements in the Borehole

P. Matuszyk, C. Torres-Verdin, I. Muga., L. Demkowicz,
D. Pardo, A. Mora, V.M. Calo

Modeling of Long Range, Shallow Water Acoustic Wave Propagation

J. Zitelli, L. Demkowicz

A New Paradigm:
DPG Method with Optimal Testing

L. Demkowicz, J. Gopalakrishan, I. Muga, J. Zitelli, J. Bramwell,
D. Pardo. V. Calo, O. Ghattas §

§ Zitelli, Muga, D, Gopalakrishnan, Pardo, Calo, *J.Comp. Phys.*, **230**, 2406-2432, 2011,
D, Gopalakrishnan, Muga, Zitelli, "Wavenumber Explicit Analysis for a DPG Method for the Multidimensional Helmholtz Equation",in preparation.

Petrov-Galerkin Method
with Optimal Test Functions

$$\Updownarrow$$

Least Squares (with a Twist)

Least squares and optimal test functions

Least squares and optimal test functions

- Variational formulation:

$$\left\{\begin{array}{l} u \in U \\ b(u,v) = l(v) \quad v \in V \end{array}\right. \quad \Leftrightarrow \quad \begin{array}{l} Bu = l \quad B : U \to V' \\ < Bu, v >= b(u,v) \end{array}$$

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- Least squares: $U_h \subset U$,

$$\frac{1}{2}\|Bu_h - l\|_{V'} \to \min_{u_h \in U_h}$$

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$$R_V : V \to V', < R_V v, \delta v >= (v, \delta v)_V$$

is an *isometry*, $\|R_V v\|_{V'} = \|v\|_V$.

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- Least squares reformulated:

$$\frac{1}{2}\|Bu_h - l\|_{V'} = \frac{1}{2}\|R_V^{-1}(Bu_h - l)\|_V^2 \to \min_{u_h \in U_h}$$

Least squares and optimal test functions

Taking Gâteaux derivative,

$$(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u_h)_V = 0 \quad \delta u_h \in U_h$$

Least squares and optimal test functions

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or

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Least squares and optimal test functions

Taking Gâteaux derivative,

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or

$$b(u_h, v_h) = l(v_h)$$

where

$$\begin{cases} v_h \in V \\ (v_h, \delta v)_V = b(\delta u_h, \delta v) \quad \delta v \in V \end{cases}$$

Least squares and optimal test functions

- Stiffness matrix is always hermitian and positive-definite.

Least squares and optimal test functions

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- The method delivers the *best approximation error* (BAE) in the "energy norm":

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Least squares and optimal test functions

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- The method delivers the *best approximation error* (BAE) in the "energy norm":

$$\|u\|_E := \|Bu\|_{V'} = \sup_{v \in V} \frac{|b(u,v)|}{\|v\|_V}$$

- The energy norm of the FE error $u - u_h$ equals the residual and can be computed,

$$\|u - u_h\|_E = \|Bu - Bu_h\|_{V'} = \|l - Bu_h\|_{V'} = \|R_V^{-1}(l - Bu_h)\|_V$$

(no need for a-posteriori error estimation, note the connection with implicit a-posteriori error estimation techniques...)

Least squares and optimal test functions

- A lot depends upon the choice of the test norm $\| \cdot \|_V$; for different test norms, we get get different methods.

Least squares and optimal test functions

- A lot depends upon the choice of the test norm $\|\cdot\|_V$; for different test norms, we get get different methods.
- Banach Closed Range Theorem $\Longrightarrow$
  If $B' : V \to V'$ is injective, and we choose

  $$\|v\|_V = \sup_{u \in U} \frac{|b(u,v)|}{\|u\|_U}$$

  then

Least squares and optimal test functions

- A lot depends upon the choice of the test norm $\|\cdot\|_V$; for different test norms, we get get different methods.
- Banach Closed Range Theorem $\implies$
  If $B' : V \to V'$ is injective, and we choose

$$\|v\|_V = \sup_{u \in U} \frac{|b(u,v)|}{\|u\|_U}$$

  then
- the energy norm coincides with the original norm in U,

$$\|u\|_E = \|u\|_U$$

Ultraweak Variational Formulation

and

Discontinuous Petrov-Galerkin Method ¶

¶Original idea due to C. Bottaso, P. Pausin, S. Micheletti, R. Sacco, 2004-2008

DPG method

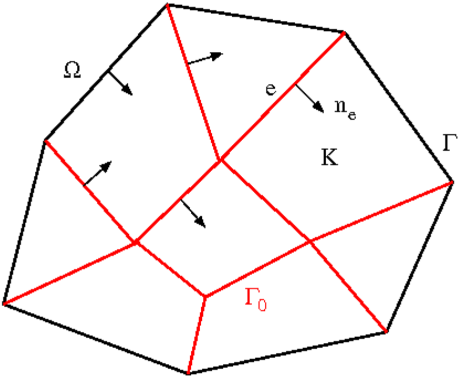
Linear acoustics in frequency domain:

$$\begin{cases} i\omega \boldsymbol{u} + \boldsymbol{\nabla} p &= \boldsymbol{0} \\ i\omega p + \text{div}\boldsymbol{u} &= 0 \end{cases}$$

with, e.g. hard boundary condition:

$$u_n = g$$

DPG method



Elements:K
Edges:e
Skeleton:$\Gamma_h = \bigcup_K \partial K$
Internal skeleton:$\Gamma_h^0 = \Gamma_h - \partial\Omega$

DPG Method

Take an element K. Multiply the equations with test functions $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K), q \in H^1(K)$:

$$\begin{cases} i\omega \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{\nabla} u \cdot \boldsymbol{v} & = 0 \\ i\omega p\, q + \mathrm{div} \boldsymbol{u}\, q & = 0 \end{cases}$$

DPG Method

Integrate over the element K:

$$\begin{cases} i\omega \int_K \boldsymbol{u} \cdot \boldsymbol{v} + \int_K \boldsymbol{\nabla} u \cdot \boldsymbol{v} & = 0 \\ i\omega \int_K p\, q + \int_K \mathrm{div} \boldsymbol{u}\, q & = 0 \end{cases}$$

DPG Method

Integrate by parts (relax) *both* equations:

$$\begin{cases} i\omega \int_K \boldsymbol{u}\cdot\boldsymbol{v} - \int_K u \cdot \mathrm{div}\boldsymbol{v} + \int_{\partial K} u v_n & = 0 \\ i\omega \int_K p\, q - \int_K \boldsymbol{u}\cdot\boldsymbol{\nabla} q + \int_{\partial K} u_n\, q\, \mathrm{sgn}(\boldsymbol{n}) & = 0 \end{cases}$$

where $u_n = \boldsymbol{u}\cdot\boldsymbol{n}_e$ and

$$\mathrm{sgn}(\boldsymbol{n}) = \begin{cases} 1 & \text{if } \boldsymbol{n} = \boldsymbol{n}_e \\ -1 & \text{if } \boldsymbol{n} = -\boldsymbol{n}_e \end{cases}$$

DPG Method

Declare traces and fluxes to be independent unknowns:

$$\begin{cases} i\omega \int_K \boldsymbol{u} \cdot \boldsymbol{v} - \int_K u \cdot \mathsf{div}\boldsymbol{v} + \int_{\partial K} \hat{u} v_n & = 0 \\ i\omega \int_K p\, q - \int_K \boldsymbol{u} \cdot \boldsymbol{\nabla} q + \int_{\partial K} \hat{u}_n\, q\, \mathsf{sgn}(\boldsymbol{n}) & = 0 \end{cases}$$

DPG Method

Use BCs to eliminate known fluxes

$$\begin{cases} i\omega \int_K \boldsymbol{u} \cdot \boldsymbol{v} - \int_K u \cdot \mathsf{div}\boldsymbol{v} + \int_{\partial K} \hat{u} v_n & = 0 \\ i\omega \int_K pq - \int_K \boldsymbol{u} \cdot \boldsymbol{\nabla} q + \int_{\partial K - \Gamma} \hat{u}_n q \mathsf{sgn}(\boldsymbol{n}) & = \int_{\partial K \cap \Gamma} g\, q \end{cases}$$

DPG Method

Sum up over all elements,

$$\begin{cases} i\omega(\boldsymbol{u},\boldsymbol{v})_\Omega - (u,\mathrm{div}\boldsymbol{v})_{\Omega_h} + < \hat{u}, v_n >_{\Gamma_h} & = 0 \\ i\omega(p,q)_\Omega - (\boldsymbol{u},\boldsymbol{\nabla} q)_{\Omega_h} + < \hat{u}_n, q >_{\Gamma_h^0} & =< g, q >_\Gamma \end{cases}$$

Trace and Flux Spaces

$$
\begin{aligned}
\Gamma_h &:= \bigcup_K \partial K \quad \text{(skeleton)} \\
\Gamma_h^0 &:= \Gamma_h - \partial\Omega \quad \text{(internal skeleton)} \\
H^{1/2}(\Gamma_h) &:= \{q|_{\Gamma_h} \,:\, q \in H^1(\Omega)\} \\
& \quad \text{with the minimum extension norm:} \\
\|q\|_{H^{1/2}(\Gamma_h)} &:= \inf\{\|Q\|_{H^1} \,:\, Q|_{\Gamma_h} = q\} \\
\tilde{H}^{-1/2}(\Gamma_h^0) &:= \{v_n|_{\Gamma_h} \,:\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div},\Omega)\} \\
& \quad \text{with the minimum extension norm:} \\
\|v_n\|_{\tilde{H}^{-1/2}(\Gamma_h^0)} &:= \inf\{\|\boldsymbol{V}\|_{\boldsymbol{H}_0(\mathrm{div},\Omega)} \,:\, \boldsymbol{V}\cdot\boldsymbol{n}|_{\Gamma_h^0} = \sigma_n\}
\end{aligned}
$$

Functional Setting

Group variables:
Solution $\boldsymbol{U} = (\boldsymbol{u}, p, \hat{u}_n, \hat{p})$:

$$\begin{array}{l} u_1, u_2, p \in L^2(\Omega_h) \\ \hat{u}_n \in \tilde{H}^{-1/2}(\Gamma_h^0) \\ \hat{p} \in \tilde{H}^{1/2}(\Gamma_h) \end{array}$$

Test function $\boldsymbol{V} = (\boldsymbol{v}, q)$:

$$\begin{array}{l} \boldsymbol{v} \in \boldsymbol{H}(\mathsf{div}, \Omega_h) \\ q \in H^1(\Omega_h) \end{array}$$

Sesquilinear form

$$\begin{aligned} b(\boldsymbol{U}, \boldsymbol{V}) &= -(u, i\omega \boldsymbol{v} + \boldsymbol{\nabla} q)_{\Omega_h} - (p, i\omega q + \mathsf{div}\boldsymbol{v})_{\Omega_h} \\ &+ < \hat{u}_n, q >_{\Gamma_h^0} + < \hat{p}, v_n >_{\Gamma_h} \end{aligned}$$

Punchlines

Local invertibility of Riesz operator
Due to the use of "broken" Sobolev spaces (discontinuous test functions), the Riesz operator is inverted elementwise! Given any (linear) problem, and any trial shape functions, we compute the corresponding optimal test functions on the fly.

Approximate optimal test functions
The locally determined optimal test functions still need to be approximated. This is done using standard Bubnov-Galerkin method and an *enriched space*. If polynomials of order p are used to approximate the unknown velocity and pressure, the approximate optimal test functions are determined using polynomials of order:

$$p + \Delta p$$

Quasi-optimal test norm

Trial norm:

$$\|(\boldsymbol{u}, p, \hat{u}_n, \hat{p})\|_U^2 = \|\boldsymbol{u}\|_{L^2}^2 + \|p\|_{L^2}^2 + \|\hat{u}\|_?^2 + \|\hat{p}\|_?^2$$

Optimal test norm (unfortunately, non-local):

$$\begin{aligned} \|(\boldsymbol{v}, q)\|_{opt}^2 &= \|i\omega \boldsymbol{v} + \boldsymbol{\nabla} q\|_{\Omega_h}^2 + \|i\omega q + \mathsf{div}\boldsymbol{v}\|_{\Omega_h}^2 \\ &+ \sup_{\hat{u}_n, \hat{p}} \frac{|<\hat{u}_n, q> + <\hat{p}, v_n>|}{(\|\hat{u}_n\|_?^2 + \|\hat{p}_?^2)^{1/2}} \end{aligned}$$

Quasi-optimal test norm (local):

$$\|(\boldsymbol{v}, q)\|_{opt}^2 = \|i\omega \boldsymbol{v} + \boldsymbol{\nabla} q\|_{\Omega_h}^2 + \|i\omega q + \mathsf{div}\boldsymbol{v}\|_{\Omega_h}^2 + \|\boldsymbol{v}\|^2 + \|q\|^2$$

Robust stability result

Theorem: (Gopalakrishnan, Muga, D, Zitelli, 2011)
Assume: Ω contractable, impedance BC
Use: the quasi-optimal norm to define the minimum energy extension norms for fluxes $\hat{u}_n$ and traces $\hat{p}$.
Then

$$\|(\boldsymbol{v}, q)\|_{opt}^2 \approx \|(\boldsymbol{v}, q)\|_{qopt}^2 \quad \text{(uniformly in } k \text{ and mesh)}$$

Consequently, we get the robust stability in the *desired norm*:

$$\left(\|\boldsymbol{u} - \boldsymbol{u}_h\|^2 + \|p - p_h\|^2 + \|\hat{u}_n - \hat{u}_{n,h}\| + \|\hat{p} - \hat{p}_h\|^2\right)^{\frac{1}{2}}$$

$$\lesssim \|(\boldsymbol{u}, p, \hat{u}_n, \hat{p}) - (\boldsymbol{u}_h, p_h, \hat{u}_{n,h}, \hat{p}_h)\|_E$$

$$= \text{BAE of } (\boldsymbol{u}, p, \hat{u}_n, \hat{p}) \text{ in energy norm}$$

$$\lesssim \text{BAE of } (\boldsymbol{u}, p, \hat{u}_n, \hat{p}) \text{ in desired norm}$$

No pollution in 1D case

In 1D, traces and fluxes and just numbers. Thus, the BAE of fluxes and traces is zero. We get,

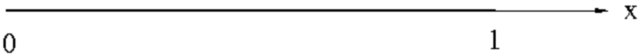
$$\left(\|u-u_h\|^2+\|p-p_h\|^2+\|\hat{u}_n-\hat{u}_{n,h}\|+\|\hat{p}-\hat{p}_h\|^2\right)^{\frac{1}{2}}$$
$$\lesssim \mathsf{inf}_{w_h,r_h}\left(\|u-w_h\|^2+\|p-r_h\|^2\right)^{\frac{1}{2}}$$

The BAE of u, p in L^2-error is pollution free.

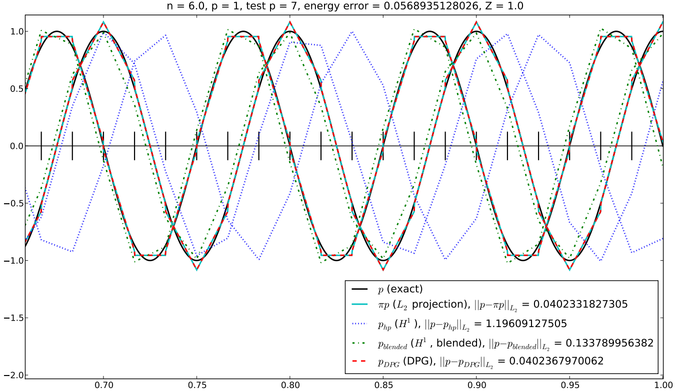
Numerical Experiments

1D experiment

Ansatz in time $e^{i\omega t}$,
Exact solution: $u = p = e^{-ikx}$ (going to the right)
BCs:
hard boundary at $x = 0$: $u(0) = 1$
impedance BC at $x = 1$: $u(1) = p(1)$
enriched space: $\Delta p = 6$

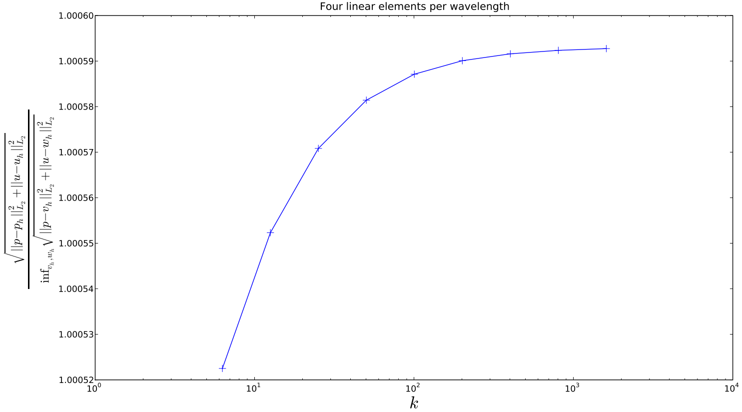


DPG vs. Standard FEs, 6 wavelenghts



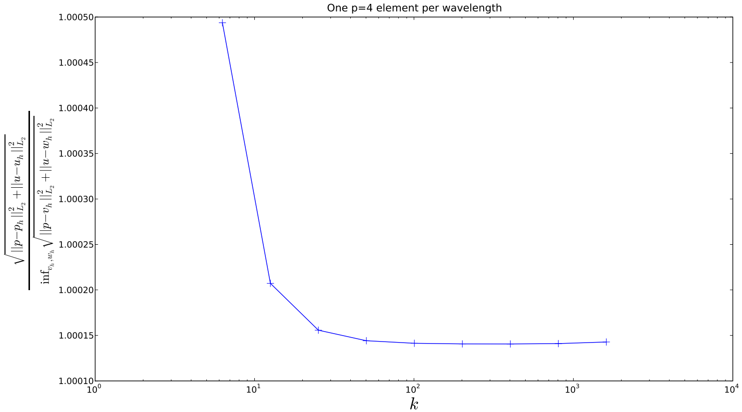
The standard H^1 conforming solution p_{hp} quickly exhibits excessive phase error; it is reduced but still present in p_{blended}

Four linear elements per wavelength



Adhering to a fixed number of elements per wavelength is sufficient to control error

One quartic element per wavelength



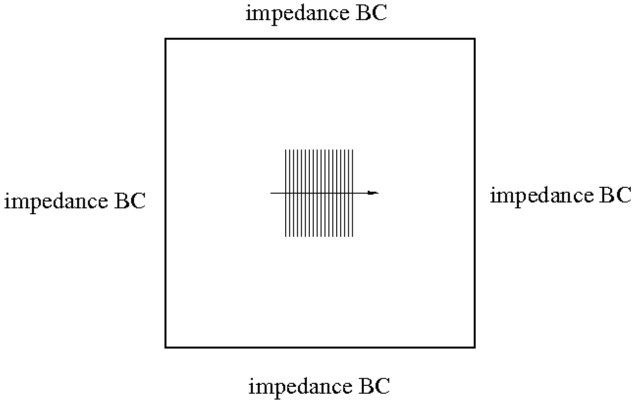
Adhering to a fixed number of elements per wavelength is sufficient to control error

Discretization:

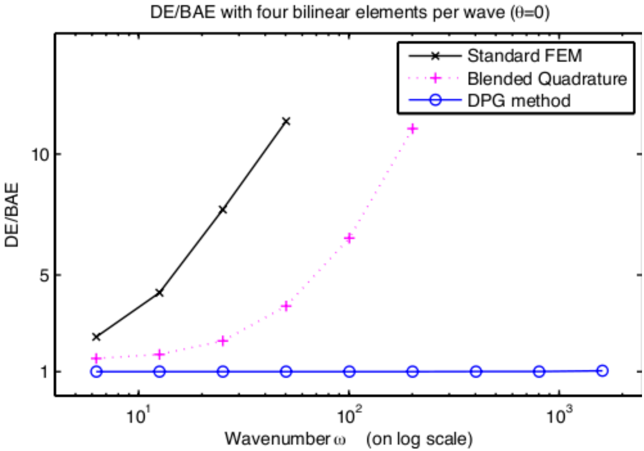
- field variables are discretized using isoparametric L^2-conforming quads of order p,
  $u_1, u_2, p \in \mathcal{P}^p \otimes \mathcal{P}^p$,
- traces are discretized using H^1-conforming elements of order $p+1$,
- fluxes are discretized using L^2-conforming elements of order $p+1$
- optimal test functions are approximated with polynomials of order $p+1+\Delta p$, i.e. $\boldsymbol{v} \in (\mathcal{P}^{p+\Delta p+1} \otimes \mathcal{P}^{p+\Delta p}) \times (\mathcal{P}^{p+\Delta p} \otimes \mathcal{P}^{p+\Delta p+1})$,
  $q \in \mathcal{P}^{p+\Delta p+1} \otimes \mathcal{P}^{p+\Delta p+1}$

2D experiment A

Exact solution: horizontal plane wave
Enriched space: $\Delta p = 2$.



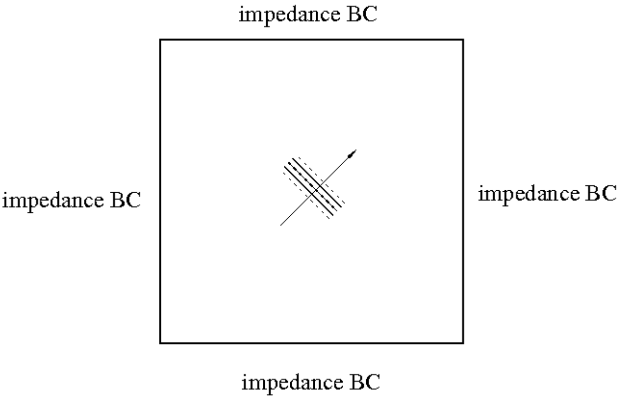
2D experiment A



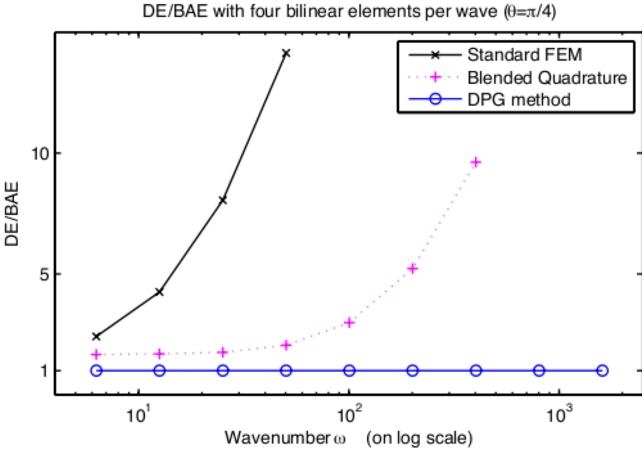
Ratio of L^2 discretization error vs BAE as a function of wave number. DPG vs standard FEs and Ainsworth-Wajid underintegration scheme.

2D experiment B

Exact solution: plane wave along diagonal
Enriched space: $\Delta p = 2$.



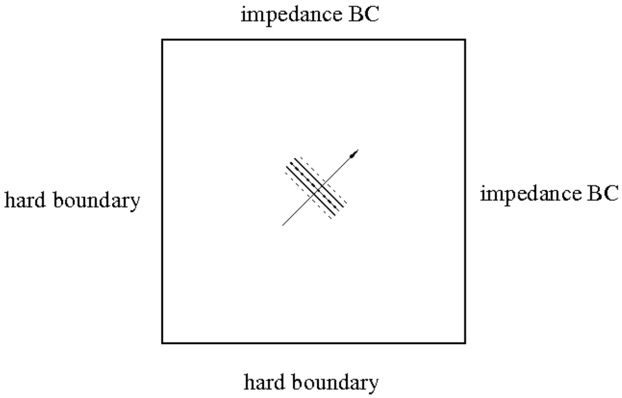
2D experiment B



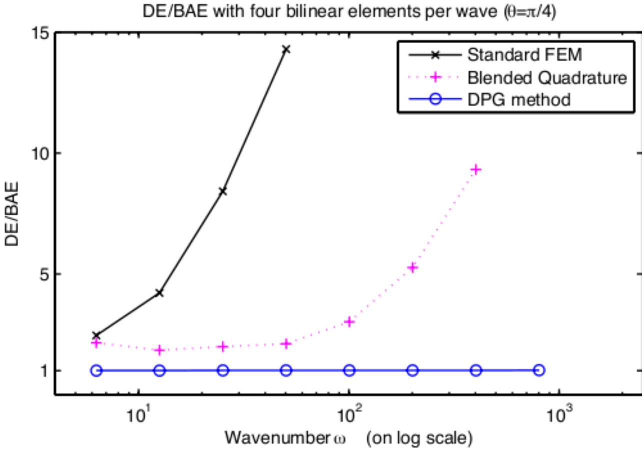
Ratio of L^2 discretization error vs BAE as a function of wave number. DPG vs standard FEs and Ainsworth-Wajid underintegration scheme.

2D experiment C

Exact solution: plane wave along diagonal
Enriched space: $\Delta p = 2$.

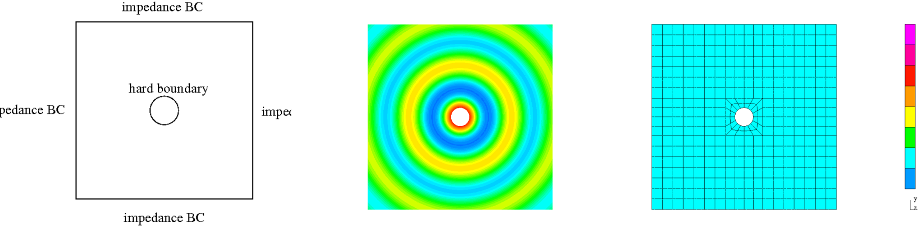


2D experiment C



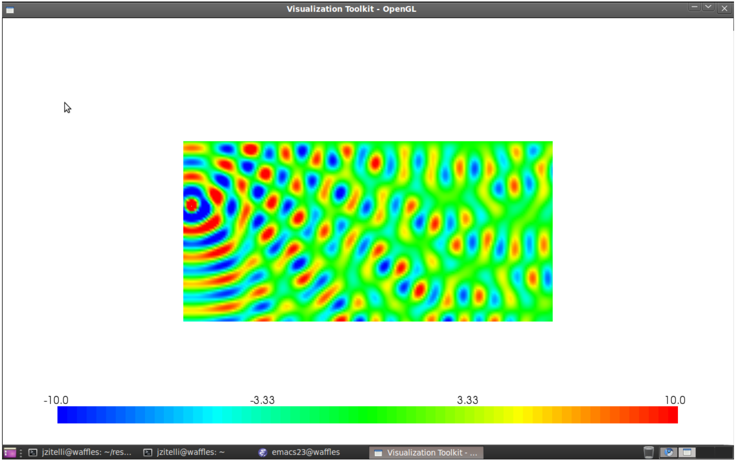
Ratio of L^2 discretization error vs BAE as a function of wave number. DPG vs standard FEs and Ainsworth-Wajid underintegration scheme.

Exact solution: outgoing cylindrical wave (Hankel functions...)
Enriched space: $\Delta p = 2$.



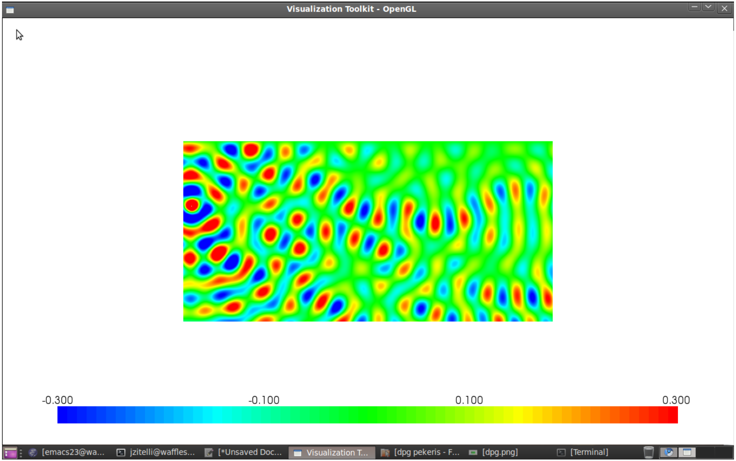
Boundary conditions, real part of pressure, initial mesh for $k = 4\pi$.

Pekeris problem, $k = 50 \cdot 2\pi$



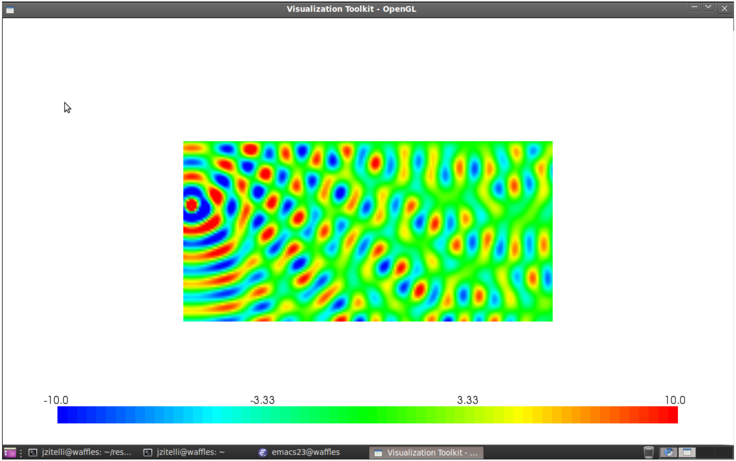
Exact solution (real part of pressure).

Pekeris problem, $k = 50 \cdot 2\pi$



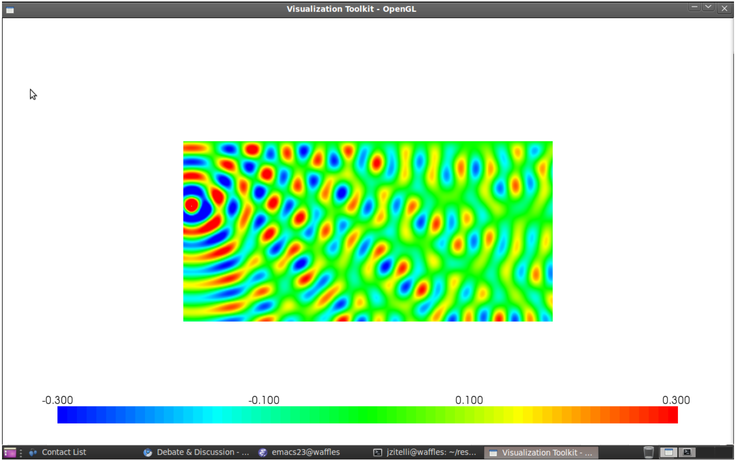
Classical FEs, four biquadratic elements per wavelength.

Pekeris problem, $k = 50 \cdot 2\pi$



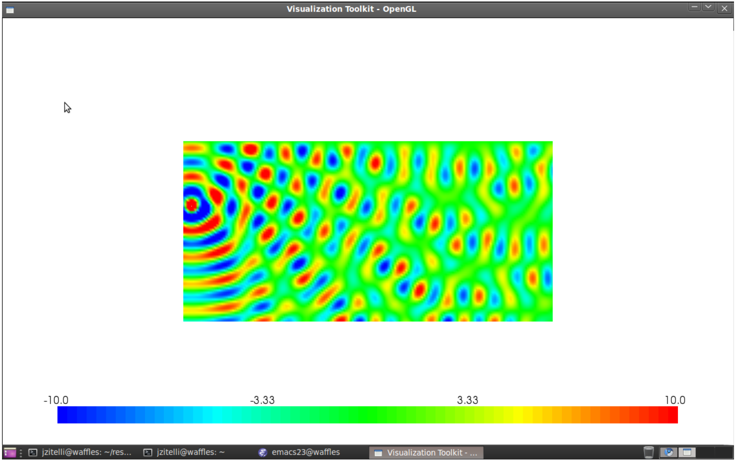
Exact solution (real part of pressure).

Pekeris problem, $k = 50 \cdot 2\pi$



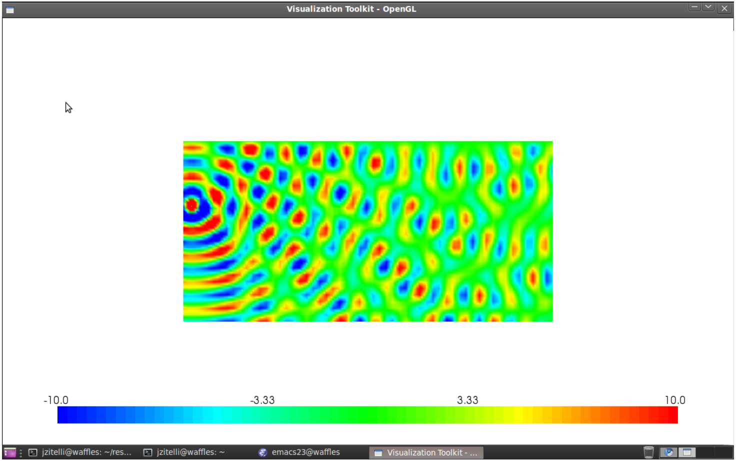
Ainsworth-Wajid quadrature, four biquadratic elements per wavelength.

Pekeris problem, $k = 50 \cdot 2\pi$



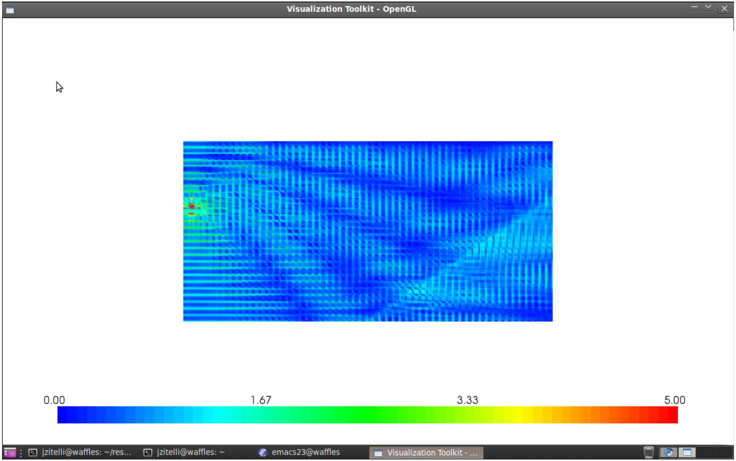
Exact solution (real part of pressure).

Pekeris problem, $k = 50 \cdot 2\pi$



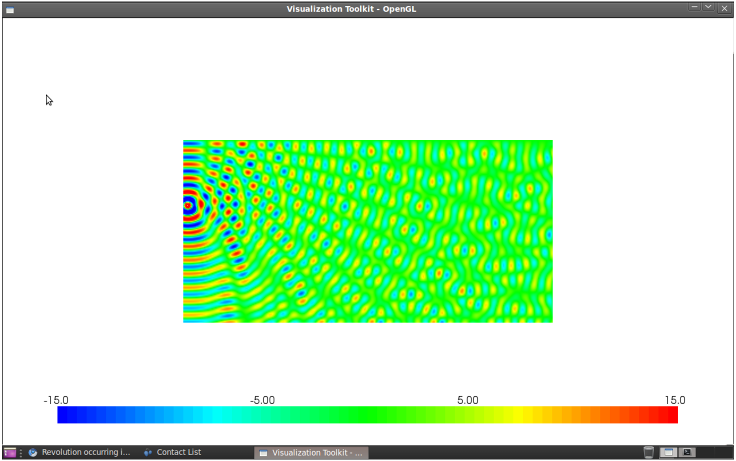
DPG method, four bilinear elements per wavelength.

Pekeris problem, $k = 50 \cdot 2\pi$



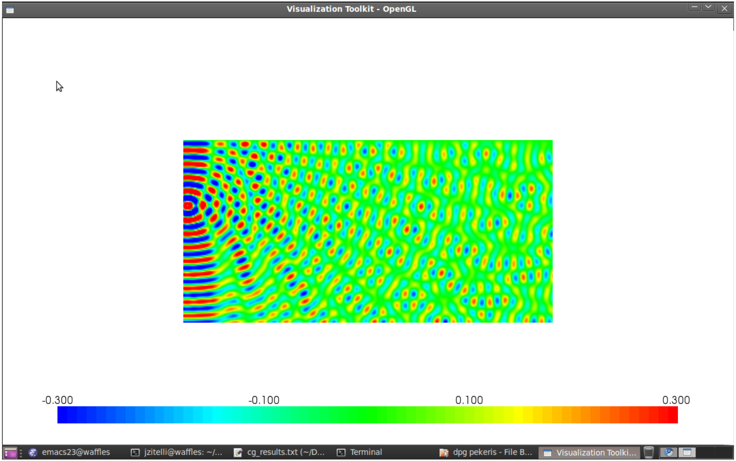
Error for the DPG method.

Pekeris problem, $k = 100 \cdot 2\pi$



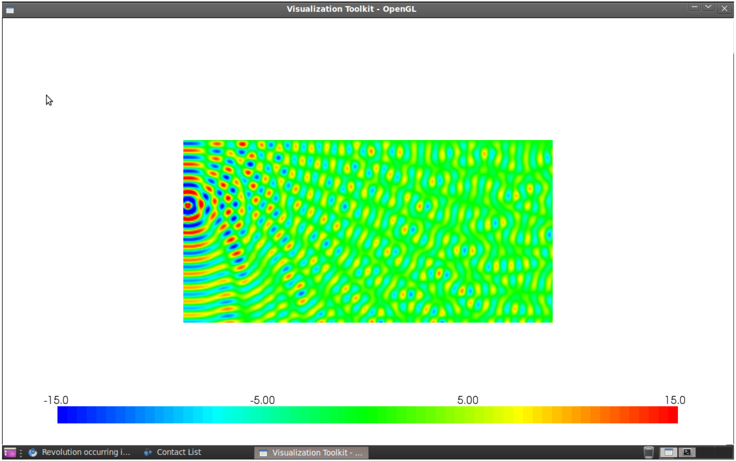
Exact solution (real part of pressure).

Pekeris problem, $k = 100 \cdot 2\pi$



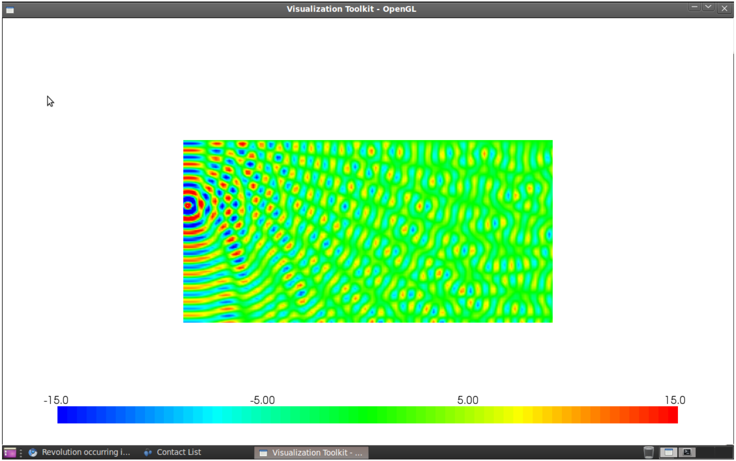
Ainsworth-Wajid quadrature, four biquadratic elements per wavelength.

Pekeris problem, $k = 100 \cdot 2\pi$



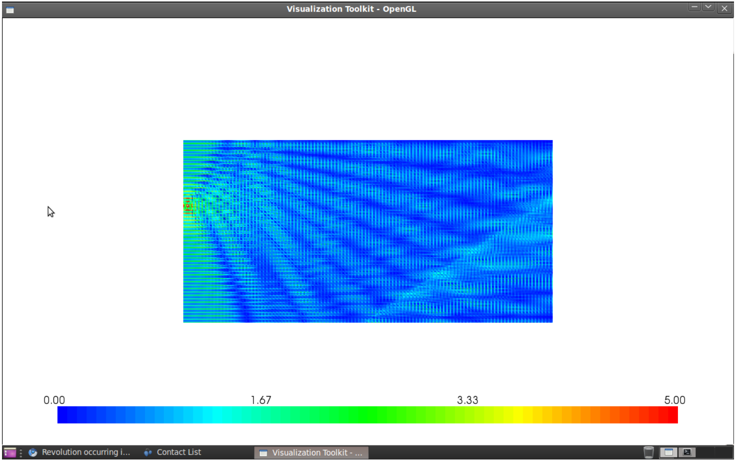
Exact solution (real part of pressure).

Pekeris problem, $k = 100 \cdot 2\pi$



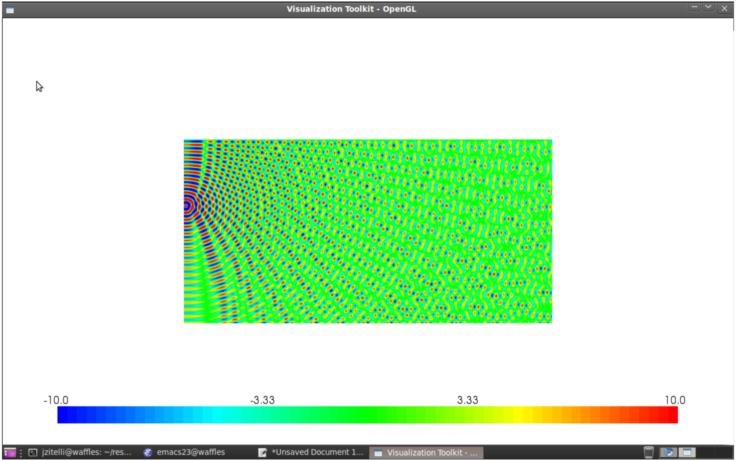
DPG method, four bilinear elements per wavelength.

Pekeris problem, $k = 100 \cdot 2\pi$



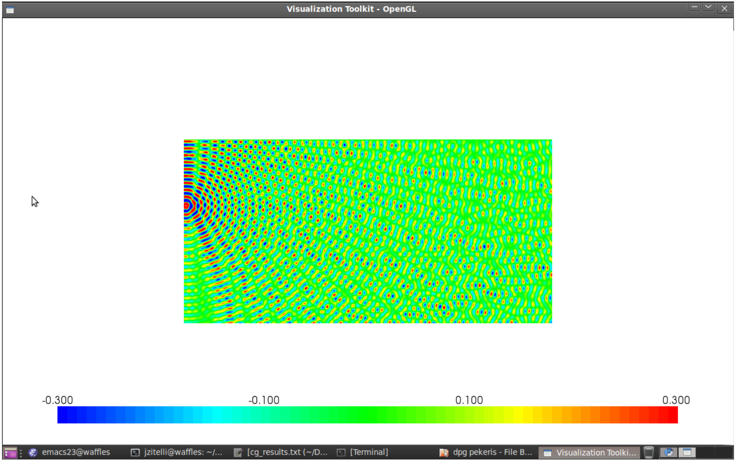
Error for the DPG method.

Pekeris problem, $k = 200 \cdot 2\pi$



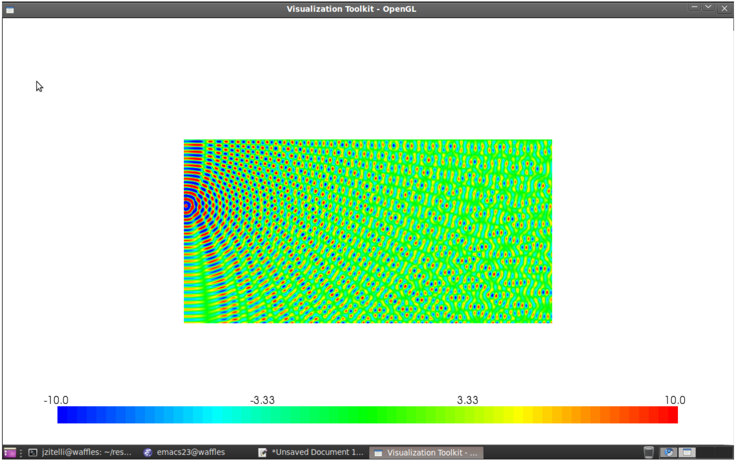
Exact solution (real part of pressure).

Pekeris problem, $k = 200 \cdot 2\pi$



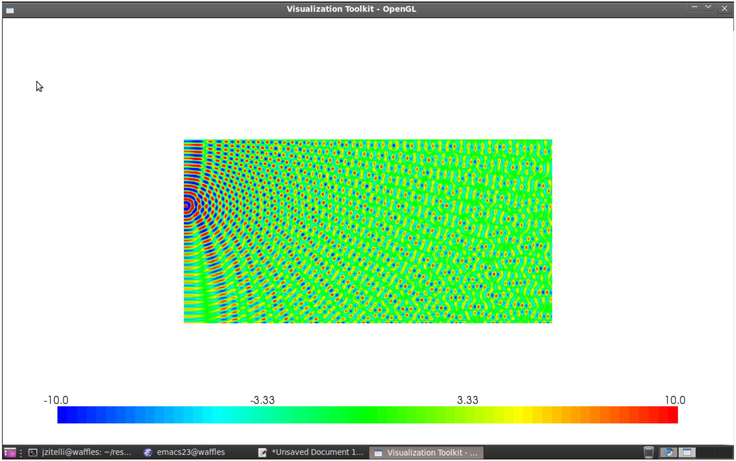
Ainsworth-Wajid quadrature, four biquadratic elements per wavelength.

Pekeris problem, $k = 200 \cdot 2\pi$



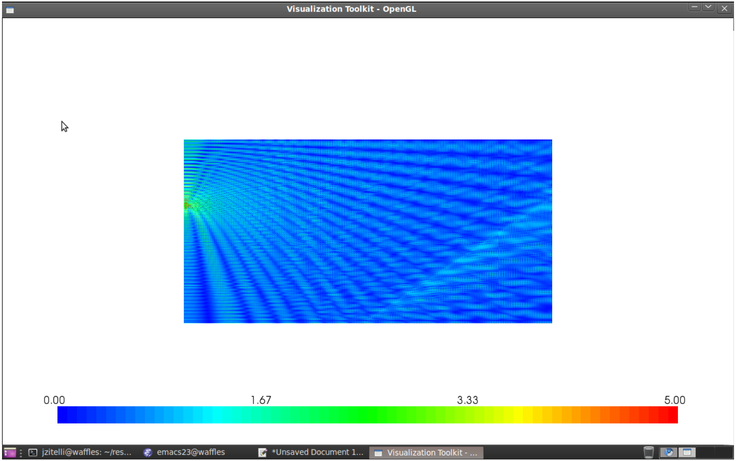
Exact solution (real part of pressure).

Pekeris problem, $k = 200 \cdot 2\pi$



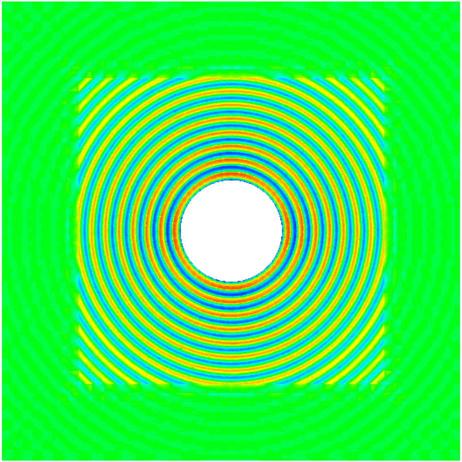
DPG method, four bilinear elements per wavelength.

Pekeris problem, $k = 200 \cdot 2\pi$

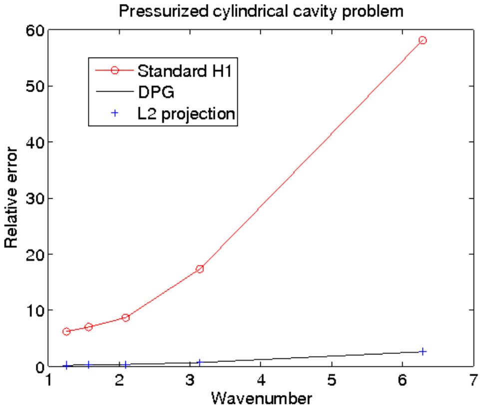


Error for the DPG method.

2D elastodynamics

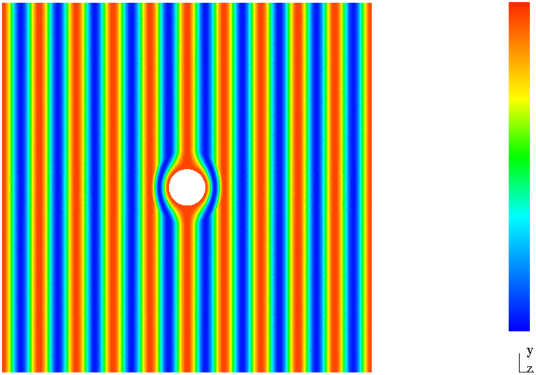


Pressurized cylindrical cavity problem with PML layer. Radial component of velocity.



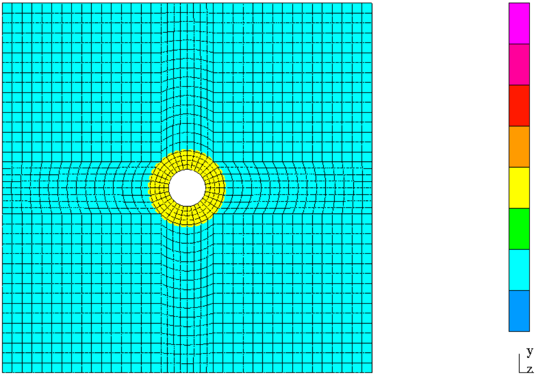
Pressurized cylindrical cavity problem with PML layer. Comparison of relative L^2 error for standard FEs and DPG with the BAE for increasing wave numbers.

2D acoustics (electromagnetics) cloaking problem



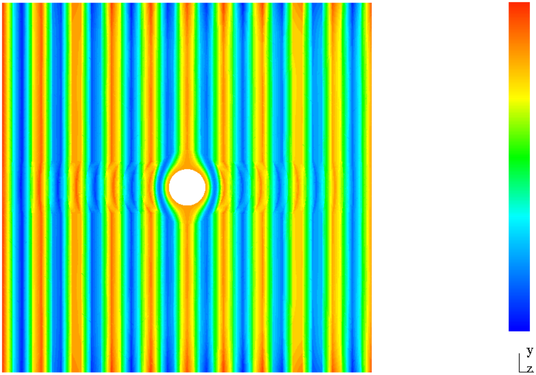
Exact solution (pressure or magnetic field)

2D acoustics (electromagnetics) cloaking problem



An hp mesh (4 bilinear elements per wavelength)

2D acoustics (electromagnetics) cloaking problem



Numerical solution (pressure or magnetic field)

Advertisements:

- Two day short course on DPG method. Cracow University of Technology, Cracow, June 25-25, 2011, (preeceding HOFEIM 2011).
- One day short course on DPG method. US Congress on Computational Mechanics, Minneapolis, July 24, 2011.

Thank You!