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Lecture Notes on
ENERGY SPACES

Institute for Computational Engineering and Sciences
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Contents

Preface

The Spring Semester 2018 marks my fourth attempt to teach Sobolev spaces or, as I prefer to call them, the *energy spaces* to graduate students from our *Computational Science, Engineering and Mathematics* (CSEM) graduate program at the *Institute for Computational Engineering and Sciences* (ICES), at The University of Texas at Austin. This presentation is the class notes on the subject.

I have been dealing with Sobolev spaces (with a different degree of understanding) throughout my whole academic career but I had to restart my study on the subject with the adventure of proving error estimates for the *Projection-Based Interpolation* (PBI), see [11, 8, 5, 12, 10] for the mathematical aspects of the technology. I somehow survived with a heavy help of Ivo Babuška and Annalisa Buffa, but I began to see strongly the need for a deeper study on the subject of fractional Sobolev spaces and, in particular, the tricky $\tilde{H}^s$ spaces. Around that time, Mark Ainsworth recommended to me the fantastic book of McLean [18] and I began to develop my lecture notes based on his presentation of Sobolev spaces (a mere 50 pages in the book).

These notes, to a large degree, is a rewrite of McLean's chapter with extra details provided. In particular, I have tied the presentation to our book with J. Tinsley Oden [20] and have attempted the presentation to be self-contained. The last chapter presents perhaps the most original part of these notes, developed with my student - Federico Fuentes, and we will attempt to publish a summary of it in a small paper. None of these results are new, we have been learning the theory of traces for the $H(\text{curl})$ space from the ground breaking papers [3, 4, 19].

A very special thanks go to Martin Costabel who has been patiently teaching me the subject, answering multiple (not always smart) E-mails and questions, and correcting my mistakes.

Last but not least, understanding the energy spaces presented here is critical for the development of the *Discontinuous Petrov Galerkin* (DPG) Method co-invented with Jay Gopalakrishnan [13].

This is a second version of the notes, the original notes were placed on the web in May 2018. I have corrected a number of mistakes pointed out by Martin, and added Theorem 4.1.12. The notes come with many exercises including most of those from McLean's book. I have completed the theory manual. Please contact me for a copy of it if you intend to use the notes for teaching.

Leszek F. Demkowicz

Austin, August 2018

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Introduction

1.1 Variational Formulations

This section reviews a number of classical and less classical variational formulations for a few of standard model problems to motivate the study of energy spaces. For a more detailed discussion and more examples, see e.g. [7], [20], Section 6.6, and [14], Chapter 1. In all following examples, Ω is a domain (= open and connected set) in $\mathbb{R}^n$ with boundary Γ split into disjoint parts Γ_1, Γ_2.

Diffusion-Convection-Reaction Problem

The classical formulation reads as follows. Find a sufficiently regular* function u that satisfies the following Partial Differential Equation (PDE) and Boundary Conditions (BCs).

$$\begin{cases} -\dfrac{\partial}{\partial x_i}(a_{ij}(x)\dfrac{\partial u}{\partial x_j}) + b_j(x)\dfrac{\partial u}{\partial x_j} + c(x)uv = f(x) & x \in \Omega \\ u(x) = u_0(x) & x \in \Gamma_1 \\ a_{ij}(x)\dfrac{\partial u}{\partial x_j} n_i = \sigma_0(x) & x \in \Gamma_2 \end{cases}$$

or, using a more compact absolute notation,

$$\begin{cases} -\text{div}\,(a\boldsymbol{\nabla} u) + b\boldsymbol{\nabla} u + cuv = f & \text{in } \Omega \\ u = u_0 & \text{on } \Gamma_1 \\ (a\boldsymbol{\nabla} u)n = \sigma_0 & \text{on } \Gamma_2 \end{cases}$$

where coefficients (material data) $a = a_{ij}(x), b = b_j(x), c = c(x)$, and right-hand sides (load data) $f = f(x), u_0 = u_0(x), \sigma_0 = \sigma_0(x)$ are given. We can identify viscous flux $\sigma = a\boldsymbol{\nabla} u$ as a separate variable and rewrite the second order equation as a system of first order equations.

$$\begin{cases} \sigma - a\boldsymbol{\nabla} u = 0 & \text{in } \Omega \\ -\text{div}\,\sigma + b\boldsymbol{\nabla} u + cuv = f & \text{in } \Omega \\ u = u_0 & \text{on } \Gamma_1 \\ \sigma n = \sigma_0 & \text{on } \Gamma_2 \end{cases}$$

*$u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$

Standard variational formulation. Let

$$U = V := \{v \in H^1(\Omega) \, : \, v = 0 \quad \text{on } \Gamma_1\}$$

Find solution $u \in \tilde{u}_0 + U$ such that

$$(a\boldsymbol{\nabla} u, \boldsymbol{\nabla} v) + (b\boldsymbol{\nabla} u, v) + (cu, v) = (f, v) + \langle \sigma_0, v\rangle_{\Gamma_2} \, ,$$

for every $v \in V$. Here the trial space U and test space V are equal and consist of functions that, along with their first derivatives, are square integrable, and vanish on Γ_1. Function $\tilde{u}_0 \in H^1(\Omega)$ is a finite-energy lift of boundary data u_0. (u, v) denotes the $L^2(\Omega)$ inner product, and $\langle \sigma_0, v\rangle_{\Gamma_2}$ is the L^2 inner product on boundary part Γ_2. Derivatives are understood in the sense of distributions and vanishing on Γ_1 in the sense of traces. These notes are supposed to help you to understand all these notions more precisely.

Eliminating boundary conditions on test functions. Each variational formulation leads to a separate Finite Element (FE) approximation. It may be convenient to eliminate the boundary conditions on test functions and redefine the test space,

$$V = H^1(\Omega) \, .$$

We are testing now with more test functions so we expect additional unknowns. Indeed, the additional unknown is the flux $\hat{\sigma}$ on the boundary. The new formulation looks as follows.

$$\begin{cases} u \in H^1(\Omega)\, , & u = u_0 \qquad \text{on } \Gamma_1 \\ \hat{\sigma} \in H^{-\frac{1}{2}}(\Gamma)\, , & \hat{\sigma} = \sigma_0 \qquad \text{on } \Gamma_2 \\ (a\boldsymbol{\nabla} u, \boldsymbol{\nabla} v) + (b\boldsymbol{\nabla} u, v) + (cu, v) - \langle \hat{\sigma}, v\rangle = (f, v) & v \in H^1(\Omega) \end{cases}$$

Above $H^{-\frac{1}{2}}(\Gamma)$ denotes the trace space of another energy space,

$$H(\text{div}, \Omega) := \{v \in (L^2(\Omega)^3 \, : \, \text{div}\, v \in L^2(\Omega)\}$$

where the divergence is again understood in the sense of distributions. We will spend a lot of time discussing these two energy spaces as well.

Ultraweak variational formulation. You may want to develop variational formulations starting with the first order system rather than the second order equation. The ultraweak variational formulation is the most "relaxed" one as it seeks solution in the $L^2(\Omega)$ space.

$$\begin{cases} \hat{u} \in H^{\frac{1}{2}}(\Gamma)\, , & \hat{u} = u_0 \qquad \text{on } \Gamma_1 \\ \hat{\sigma} \in H^{-\frac{1}{2}}(\Gamma)\, , & \hat{\sigma} = \sigma_0 \qquad \text{on } \Gamma_2 \\ u \in L^2(\Omega),\, \sigma \in (L^2(\Omega))^3 & \\ (a^{-1}\sigma, \tau) + (u, \text{div}\tau) - \langle \hat{u}, \tau n\rangle \quad = 0 & \tau \in H(\text{div}, \Omega) \\ (\sigma, \boldsymbol{\nabla} v) - \langle \hat{\sigma}, v\rangle + (b\boldsymbol{\nabla} u + cu, v) = (f, v) & v \in H^1(\Omega) \end{cases}$$

Above a^{-1} denotes the inverse of diffusion matrix a, and $H^{\frac{1}{2}}(\Gamma)$ denotes the trace space of space $H^1(\Omega)$, to be discussed. Note that we have now two unknowns that live on the boundary and two unknowns defined on Ω.

Time Harmonic Maxwell Equations

We seek electric (complex-valued) electric field $E = E(x)$ and magnetic field $H(x)$ satisfying:

- Faraday's law in Ω,
$$\frac{1}{\mu}\boldsymbol{\nabla} \times E = -i\omega H \,,$$
- Ampère's law in Ω,
$$\boldsymbol{\nabla} \times H = J^{\text{imp}} + \sigma E + i\omega\epsilon E \,.$$
- BCs on the electric field on Γ_1,
$$n \times E = n \times E_0 \,,$$
- BCs on the magnetic field on Γ_2,
$$n \times H = n \times H_0 \,,$$

Above, μ, ϵ, σ denote permeability, permittivity and conductivity (functions of position x), ω is the angular frequency and $J^{\text{imp}} = J^{\text{imp}}(x)$ is a prescribed impressed current.

Eliminating magnetic field H, we obtain a second order curl-curl problem,

$$\begin{cases} \boldsymbol{\nabla} \times (\frac{1}{\mu}\boldsymbol{\nabla} \times E) - (\omega^2\epsilon - i\omega\sigma)E = -i\omega J^{\text{imp}} & \text{in } \Omega \\ n \times E = n \times E_0 & \text{on } \Gamma_1 \\ n \times (\frac{1}{\mu}\boldsymbol{\nabla} \times E) = n \times (-i\omega H_0) := i\omega J_S^{\text{imp}} & \text{on } \Gamma_2 \end{cases}$$

where J_S^{imp} is termed to be the impressed surface current.

Standard variational formulation looks as follows.

$$\begin{cases} E \in H(\text{curl}, \Omega), \qquad n \times E = n \times E_0 & \text{on } \Gamma_1 \\ (\frac{1}{\mu}\boldsymbol{\nabla} \times E, \boldsymbol{\nabla} \times F) - ((\omega^2\epsilon - i\omega\sigma)E, F) = -i\omega(J^{\text{imp}}, F) + i\omega\langle J_S^{\text{imp}}, F\rangle_{\Gamma_2} \end{cases}$$

for every test function $F \in H(\text{curl}, \Omega) \,:\, F_t = 0 \quad \text{on } \Gamma_1$. Here $F_t = -n \times (n \times F)$ is the (standard) tangential component of F on the boundary. We have arrived at a new energy space,

$$H(\text{curl}, \Omega) := \{E \in L^2(\Omega) \,:\, \boldsymbol{\nabla} \times E \in (L^2(\Omega))^3\}$$

where the curl operator is understood in the distributional sense. The BCs above are understood again in the sense of traces. The $H(\text{curl}, \Omega)$ energy space and the corresponding two traces E_t and $n \times E = n \times E_t$ are a central focus of these notes.

Eliminating BCs on the test functions is again possible. It leads to a new unknown - the magnetic field (impressed surface current) on the boundary.

$$\begin{cases} E \in H(\text{curl}, \Omega), & n \times E = n \times E_0 \quad \text{on } \Gamma_1 \\ \hat{H} \in H^{-\frac{1}{2}}(\text{curl}, \Gamma)\,, & n \times \hat{H} = n \times H_0 \quad \text{on } \Gamma_2 \\ (\frac{1}{\mu}\boldsymbol{\nabla} \times E, \boldsymbol{\nabla} \times F) - i\omega\langle n \times \hat{H}, F\rangle - ((\omega^2\epsilon - i\omega\sigma)E, F) = -i\omega(J^{\text{imp}}, F) & F \in H(\text{curl}, \Omega) \end{cases}$$

The new energy space $H^{-\frac{1}{2}}(\text{curl}, \Gamma)$, perhaps the most difficult subject of these notes, is the (first) trace space of $H(\text{curl}, \Omega)$.

Ultraweak variation formulation. We finish with the most relaxed formulation derived directly from the first order system, see [6] for details.

$$\begin{cases} \hat{E} \in H^{-\frac{1}{2}}(\text{curl}, \Gamma)\,, & n \times \hat{E} = n \times E_0 \quad \text{on } \Gamma_1 \\ \hat{H} \in H^{-\frac{1}{2}}(\text{curl}, \Gamma)\,, & n \times \hat{H} = n \times H_0 \quad \text{on } \Gamma_2 \\ E, F \in (L^2(\Omega))^3 & \\ (E, \boldsymbol{\nabla} \times F) - \langle n \times \hat{E}, F\rangle + i\omega(\mu H, F) = 0 & F \in H(\text{curl}, \Omega) \\ (H, \boldsymbol{\nabla} \times F) - \langle n \times \hat{H}, F\rangle - ((\sigma + i\omega\epsilon)E, F) = (J^{\text{imp}}, F) & F \in H(\text{curl}, \Omega) \end{cases}$$

Notice the new unknown - boundary trace $\hat{E}$ of the electric field.

REMARK 1.1.1 The presentation on Maxwell equations is very simplified. In reality, we have to satisfy two more equations: Gaussian law for magnetic field and so-called continuity equation. For $\omega \neq 0$, the extra equations are linearly dependent and they are automatically satisfied at the continuous level (in some sense depending upon the variational formulation and functional setting). The whole art of discretization of Maxwell equations is to assure that, with a proper discretization, these extra equations are also well approximated on the discrete level. This leads to the concept of differential complex and exact sequence discussed in Section 4.1. For more details on the subject, see [9, 14]. ∎

The next two exercises represent a bit a “cart before horses” as they require an elementary knowledge of energy spaces in one space dimension. They are supposed to motivate studying these notes and refresh your knowledge of fundamental results from Functional Analysis including the *Closed Range Theorem* (for both continuous and closed operators) and the *Babuška-Nečas Theorem*.

Exercises

Exercise 1.1.1 Study the simplest boundary value problem,

$$u' = f \quad \text{in } (0, l)\,,$$

with each of the following boundary conditions:

(i) Inflow BC: $u(0) = 0$.

(ii) Outflow BC: $u(l) = 0$.

(iii) Inflow and outflow BC together: $u(0) = 0, u(l) = 0$.

(iv) No BCs at all.

Discuss the well-posedness (existence, uniqueness, possible compatibility conditions for f, continuous dependence upon data f) within each of the following formalisms:

1. closed operator setting in L^2,
2. continuous operator setting in $\mathcal{L}(H^1, L^2)$,
3. trivial variational formulation,
4. weak variational formulation.

Discuss relations between the different formulations and the corresponding stability constants using Closed Range Theorem (for both closed and continuous operators). Identify clearly dual (transpose) operators and discussed their role. Consult [20], Section 6.6.

Exercise 1.1.2 Repeat Exercise 1.1.1 for another "baby problem",

$$-u'' + u = f \quad \text{in } (0, l)\,,$$

with the boundary conditions:

$$u(0) = 0 \qquad u'(l) = 0\,.$$

Discuss first the well-posedness using the "strong formulations": closed operator setting in L^2, and continuous operator setting in $\mathcal{L}(H^2, L^2)$. Next, identify the derivative as a new unknown, $\sigma = u'$, and reformulate the problem using the first order setting. Following the discussion in Section 1.1, consider then the four possible variational formulations: trivial, classical, mixed, and ultraweak. Discuss the relation between the corresponding stability constants.

2

Preliminaries

2.1 L^p Functions

This section reviews some fundamental facts about the L^p functions. Let $f \in L^p(\Omega)$ and $v \in L^{p_*}(\Omega)$ for some $p \in (1,\infty),\ \ 1/p + 1/p^* = 1$. Hölder inequality states that

$$|\int_\Omega fv| \leq \|f\|_{L^p(\Omega)} \|v\|_{L^{p_*}(\Omega)}\,,$$

i.e., linear functional in v defined by the left-hand side is continuous on $L^{p_*}(\Omega)$ with the norm bounded by (in fact equal to) the L^p-norm of function f. It turns out that the converse of this statement is true as well. If a function f generates a bounded functional in v then f must be an L^p function with a bound on its norm.

THEOREM 2.1.1 (Duality argument)

Let $p \in [1,\infty]$, and let f be a measurable, complex-valued function defined on a measurable set $\Omega \subset \mathbb{R}^n$. Assume that

$$|\int_\Omega fv\,dx| \leq M\|v\|_{L^{p_*}(\Omega)} \qquad \forall v \in L^{p_*}(\Omega)\,,$$

with some constant M. Then $f \in L^p(\Omega)$, and

$$\|f\|_{L^p(\Omega)} \leq M\,.$$

PROOF

Case: $p \in (1,\infty)$.

Step 1: $f \geq 0$, $m(\Omega) < \infty$.

Suppose to the contrary that $\|f\|_{L^p(\Omega)} = \infty$. Define a sequence of functions $f_n := \min\{f, n\}$. Then f_n is measurable and bounded. Testing with $v = f_n^{p-1}$, we obtain,

$$\int_\Omega f_n^p = \int_\Omega f_n f_n^{p-1} \leq \int_\Omega f f_n^{p-1} \leq M \underbrace{\left(\int_\Omega f_n^p\right)^{1-\frac{1}{p}}}_{\|v\|_{L^{p_*}(\Omega)}}$$

and, therefore,

$$\left(\int_\Omega f_n^p\right)^{1/p} \le M\,.$$

But, by the monotone convergence theorem ([20], Lemma 3.5.1),

$$\int_\Omega f_n^p \to \int_\Omega f^p = \infty \quad \text{as } n \to \infty\,,$$

a contradiction.

Step 2: $f \ge 0$, Ω arbitrary.

Let χ_n be the indicator function of $\Omega_n := B(0,n) \cap \Omega$. We have

$$|\int_{\Omega_n} fv| = |\int_\Omega f\chi_n v| \le M\|\chi_n v\|_{L^p(\Omega)} = M\|v\|_{L^p(\Omega_n)}\,.$$

By Step 1,

$$\int_\Omega (f\chi_n)^p = \int_\Omega \chi_n f^p = \int_{\Omega_n} f^p \le M^p\,.$$

But again, by the monotone convergence theorem,

$$\int_\Omega (f\chi_n)^p \to \int_\Omega f^p\,,$$

and the bounds holds in the limit.

Step 2: f arbitrary, Ω arbitrary.

Define

$$\phi = \begin{cases} 1 & \text{if } f = 0 \\ \frac{|f|}{f} & \text{if } f \ne 0\,. \end{cases}$$

We have,

$$|\int_\Omega |f|\,\phi v| = |\int_\Omega f\phi^{-1}\phi v| = |\int_\Omega fv| \le M\|v\|_{L^{p*}(\Omega)} = M\|\phi v\|_{L^{p*}(\Omega)}\,.$$

By Step 2, $|f| \in L^p(\Omega)$ and, therefore, $f \in L^p(\Omega)$ as well, and the bound holds.

Case: $p = 1$.

Just test with

$$v = \begin{cases} \overline{f}/|f| & f \ne 0 \\ 1 & f = 0\,. \end{cases}$$

Case: $p = \infty$.

Let E be an arbitrary measurable subset of Ω. Test with the same v as above but premultiplied with indicator function χ_E to obtain:

$$\frac{1}{m(E)}\int_E |f| \le M\,.$$

This implies that $\|f\|_{L^\infty(\Omega)} \le M$. Indeed, if there were $|f(x)| > M$ on some subset E of positive measure, then the average of $|f|$ over the very set would be strictly greater than M. ∎

REMARK 2.1.1 Contrary to Theorem 2.1.1, the fundamental representation theorem for the dual of L^{p_*} ([20], Theorem 5.12.1)

$$(L^{p_*})' \sim L^p \, ,$$

does not hold for $p_* = \infty$ as there exist bounded linear functionals on $L^\infty(\Omega)$ that are not generated by L^1-functions. But here the job is easier, we confine ourselves to functionals generated by functions to deduce only the integrability properties of the generating function. ∎

Distribution function. Let $\Omega \subset \mathbb{R}^n$ be an open set, and u be a real- or complex-valued function such that $|u(x)|$ is finite almost everywhere. Let $\sigma > 0$. We introduce the *distribution function* of function u defined by:

$$m(\sigma, u) := \text{meas}(\{x \in \Omega \, : \, |u(x)| > \sigma\}) \tag{2.1}$$

where meas denotes the Lebesgue measure. Function $m(\sigma, u)$, defined on $(0, \infty)$ takes values in $[0, \infty]$, i.e., it is an extended real-valued function. Clearly, $m(\sigma, u)$ is (weakly) decreasing. It can be proved that $m(\sigma, u)$ is continuous from the right, Additionally, if $u \in L^p(\Omega)$, $p \in [1, \infty]$, then

$$\|u\|_{L^p(\Omega)} = \left(p \int_0^\infty \sigma^p m(\sigma, u) \, \frac{d\sigma}{\sigma} \right)^{1/p} \qquad 1 \le p < \infty$$
$$\|u\|_{L^\infty(\Omega)} = \inf\{\sigma \, : \, m(\sigma, u) = 0\} \, ,$$

see Exercise 2.1.1. If no confusion occurs, we will abbreviate symbol "meas" for Lebesgue measure to a single letter m.

***p*-mean modulus of continuity.** Let $u \in L^p(\mathbb{R}^n)$, $p \in [1, \infty)$. We define the p-mean modulus of continuity of function u by:

$$\omega_p(t, u) := \sup_{|h| \le t} \left(\int_{\mathbb{R}^n} |u(x+h) - u(x)|^p \, dx \right)^{1/p} . \tag{2.2}$$

Note that case $p = \infty$ is excluded. This is because of the crucial density result:

$$\overline{C_0(\Omega)}^{L^p} = L^p(\Omega)$$

which holds only for $p \in [1, \infty)$, comp. Exercises 4.9.1 - 4.9.4 in [20]. Here $C_0(\Omega)$ denotes the space of functions defined on Ω that are continuous and with a compact support contained in Ω.

Take an arbitrary $\epsilon > 0$. By the density result, there exists a function $g \in C_0(\Omega)$ such that

$$\|g - u\|_{L^p} \le \frac{\epsilon}{3} \, .$$

Recall that any continuous function defined on a compact set is automatically *uniformly continuous*, see Exercise 2.1.2. Thus, for any $\epsilon_1 > 0$ there exists a δ such that

$$|h| < \delta \quad \Rightarrow \quad |g(x+h) - g(x)| < \epsilon_1 \, .$$

Consequently,

$$\left(\int_{\mathbb{R}^n} |g(x+h) - g(x)|^p \, dx\right)^{1/p} \le (\epsilon_1^p \, m(\text{supp } g))^{1/p} = \epsilon_1 m(\text{supp } g)^{1/p}$$

Let $T_h g(x) := g(x+h)$. Take $\epsilon_1 = \frac{\epsilon}{3} \, m(\text{supp } g)^{1/p}$. We have thus,

$$\|T_h g - g\|_{L^p} \le \frac{\epsilon}{3} \qquad \forall \, |h| < \delta = \delta(\epsilon_1(\epsilon)) \, .$$

Finally,

$$\|T_h u - u\|_{L^p} \le \underbrace{\|T_h u - T_h g\|_{L^p}}_{=\|u-g\|_{L^p}} + \|T_h g - g\|_{L^p} + \|g - u\|_{L^p} \le \epsilon \, .$$

We have arrived at the following result.

PROPOSITION 2.1.1

Let $u \in L^p(\Omega)$, $p \in [1, \infty)$. Then

$$\omega_p(t, u) \to 0 \quad \text{as } t \to 0 \, .$$

The following result is a consequence of integral form of Minkowski inequality, see [20], Proposition 5.12.1.

THEOREM 2.1.2 (Hardy's Inequalities)

Let $\alpha > 0$ and $p \in [1, \infty)$. The following inequalities hold:

$$\left[\int_0^\infty \left(x^{-\alpha} \int_0^x |f(y)| \, \frac{dy}{y}\right)^p \, \frac{dx}{x}\right]^{\frac{1}{p}} \le \frac{1}{\alpha} \left[\int_0^\infty |y^{-\alpha} f(y)|^p \, \frac{dy}{y}\right]^{\frac{1}{p}} \tag{2.3}$$

and,

$$\left[\int_0^\infty \left(x^{\alpha} \int_x^\infty |f(y)| \, \frac{dy}{y}\right)^p \, \frac{dx}{x}\right]^{\frac{1}{p}} \le \frac{1}{\alpha} \left[\int_0^\infty |y^{\alpha} f(y)|^p \, \frac{dy}{y}\right]^{\frac{1}{p}} \tag{2.4}$$

PROOF Inequality (2.3):

$$\left[\int_0^\infty \left(x^{-\alpha}\int_0^x |f(y)|\,\frac{dy}{y}\right)^p \frac{dx}{x}\right]^{\frac{1}{p}} = \left[\int_0^\infty x^{-\alpha p}\left(\int_0^1 |f(xt)|\,\frac{dt}{t}\right)^p \frac{dx}{x}\right]^{\frac{1}{p}} \qquad \text{(change of variable: } y = xt\text{)}$$

$$\le \int_0^1 \left[\int_0^\infty x^{-\alpha p}|f(xt)|^p\,\frac{dx}{x}\right]^{\frac{1}{p}} \frac{dt}{t} \qquad \text{(Integral Minkowski Inequality)}$$

$$= \int_0^1 \left[\int_0^\infty (t^{-1}y)^{-\alpha p}|f(y)|^p\,\frac{dy}{y}\right]^{\frac{1}{p}} \frac{dt}{t} \qquad \text{(change of variable: } xt = y,\ \tfrac{dx}{x} = \tfrac{dy}{y}\text{)}$$

$$= \underbrace{\int_0^1 t^{\alpha-1}\,dt}_{=1/\alpha} \left[\int_0^\infty |y^{-\alpha}f(y)|^p\,\frac{dy}{y}\right]^{\frac{1}{p}}$$

Proof of inequality (2.4) is fully analogous. ∎

Exercises

Exercise 2.1.1 Prove that distribution function (2.1) is continuous from the right, and

$$\|u\|_{L^p(\Omega)} = \left(p\int_0^\infty \sigma^p m(\sigma,u)\,\frac{d\sigma}{\sigma}\right)^{1/p} \qquad 1 \le p < \infty$$

$$\|u\|_{L^\infty(\Omega)} = \inf\{\sigma \,:\, m(\sigma,u) = 0\}\,.$$

Exercise 2.1.2 Prove that any continuous function defined on a compact set must be uniformly continuous.

2.2 Convolutions

Let u, v denote complex-valued functions defined on the whole $\mathbb{R}^n$. The *convolution* of functions u and v is defined as:

$$(u * v)(x) := \int_{\mathbb{R}^n} u(x-y)v(y)\,dy\,.$$

We are implicitly assuming that product $u(x-\cdot)v(\cdot) \in L^1(\mathbb{R}^n)$. A simple change of variables shows that convolution is symmetric,

$$\begin{aligned} (u*v)(x) &= \int_{\mathbb{R}^n} u(x-y)v(y)\,dy \\ &= \int_{\mathbb{R}^n} u(z)v(x-z)\,dz \qquad (z = x-y) \\ &= (v*u)(x)\,. \end{aligned}$$

More precisely, if either the left- or right-hand side is well defined, the so is the other side, and they are equal. The definition can be extended to three functions,

$$u*v*w := (u*v)*w\,.$$

As the operation is associative (Exercise 2.2.1), i.e.,

$$(u*v)*w = u*(v*w)$$

we are justified to use the notation without any parentheses indicating the order of computing the convolutions. By induction, the notion extends to any finite number of functions.

The following theorem formulates sufficient conditions for the convolution to be well defined, and its continuity properties.

THEOREM 2.2.1

Let

$$\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}, \qquad p,q,r \in [1,\infty]\,,$$

and $u \in L^p(\mathbb{R}^n)$, $v \in L^q(\mathbb{R}^n)$. Then $u*v$ exists a.e. and

$$\|u*v\|_{L^r} \le \|u\|_{L^p}\,\|v\|_{L^q}\,.$$

PROOF Let $u \in L^p(\mathbb{R}^n), v \in L^q(\mathbb{R}^n)$ and $\phi \in L^{r_*}(\mathbb{R}^n)$ be a "testing function" where p,q,r satisfy the relation above. Let $\psi(x) := \phi(-x)$. The result from Exercise 2.2.2 implies that

$$|\langle u*v,\phi\rangle| \le |(u*v*\psi)(0)| \le \|u\|_{L^p}\,\|v\|_{L^q}\,\|\psi\|_{L^{r^*}} = \|u\|_{L^p}\,\|v\|_{L^q}\,\|\phi\|_{L^{r^*}}$$

Thus, by Theorem 2.1.1, $u*v \in L^r(\mathbb{R}^n)$ and the estimate holds. ∎

In the case of $r=\infty$, we can obtain a stronger result.

THEOREM 2.2.2

Let $p \in [1,\infty]$, and $u \in L^p(\mathbb{R}^n), v \in L^{p_*}(\mathbb{R}^n)$. Then *(i)* $u * v$ *is uniformly continuous in* $\mathbb{R}^n$.

(ii) *For* $p \in (1, \infty)$ *we control additionally behavior of* $u * v$ *at infinity,*

$$|(u * v)(x)| \to 0 \quad \text{as } |x| \to \infty\,.$$

PROOF

Case: $p < \infty$. We have,

$$|(u * v)(x) - (u * v)(y)| = |\int_{\mathbb{R}^n} (u(x - z) - u(y, z))v(z)\,dz| \le \omega_p(|x - y|, u)\,\|v\|_{L^{p_*}}$$

and the uniform continuity follows from Proposition 2.1.1.

Case: $p = \infty$. Switch u with v.

In order to prove the second part, we begin by noticing that the Lebesgue Theorem implies that

$$\int_{|y|>R} |u(y)|^p\,dy \to 0 \quad \text{as } R \to \infty\,.$$

Consequently,

$$\forall\,\epsilon \quad \exists\,R_0 > 0 \quad \forall\,R > R_0 \qquad \int_{|y|>R} |u(y)|^p\,dy < \epsilon^p \text{ and } \int_{|y|>R} |v(y)|^{p_*}\,dy < \epsilon^{p_*}\,.$$

Note that the argument breaks down for $p = \infty$. We have now,

$$\begin{aligned}
|(u * v)(x)| &\le |\int_{|y|\le R} u(x - y)v(y)\,dy| + |\int_{|y|>R} u(x - y)v(y)\,dy| \\
&\le \left(\int_{|y|\le R} |u(x - y)|^p\,dy\right)^{1/p} \|v\|_{L^{p_*}(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} \left(\int_{|y|>R} |v(y)|^{p_*}\,dy\right)^{1/p_*} \\
&\le \epsilon\|v\|_{L^{p_*}(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)}\,\epsilon
\end{aligned}$$

provided $|x| \ge 2R$. Notice that $|x - y| \ge R$ for $|x| \ge 2R$ and $|y| \le R$. ∎

Exercises

Exercise 2.2.1 Associativity of convolutions. Prove that

$$(u * v) * w = u * (v * w)$$

for $u, v, w \in L^1(\mathbb{R}^n)$.

Exercise 2.2.2 ([18], Exercise 3.3) Let $u_j \in L^{p_j}(\mathbb{R}^n)$, $j = 1, \ldots, m$, $\frac{1}{p_1} + \ldots + \frac{1}{p_m} = m - 1$. Prove the bound:

$$|(u_1 * \ldots * u_m)(x)| \le \|u_1\|_{L^{p_1}(\mathbb{R}^n)} \cdots \|u_m\|_{L^{p_m}(\mathbb{R}^n)}$$

for any $x \in \mathbb{R}^n$.

Outline of the proof: Let $f_j \in L^1(\mathbb{R}^n),\ j = 1, \ldots, m$ be non-negative functions with compact supports and unit L^1-norms. Let $0 \leq \lambda_j \leq 1,\ j = 1, \ldots, m$. Fix $x \in \mathbb{R}^n$ and consider the function:

$$g(\lambda) := (f_1^{\lambda_1} * \cdots * f_m^{\lambda_m})(x) \qquad \lambda := (\lambda_1, \ldots, \lambda_m)$$

with $f(x)^0 := 1$ for any value of $f(x)$.

(i) Show that $g(\tilde{e}_j) = 1$ where

$$\tilde{e}_j := (1, \ldots, \underbrace{0}_{j}, \ldots, 1) \in \mathbb{R}^m .$$

(ii) Use the fact that, for any positive a_j and non-negative λ_j, μ_j,

$$\prod_{j=1}^{m} a_j^{(1-t)\lambda_j + t\mu_j} = \exp\left((1-t)\sum_{j=1}^{m} \ln a_j^{\lambda_j} + t\sum_{j=1}^{m} \ln a_j^{\mu_j}\right)$$

to show that the function $g : [0,1]^m \to [0, \infty)$ is convex.

(iii) Deduce that $g(\lambda) \leq 1$ if $\lambda_1 + \ldots + \lambda_m = m - 1$. *Hint:* $\lambda = \sum_{j=1}^{m}(1 - \lambda_j)\tilde{e}_j$.

(iv) Deduce the final result.

Exercise 2.2.3 Let $u, v \in L^1(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$. Show that,

$$\text{supp}\ u * v \subset \text{supp}\ u + \text{supp}\ v := \{x + y\ :\ x \in \text{supp}\ u, y \in \text{supp}\ v\}$$

if supp u or supp v is bounded.

2.3 Differentiation

Multiindex notation. We shall use the standard multiindex notation for partial derivatives. Let u be a complex-valued function defined on an open set $\Omega \subset \mathbb{R}^n$. Partial derivatives of u will be denoted by:

$$\partial^\alpha u(x) := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x^{\alpha_n}}(x)$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, \quad |\alpha| = \alpha_1 + \ldots + \alpha_n .$$

Differentials. Let $x \in \Omega$. The first differential of function u at x, denoted $d_x u$, is a linear functional on $\mathbb{R}^n$, $d_x u \in (\mathbb{R}^n)^*$,

$$(d_x u)(y) = (d_x u)(\sum_{i=1}^n y_i e_i) = \sum_{i=1}^n \underbrace{(d_x u)(e_i)}_{= \frac{\partial u}{\partial x_i}(x)} y_i = \sum_{i=1}^n \frac{\partial u}{\partial x_i}(x)\, y_i \,.$$

The second differential of function u at x, denoted $d_x^2 u$, is a bilinear, symmetric functional on $\mathbb{R}^n$, $d_x^2 u \in M^2_{\text{sym}}(\mathbb{R}^n)$,

$$\begin{aligned}(d_x^2 u)(y,y) = (d_x^2 u)(\sum_{i=1}^n y_i e_i, \sum_{j=1}^n y_j e_j) &= \sum_{i=1}^n \sum_{j=1}^n y_i y_j \underbrace{(d_x^2 u)(e_i, e_j)}_{= \frac{\partial^2 u}{\partial x_i \partial x_j}(x)} \\ &= \sum_{|\alpha|=2} \frac{2!}{\alpha_1! \cdots \alpha_n!}\ \partial^\alpha u(x)\ y_1^{\alpha_1} \cdots y_n^{\alpha_n} \\ &= \sum_{|\alpha|=2} \frac{2!}{\alpha!}\ \partial^\alpha u(x)\ y^\alpha\end{aligned}$$

where

$$y^\alpha := y_1^{\alpha_1} \cdots y_n^{\alpha_n}, \qquad \alpha! := \alpha_1! \cdots \alpha_n! \quad .$$

Notice how the multiindex notation helps us to avoid using two separate indices i and j. The k-th differential is a k-linear, symmetric functional on $\mathbb{R}^n$, $d_x^k u \in M^k_{\text{sym}}(\mathbb{R}^n)$,

$$(d_x^k u)(\underbrace{y, \ldots, y}_{n \text{ times}}) = \sum_{|\alpha|=k} \frac{k!}{\alpha!}\ \partial^\alpha u(x)\ y^\alpha \,.$$

We can write now a particular version of the Taylor formula in a very compact form,

$$u(x+y) = \sum_{k=0}^m \frac{1}{k!} d_x^k(y, \ldots, y) + \frac{1}{k!} \int_0^1 (1-t)^k d_{x+ty}^{m+1}(y, \ldots, y)\, dt \,.$$

If we define the k-th derivative of u, denoted $u^{(k)}$, as the function that for each $x \in \Omega$, prescribes the corresponding k-th order differential at x,

$$u^{(k)}\ :\ \Omega \ni x \to d_x^k u \in M^k_{\text{sym}}(\mathbb{R}^n)\,,$$

then we can rewrite the Taylor formula in a form resembling its 1D version,

$$u(x+y) = \sum_{k=0}^m \frac{1}{k!} u^{(k)}(x)(y, \ldots, y) + \frac{1}{k!} \int_0^1 (1-t)^k u^{(k+1)}(x+ty)(y, \ldots, y)\, dt \,. \tag{2.5}$$

THEOREM 2.3.1 (Differentiation and convolution commute)

Let $k \in \mathbb{N}$, $p \in [1, \infty]$. Let $u \in C_0^k(\mathbb{R}^n)$ and $v \in L^p(\mathbb{R}^n)$. Then

(i) $u * v \in C^k(\mathbb{R}^n)$ *and,*

(ii) $\partial^\alpha (u * v) = (\partial^\alpha u) * v \qquad \forall |\alpha| \le k.$

PROOF

Case: $k = 0$ has already been proved, see Theorem 2.2.2.

Case: $k = 1$. Denote the finite difference corresponding to partial derivative by:

$$(\Delta_{l,h}u)(x) := \frac{u(x + he_l) - u(x)}{h} . \tag{2.6}$$

We have,

$$\frac{(u * v)(x + he_l) - (u * v)(x)}{h} = \int_{\mathbb{R}^n} \frac{u(x + he_l - z) - u(x - z)}{h} v(z)\, dz ,$$

i.e., the finite difference and convolution commute,

$$\Delta_{l,h}(u * v) = (\Delta_{l,h}u) * v .$$

We have thus the estimate,

$$|\Delta_{l,h}(u * v)(x) - (\partial_l u * v)(x)| \le |((\Delta_{l,h}u - \partial_l u) * v)(x)| \le \|\Delta_{l,h}u - \partial_l u\|_{L^{p*}} \|v\|_{L^p} .$$

As derivatives of function u are continuous and have compact support, we have a global, pointwise bound

$$|\partial_l u(x)| \le C$$

for some C. By the Mean-Value Theorem, finite difference (2.6) equals derivative $\partial_l u(x + \xi he_l)$ for some $\xi \in [0, 1]$ and, therefore, it is bounded by constant C as well. Consequently,

$$|\frac{u(x + he_l) - u(x)}{h} - \partial_l u(x)| \le 2C .$$

We have thus a trivial dominating function and, by the Lebesgue Theorem, pointwise convergence $\Delta_{l,h} \to \partial_l u$ implies global convergence,

$$\int_{\mathbb{R}^n} |\Delta_{l,h}u - \partial_l u|^{p_*} = \int_{\text{supp } u} |\Delta_{l,h}u - \partial_l u|^{p_*} \to 0 \quad \text{as } h \to 0 .$$

Case: $k > 1$. Use induction. ∎

Theorem 2.3.1 leads to the concept of *smoothing by convolution.* Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be an arbitrary non-negative function with a support in the unit ball and unit integral,

$$\psi \ge 0, \qquad \psi(x) = 0 \text{ for } |x| > 1, \qquad \int_{\mathbb{R}^n} \psi = 1 .$$

We scale ψ with an $\epsilon > 0$,

$$\psi_\epsilon(x) := \epsilon^{-n}\psi(\epsilon^{-1}x), \qquad \psi_\epsilon(x) = 0 \text{ for } |x| > \epsilon, \qquad \int_{\mathbb{R}^n} \psi_\epsilon = 1 . \tag{2.7}$$

By Theorem 2.3.1, $\psi_\epsilon * u$ is C^∞ and we have the following convergence result.

THEOREM 2.3.2

Let $u \in L^p(\mathbb{R}^n)$, $p \in [1, \infty)$. *Then* *(i)* $\|\psi_\epsilon * u\|_{L^p} \leq \|u\|_{L^p}$ *and,*

(ii) $\|\psi_\epsilon * u - u\|_{L^p} \leq \omega_p(\epsilon, u)$ *.*

PROOF The first assertion follows immediately from Theorem 2.2.1,

$$\|\psi_\epsilon * u\|_{L^p} \leq \|\psi_\epsilon\|_{L^1} \, \|u\|_{L^p} = \|u\|_{L^p} \, .$$

We have,

$$(\psi_\epsilon * u)(x) - u(x) = (u * \psi_\epsilon)(x) - u(x) \int_{\mathbb{R}^n} \psi_\epsilon(y)\, dy = \int_{|y|\leq\epsilon} [u(x-y) - u(x)]\, \psi_\epsilon(y)\, dy \, .$$

$$\begin{aligned}
|\langle \psi_\epsilon * u - u, \phi\rangle| &= |\int_{\mathbb{R}^n} \int_{|y|\leq\epsilon} |u(x-y) - u(x)|\, \psi_\epsilon(y)\, dy\, \phi(x)\, dx \\
&= \int_{|y|\leq\epsilon} \int_{\mathbb{R}^n} |u(x-y) - u(x)|\, \psi_\epsilon(y)\, \phi(x)\, dx\, dy \\
&\leq \int_{|y|\leq\epsilon} \omega_p(\epsilon, u)\, \|\phi\|_{L^{p*}}\, \psi_\epsilon(y)\, dy \\
&= \omega_p(\epsilon, u)\, \|\phi\|_{L^{p*}} \, .
\end{aligned}$$

Use Theorem 2.1.1 to finish the argument. ∎

Recall that continuous functions with compact support are dense in $L^p(\Omega)$ for $p \in [1, \infty)$. We can upgrade the density result now to C^∞-functions.

COROLLARY 2.3.1

$C_0^\infty(\Omega)$ *is dense in* $L^p(\Omega)$ *for* $p \in [1, \infty)$.

PROOF Select a sequence of compact sets K_j such that

$$K_1 \subset\subset K_2 \subset\subset \ldots K_j \subset\subset K_{j+1} \subset\subset \ldots \quad \Omega \qquad \Omega = \bigcup_{j=1}^{\infty} K_j \, .$$

You can take for instance,

$$K_j := \{x \in \Omega \, : \, d(x, \mathbb{R}^n - \Omega) \geq \frac{1}{j}, \, |x| \leq j\} \, .$$

Let χ_j be the indicator function of set K_j. Consider,

$$u_j := u\chi_j, \quad u_{j,\epsilon} := \psi_\epsilon * u_j \, .$$

Note that the convolution used to define $u_{j,\epsilon}$ is well-defined for sufficiently small ϵ. We have,

$$\|u_{j,\epsilon} - u\|_{L^p} \leq \|u_{j,\epsilon} - u_j\|_{L^p} + \|u_j - u\|_{L^p} \, .$$

The second term converges to zero by the standard Lebesgue Theorem argument, the first one by Theorem 2.3.2. ***Question:*** Where have we used the assumption that $p \neq \infty$? ∎

We finish this section with an approximation result for indicator functions.

THEOREM 2.3.3 (External Approximation of Indicator Function)

Let $F \subset \mathbb{R}^n$ be an arbitrary closed set. There exists a function $\chi_\epsilon \in C^\infty(\mathbb{R}^n)$ such that

$$\begin{array}{ll} \chi_\epsilon(x) = 1 & x \in F \\ 0 \leq \chi_\epsilon(x) \leq 1, \quad |\partial^\alpha \chi_\epsilon(x)| \leq C(\alpha)\,\epsilon^{-|\alpha|} & 0 < d(x,F) < \epsilon \\ \chi_\epsilon(x) = 0 & d(x,F) \geq \epsilon\,. \end{array}$$

PROOF Let $v_\epsilon \in L^\infty(\mathbb{R}^n)$ be defined by:

$$v_\epsilon(x) = \begin{cases} 1 & d(x,F) < \epsilon \\ 0 & d(x,F) \geq \epsilon\,. \end{cases}$$

Let ψ_ϵ be a function like in (2.7). Set $\chi_\epsilon := \psi_{\epsilon/4} * v_{\epsilon/2}$. Then function χ_ϵ satisfies the vanishing conditions listed above (explain, why?) and we have the estimate:

$$\begin{aligned} |(\partial^\alpha \chi_\epsilon)(x)| &= |(\partial^\alpha \psi_{\epsilon/4} * v_{\epsilon/2})(x)| = |(v_{\epsilon/2} * \partial^\alpha \psi_{\epsilon/4})(x)| \\ &= \int_{\mathbb{R}^n} v_{\epsilon/4}(x-y) \partial^\alpha \psi_{\epsilon/4}(y)\,dy| \\ &\leq \int_{|y| \leq \epsilon/4} |\partial^\alpha \psi_{\epsilon/4}(y)|\,dy \leq \underbrace{C \|\partial^\alpha \psi\|_{L^\infty}}_{\text{depends upon } \alpha} \epsilon^{-n} \epsilon^{-|\alpha|} \epsilon^n \end{aligned}$$

as,

$$(\partial^\alpha \psi_\epsilon)(x) = \epsilon^{-n} \epsilon^{-|\alpha|} (\partial^\alpha \psi)(\epsilon^{-1} x)\,.$$

∎

COROLLARY 2.3.2

The dependence of the bounding constant upon index α is unavoidable. Consider the 1D case illustrated in Fig. 2.1. If all derivatives $\psi^{(m)}(\xi)$ of function ψ are bounded uniformly by some constant C, we have the estimate:

$$|\psi(x) - \underbrace{\sum_{k=1}^{m} \frac{1}{k!} \psi^{(m)}(0)\, x^m}_{=0}| = |\psi^{(m+1)}(\xi)| x^{m+1} \leq C x^{m+1} \qquad x \in (0,1)$$

with ξ being some intermediate point between 0 and x. Upon passing with $m \to \infty$, we get $\psi(x) = 0$, a contradiction.

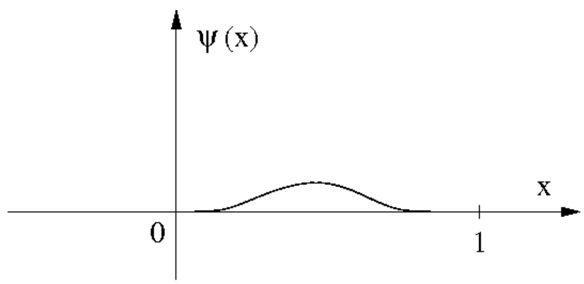

Figure 2.1
Function ψ.

Exercises

Exercise 2.3.1 ([18], Exercise 3.5) Derive Taylor formula (2.5).

Outline of the proof:

(i) Use integration by parts and induction to derive Taylor formula in one dimension:

$$f(s) = \sum_{j=0}^{m} \frac{f^{(j)}(0)}{j!} s^j + \frac{s^{m+1}}{m!} \int_0^1 (1-t)^m f^{(m+1)}(ts)\, dt\,.$$

(ii) Take $f(s) := u(x+sy)$ and apply the 1D formula.

Exercise 2.3.2 Differentiation of product of functions. Prove the formula:

$$\partial^\alpha (fg) = \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} \partial^\gamma f \; \partial^{\alpha-\gamma} g\,.$$

2.4 Distributions

L^1_{loc} functions. Let $\Omega \subset \mathbb{R}^n$ be an open set. Consider a measurable function $u : \Omega \to \mathbb{C}$. The following two conditions are equivalent to each other, see Exercise 2.4.2.

$$\begin{gathered} \forall\, x \in \Omega \quad \exists B = B(x,\epsilon) \quad u \in L^1(B) \\ \forall \text{ compact } K \subset \Omega \quad u \in L^1(K) \end{gathered} \tag{2.8}$$

We say that function u is *locally integrable* and denote the space of such functions by $L^1_{loc}(\Omega)$. We will use the smoothing by convolution technique to prove the following fundamental result.

THEOREM 2.4.1

Let $u \in L^1_{loc}(\Omega)$ satisfy the orthogonality condition:

$$\int_\Omega u\phi = 0 \qquad \forall\, \phi \in C_0^\infty(\Omega)\,.$$

Then $u = 0$ a.e. in Ω.

PROOF Let $x \in \Omega$. Consider a ball $B(x, 2\delta) \subset \Omega$ and take ψ_ϵ like in (2.7). Let

$$u_B := \begin{cases} u & x \in B(x,2\delta) \\ 0 & \text{otherwise} \end{cases}\,, \qquad u_B \in L^1(\mathbb{R}^n)\,.$$

Then, for $x \in B(0, \delta)$,

$$(\psi_\epsilon * u_B)(x) = \int_{\mathbb{R}^n} u_B(y)\psi_\epsilon(x-y)\,dy = \int_\Omega u(y)\psi_\epsilon(x-y)\,dy = 0 \quad \forall \epsilon < \delta\,,$$

and

$$\|(\underbrace{\psi_\epsilon * u_B}_{=0} - u_B\|_{L^1(B(x,\delta))} \le \|\psi_\epsilon * u_B - u_B\|_{L^1(\mathbb{R}^n)} \le \omega_1(\epsilon, u_B) \to 0 \quad \text{as } \epsilon \to 0\,.$$

Consequently, $u = 0$ a.e. in $B(x, \delta)$, i.e.

$$m(\{x \in B(x,\delta)\,:\, u(x) \neq 0\}) = 0\,.$$

This implies that $u = 0$ a.e. in every compact subset of Ω (explain, why). As Ω can be represented as a union of a countable family of compact subsets (see proof of Corollary 2.3.1), this in turn implies that $u = 0$ a.e. in Ω as well. ∎

Note that the result *does not* follow from a duality argument for spaces L^∞ as the test functions are not dense in L^∞.

Test functions and distributions. Space of infinitely differentiable functions with compact support in Ω, $C_0^\infty(\Omega)$, equipped with a very special topology of the *locally convex (l.c.) inductive topological limit*, see [20], pp. 366-371, is called the *space of (Schwartz) test functions* and denoted by $\mathcal{D}(\Omega)$. In principle, every time we use the symbol $\mathcal{D}(\Omega)$ instead of $C_0^\infty(\Omega)$, we emphasize the importance of the topology. Space $\mathcal{D}(\Omega)$ is not *first countable*, i.e. one cannot introduce it with countable bases of neighborhoods. In fact, it is not even *sequential*, i.e. notions of sequential continuity and continuity for functions are not equivalent. However, see [20], p. 414, a *linear functional* defined on $\mathcal{D}(\Omega)$ is continuous iff it is sequentially continuous. As the topological dual of the space of test functions, denoted $\mathcal{D}'(\Omega)$, is identified as the *space of distributions* on Ω, it is sufficient to work only with the notion of sequential continuity for distributions. It turns out that a sequence of test functions ϕ_j converges to zero in $\mathcal{D}(\Omega)$ iff there exists a compact set $K \subset \Omega$ such that

$$\text{supp}\,\phi_j \subset K,\ j = 1, \ldots \quad \text{and} \quad \sup_K |\partial^\alpha \phi_j| \to 0 \quad \forall \alpha\,,$$

see [20], Proposition 5.3.2. Consequently, a linear functional $f \in (C_0^\infty(\Omega))^*$ is a distribution if

$$\langle f, \phi_j \rangle := f(\phi_j) \to 0 \quad \text{for any } \phi_j \stackrel{\mathcal{D}(\Omega)}{\to} 0 \,.$$

Once we have introduced the dual space $\mathcal{D}'(\Omega)$, we need to decide about the topology for $\mathcal{D}'(\Omega)$. Recall that for a normed space V, its topological dual V' can be equipped with the strong topology introduced by the (dual) norm,

$$\|f\|_{V'} := \sup_v \frac{|\langle f, v \rangle|}{\|v\|_V} \,,$$

weak topology introduced by functionals from the bidual space V'', or weak* topology generated by V. For reflexive spaces, $V \sim V''$ and the weak and weak* topologies are the same. So, it should not be surprising that the topology in $\mathcal{D}'(\Omega)$ can be introduced in more than one way. In these notes we shall equip $\mathcal{D}'(\Omega)$ with the weak* topology and restrict ourselves to sequences only, i.e.,

$$f_j \to 0 \text{ in } \mathcal{D}'(\Omega) \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \langle f_j, \phi \rangle \to 0 \quad \forall \phi \in \mathcal{D}(\Omega) \,.$$

Occasionally, we shall also need a larger space of test functions, a *locally convex topological vector space (lctvs)*, $\mathcal{E}(\Omega) = C^\infty(\Omega)$, with the topology introduced by the family of seminorms

$$\sup_K |\partial^\alpha \phi|,$$

for any compact $K \subset \Omega$ and multiindex α. As the topology can be introduced with a countable set of compact sets, see proof of Collorary 2.3.1, space $\mathcal{E}(\Omega)$ is first countable and, consequently, continuity is equivalent to sequential continuity for any functional defined on $\mathcal{E}(\Omega)$, linear or not.

As the space $\mathcal{D}(\Omega)$ is dense in space $\mathcal{E}(\Omega)$, see Exercise 2.4.3, and

$$\phi_j \to 0 \text{ in } \mathcal{D}(\Omega) \quad \Rightarrow \quad \phi_j \to 0 \text{ in } \mathcal{E}(\Omega) \,,$$

see Exercise 2.4.4, i.e. space $\mathcal{D}(\Omega)$ is continuously embedded in space $\mathcal{E}(\Omega)$, we have automatically the continuous embedding for the dual spaces as well,

$$\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega) \,.$$

The embedding symbol communicates two facts.

(i) Let $\iota : \mathcal{D}(\Omega) \to \mathcal{E}(\Omega)$ be the continuous injection of $\mathcal{D}(\Omega)$ into $\mathcal{E}(\Omega)$. The transpose ι^T maps then dual $\mathcal{E}'(\Omega)$ into dual $\mathcal{D}'(\Omega)$,

$$\iota^T \,:\, \mathcal{E}'(\Omega) \ni f \to f \circ \iota \in \mathcal{D}'(\Omega) \,.$$

The density result implies that the transpose operator is *injective*, i.e.

$$\langle f, \phi \rangle = 0 \, \forall \phi \in \mathcal{D}(\Omega) \quad \Rightarrow \quad \langle f, \phi \rangle = 0 \, \forall \phi \in \mathcal{E}(\Omega) \,.$$

Only then we can talk about the “embedding”.

(ii) The transpose operator is always automatically continuous. In our case,

$$f_n \to 0 \text{ in } \mathcal{E}'(\Omega) \quad \Rightarrow \quad f_n \to 0 \text{ in } \mathcal{D}'(\Omega)$$

which is automatically satisfied for the weak* topologies. Indeed, the statement above means:

$$\langle f_n, \phi\rangle \to 0 \,\forall\phi \in \mathcal{E}(\Omega) \quad \Rightarrow \quad \langle f_n, \phi\rangle \to 0 \,\forall\phi \in \mathcal{D}(\Omega)$$

which is trivially satisfied since $\mathcal{D}(\Omega) \subset \mathcal{E}(\Omega)$.

We shall characterize space $\mathcal{D}'(\Omega)$ in a moment as the space of distributions with compact support, see Theorem 2.4.2.

Regular and irregular distributions. With every function $u \in L^1_{loc}(\Omega)$ we associate the corresponding linear functional $R_u \in (C_0^\infty(\Omega))^*$ defined by the Lebesgue integral,

$$\langle R_u, \phi\rangle := \int_\Omega u\phi\,.$$

Note that the integral is finite (Explain, why?) so the functional R_u is well-defined. Let $\phi_n \to 0$ in $\mathcal{D}(\Omega)$ and K be a compact set such that supp $\phi_j \subset K$. Then

$$|\int_\Omega u\phi_j| = |\int_K u\phi_j| \le \|u\|_{L^1(K)}\,\|\phi_j\|_{L^\infty(K)} \to 0 \quad \text{as } j \to 0\,,$$

so R_u is a distribution. Distributions generated by L^1_{loc}-functions are called *regular*. Any distribution that is not regular is identified as an *irregular distribution*. The first and perhaps the most important example of an irregular distribution is the famous *Dirac's delta*. We define the Dirac's delta a a point $x_0 \in \Omega$ by:

$$\mathcal{D}(\Omega) \ni \phi \to \langle \delta_{x_0}, \phi\rangle := \phi(x_0) \in \mathbb{C}\,.$$

Note that δ_{x_0} is trivially continuous on $\mathcal{D}(\Omega)$. In the case of $x_0 = 0$, we drop the index and use the simplified notation $\delta := \delta_0$. Suppose now that δ_{x_0} is a regular distribution, i.e., there exists a function $u \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega u\phi = \phi(x_0) \quad \forall\phi \in C_0^\infty(\Omega)\,.$$

Consequently,

$$\int_{\Omega-\{x_0\}} u\phi = 0 \quad \forall\phi \in C_0^\infty(\Omega - \{x_0\})\,.$$

By Theorem 2.4.1, $u = 0$ a.e. in $\Omega - \{x_0\}$ and, therefore, $u = 0$ a.e. in Ω, a contradiction since δ_{x_0} is not zero. In most of engineering and physics textbooks we encounter the integral symbol for Dirac's delta:

$$\langle \delta_{x_0}, \phi\rangle = \int_\Omega \delta(x - x_0)\phi(x)\,dx\,.$$

The integral above has nothing to do with the Lebesgue integral and should be understood simply as another symbol for the delta functional that, as we have just shown, cannot be generated by a function. In stronger words, the use of this symbol is mathematically illegal and should be avoided. We will give many more examples of irregular distributions.

Space $L^1_{loc}(\Omega)$ is continuously embedded in $\mathcal{D}'(\Omega)$, i.e., operator

$$R \,:\, L^1_{loc}(\Omega) \ni u \to Ru \in \mathcal{D}'(\Omega)$$

is injective and continuous. Injectivity follows from Theorem 2.4.1. In order to understand the continuity, we have first to "topologize" space $L^1_{loc}(\Omega)$. We equip $L^1_{loc}(\Omega)$ with l.c.t. introduced by the family of seminorms:

$$p_K(u) := \int_K |u|, \quad \text{compact } K \subset \Omega\,.$$

As we can restrict ourselves to a countable set of compact sets, this is a first countable topology. The continuity of the embedding now easily follows,

$$\int_K u_n \to 0 \quad \forall \text{ compact } K \subset \Omega \quad \Rightarrow \quad \int_\Omega u_n \phi \to 0 \quad \forall \phi \in \mathcal{D}'(\Omega)\,.$$

Restriction of a distribution. Many definitions for functions can be extended "by duality" to distributions. We start with the concept of a restriction. Let G be an open subset of Ω and $f \in \mathcal{D}'(\Omega)$ a distribution on Ω. The *restriction* of f to G, $f|_G \in \mathcal{D}'(G)$ is defined by

$$\langle f|_G, \phi \rangle := \langle f, \tilde{\phi} \rangle, \quad \phi \in \mathcal{D}(G)$$

where $\tilde{\phi}$ denotes the zero extension of test function ϕ to set Ω. Note that the zero extension of ϕ belongs to $\mathcal{D}(\Omega)$, and

$$\phi_n \to 0 \text{ in } \mathcal{D}(G) \quad \Rightarrow \quad \tilde{\phi}_n \to 0 \text{ in } \mathcal{D}(\Omega)\,.$$

Consequently, the restriction $f|_\Omega$ is well-defined.

Support of a distribution. Having defined the concept of restriction of a distribution, we can extend the notion of support from smooth functions to distributions. We cannot do it directly using the pointwise values of a distribution as they are undefined. Let $f \in \mathcal{D}'(\Omega)$. We define the support of f, denoted supp f to be the smallest (relatively) closed set in Ω such that the restriction of f to difference $\Omega -$ supp f vanishes. Explain why such *smallest* relatively closed set always exists. Equivalently,

$$\text{supp } f = \Omega - G$$

where G is the largest (relatively) open subset of Ω such that $f|_G = 0$. Note that, since Ω is open, G is relatively open in G iff G is open. For instance, according to the definition, Dirac's delta δ_x is supported at the single point x.

If $u \in L^1_{loc}(\Omega)$, we define the *essential support* of u, denoted ess supp$f$, to be the smallest (relatively) closed subset of Ω such that the restriction of u to difference $\Omega -$ ess supp$f$ is zero a.e. in $\Omega -$ ess supp$f$.

One can prove that the essential support of function u coincides with the support of the corresponding regular distribution R_u, see Exercise 2.4.5.

We are ready now to characterize space $\mathcal{E}'(\Omega)$ as distributions with compact support.

THEOREM 2.4.2 (Characterization of $\mathcal{E}'(\Omega)$)

We have:

$$\mathcal{E}'(\Omega) = \{u \in \mathcal{D}'(\Omega) \, : \, \text{supp } u \text{ is compact, and supp } u \subset \Omega\} \, .$$

PROOF

Case: $\supset$. Use Theorem 2.3.3 to fetch a function $\chi \in C_0^\infty(\Omega)$ such that $\chi = 1$ in an ϵ-neighborhood of supp u, and define:

$$\langle \tilde{u}, \phi \rangle := \langle u, \chi\phi \rangle \quad \forall \, \phi \in \mathcal{E}(\Omega) \, .$$

Let now $\phi \in \mathcal{E}(\Omega)$ and $\phi_j \in \mathcal{E}(\Omega)$ be a sequence converging to ϕ in $\mathcal{E}(\Omega)$. Then $\chi\phi_j \to \chi\phi$ in $\mathcal{D}(\Omega)$ and, therefore,

$$\langle \tilde{u}, \phi_j \rangle = \langle u, \chi\phi_j \rangle \to \langle u, \chi\phi \rangle = \langle \tilde{u}, \phi \rangle \, ,$$

i.e., $\tilde{u} \in \mathcal{E}'(\Omega)$. Take an arbitrary test function $\phi \in \mathcal{D}(\Omega)$. Then $(1-\chi)\phi \in \mathcal{D}(\Omega)$ and $(1-\chi)\phi = 0$ in the ϵ-neigborhood of supp u, so

$$\langle u, (1-\chi)\phi \rangle = 0 \, .$$

But this means that

$$\langle \tilde{u}, \phi \rangle = \langle u, \chi\phi \rangle = \langle u, \phi \rangle \, ,$$

i.e., $u = \tilde{u} \in \mathcal{E}'(\Omega)$.

Case: $\subset$. Suppose supp u is *not* compact. Take a sequence of compact sets like in the proof of Corollary 2.3.1,

$$K_1 \subset\subset K_2 \subset\subset \ldots \Omega \quad \Omega = \bigcup_{j=1}^{\infty} K_j \, .$$

If the support of u is not compact then

$$u|_{\Omega - K_j} \neq 0 \quad \forall j$$

(explain,why). Consequently, for every j, there exists a test function $\phi_j \in \mathcal{D}(\Omega)$ such that

$$\text{supp } \phi_j \subset \Omega - K_j, \quad \langle u, \phi_j \rangle \neq 0 \, .$$

We can normalize functions ϕ_j with $\langle u, \phi_j \rangle$ to obtain $\langle u, \phi_j \rangle = 1$. But, by the definition of topology in $\mathcal{E}(\Omega)$ (explain why),

$$\phi_j \to 0 \text{ in } \mathcal{E}(\Omega) \quad \Rightarrow \quad \langle u, \phi_j \rangle \to 0$$

a contradiction. ∎

Derivative of a distribution. Let $u \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}^n$. Motivated with the integration by parts formula, we define:

$$\langle \partial^\alpha u, \phi \rangle := (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle , \quad \phi \in \mathcal{D}(\Omega) .$$

Note that

$$\phi_j \stackrel{\mathcal{D}(\Omega)}{\longrightarrow} 0 \quad \Rightarrow \quad \partial^\alpha \phi_j \stackrel{\mathcal{D}(\Omega)}{\longrightarrow} 0 ,$$

so $\partial^\alpha u \in \mathcal{D}'(\Omega)$. Distributions, like C^∞ functions, have derivatives of any order.

Example 2.4.1

Consider one-dimensional case: $\Omega = (a,b)$. Let $u : (a,b) \to \mathbb{C}$ be a function consisting of two smooth branches:

$$u(x) := \begin{cases} u_1(x) & x \in (a, x_0) \\ \text{anything at} & x = x_0 \\ u_2(x) & x \in (x_0, b) \end{cases}$$

where $u_1 \in C^1[a, x_0]$ and $u_2 \in C^1[x_0, b]$. Let R_u be the regular distribution associated with function u. Definition of distributional derivative and elementary calculations show that (see e.g. [9], p.32)

$$\langle \frac{d}{dx} R_u, \phi \rangle = \langle R_{u'}, \phi \rangle + [u(x_0)] \phi(x_0)$$

or, in the argumentless notation,

$$\frac{d}{dx} R_u = R_{u'} + [u(x_0)] \delta_{x_0} .$$

Above, $\frac{d}{dx}$ denotes the distributional derivative, u' stands for the classical, pointwise derivative of u defined everywhere except for x_0, and $[u(x_0)] := u_2(x_0) - u_1(x_0)$ is the jump of u at point x_0. In particular, if function u is globally continuous, i.e., $[u(x_0)] = 0$, then

$$\frac{d}{dx} R_u = R_{u'} .$$

Explain why the value of u at x_0 does not matter. $\square$

Example 2.4.2

Example 2.4.1 generalizes to multidimensions. Let $\Omega \subset \mathbb{R}^n$ be a open set partitioned into two open subsets Ω_i, $i = 1, 2$, separated by interface Γ_0. More precisely,

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \quad \Omega_1 \cap \Omega_2 = \emptyset, \quad \Gamma_0 = \partial\Omega_1 \cap \partial\Omega_2 .$$

Let u be again a scalar-valued function consisting of two smooth branches:

$$u(x) := \begin{cases} u_1(x) & x \in \Omega_1 \\ \text{anything} & x \in \Gamma_0 \\ u_2(x) & x \in \Omega_2 . \end{cases}$$

where $u_i \in C^1(\overline{\Omega}_i)$, $i = 1, 2$. Let R_u again detote the regular distribution associated with function u. Definition of distributional derivative and elementary integration by parts formula lead to:

$$\langle \frac{\partial}{\partial x_i} R_u, \phi \rangle = \langle R_{u_{,i}}, \phi \rangle + \underbrace{\int_{\Gamma_0} [u] n_i \, \phi \, dS}_{=:\langle \delta_{\Gamma_0, [u] n_i}, \phi \rangle} \tag{2.9}$$

or, in the argumentless notation,

$$\frac{\partial}{\partial x_i} R_u = R_{u_{,i}} + \delta_{\Gamma_0, [u] n_i} \,.$$

Here, $\frac{\partial}{\partial x_i}$ denotes the distributional derivative of R_u, $u_{,i}$ stands for the classical, pointwise partial derivative $\frac{\partial u}{\partial x_i}$ defined everywhere except for $x \in \Gamma_0$, $[u](x) := u_2(x) - u_1(x)$, $x \in \Gamma_0$ is the jump of u across Γ_0, and n_i are components of unit normal to Γ_0 directed from Ω_1 into Ω_2. Finally, $\delta_{\Gamma_0, [u] n_i}$ is an irregular distribution defined by the boundary integral, dependent upon jump of u and normal n, comp. Exercise 2.4.8. Again, if function u is globally continuous, i.e., $[u] = 0$, then

$$\frac{\partial}{\partial x_i} R_u = R_{u_{,i}} \,.$$

$\square$

Complex conjugate $\bar{u}$ of a distribution u is defined by:

$$\langle \overline{u}, \phi \rangle := \overline{\langle u, \overline{\phi} \rangle} \,, \quad \phi \in \mathcal{D}(\Omega) \,.$$

Convolution of a distribution with a test function. Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $\psi, \phi \in \mathcal{D}(\mathbb{R}^n)$. We have,

$$\begin{aligned}
\int_{\mathbb{R}^n} (f * \psi)(z) \phi(z) \, dz &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(z-y) \psi(y) \, dy \, \phi(z) \, dz \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} f(z-y) \psi(y) \phi(z) \, dz \, dy && \text{(Fubini)} \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x) \psi(y) \phi(x+y) \, dx \, dy && (z = x + y) \\
&= \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} \psi(y) \phi(x+y) \, dy \, dx && \text{(Fubini)} \\
&= \int_{\mathbb{R}^n} f(x) \underbrace{\int_{\mathbb{R}^n} \psi(-y) \phi(x-y) \, dy}_{=((\check{\psi} * \phi)(x)} \, dx && (y = -y)
\end{aligned}$$

where $\check{\psi}(x) = \psi(-x)$. This suggests to define the convolution of distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ with test function $\psi \in \mathcal{D}(\mathbb{R}^n)$ as:

$$\langle f * \psi, \phi \rangle := \langle f, \check{\psi} * \phi \rangle \,, \quad \phi \in \mathcal{D}(\mathbb{R}^n) \,.$$

The convolution is well defined because the operation:

$$\mathcal{D}(\mathbb{R}^n) \ni \phi \to \check{\psi} * \phi \in \mathcal{D}(\mathbb{R}^n) \tag{2.10}$$

is well defined and continuous, see Exercise 2.4.11. Note also that the convolution of a distribution and a test function is actually a function. Indeed,

$$\langle f * \psi, \phi \rangle = \langle f, \int_{\mathbb{R}^n} \psi(y - \cdot)\phi(y)\, dy \rangle = \int_{\mathbb{R}^n} \langle f, \psi(y - \cdot) \rangle \phi(y)\, dy\, .$$

For example, it is easy to check that the convolution with Dirac's delta reduces to the identity operator,

$$\langle \delta * \psi, \phi \rangle = \langle \delta, \check{\psi} * \phi \rangle = (\check{\psi} * \phi)(0) = \int_{\mathbb{R}^n} \check{\psi}(-y)\phi(y)\, dy = \int_{\mathbb{R}^n} \psi(y)\phi(y)\, dy = \langle \psi, \phi \rangle\, .$$

Product of a distribution with a C^∞ function. Let $\psi \in C^\infty(\Omega)$ and $u \in \mathcal{D}'(\Omega)$. One more time we "pass the job to the test function" to define:

$$\langle \psi u, \phi \rangle := \langle u, \psi\phi \rangle\, , \quad \phi \in \mathcal{D}(\Omega)\, .$$

Note that the product is well defined because a) $\psi\phi \in \mathcal{D}(\Omega)$, b) $\phi_j \to 0$ in $\mathcal{D}(\Omega)$ implies that $\psi\phi_j \to 0$ in $\mathcal{D}(\Omega)$ as well.

We conclude this section with a result characterizing Dirac's delta. Support of δ_x contains a single point, supp $\delta_x = \{x\}$. It turns out that, conversely, a distribution with the support at a single point, must be a linear combination of δ_x and its derivatives.

THEOREM 2.4.3

Let $u \in \mathcal{D}'(\Omega)$ be a distribution with the support in a single point, supp $u \subset \{x\}$, $x \in \Omega$. *Then u must be of the following form:*

$$u = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha \delta_x, \quad \text{for some } m \in \mathbb{N}\, ,$$

where coefficients a_α are given by:

$$a_\alpha = \frac{(-1)^{|\alpha|}}{\alpha!} \langle u, \phi \rangle \quad \text{where } \phi(t) = (t - x)^\alpha\, .$$

PROOF Choose sufficiently small ϵ such that $K := \overline{B}(x, \epsilon) \subset \Omega$. By Exercise 2.4.6, there exists a positive integer m such that

$$|\langle u, \phi \rangle| \le C(m, K) \sum_{|\alpha| \le M} \sup_K |\partial^\alpha \phi|\, , \quad \forall\, \phi \in \mathcal{D}(\Omega) \text{ such that supp } \phi \subset K\, .$$

Take now a test function $\phi \in \mathcal{E}(\Omega)$ (distributions with compact support belong to $\mathcal{E}'(\Omega)$) and consider the Taylor's expansion (2.5) of ϕ,

$$\phi(y) = \underbrace{\sum_{k=0}^{m} \frac{1}{k!} \phi^{(k)}(x)(y - x, \ldots, y - x)}_{:=\phi_1(y)} + \underbrace{\frac{1}{k!} \int_0^1 (1 - t)^k \phi^{(k+1)}(x + ty)(y - x, \ldots, y - x)\, dt}_{:=\phi_2(y)}\, .$$

The action of u on ϕ_1 yields:

$$\langle u, \phi_1 \rangle = \sum_{|\alpha| \le m} \frac{1}{\alpha!} \underbrace{\langle u, \partial^\alpha \phi(x)(\cdot - x)^\alpha \rangle}_{\partial^\alpha \phi(x) \langle u, (\cdot - x)^\alpha \rangle} \qquad (\phi^{(j)}(x)(y-x, \dots, y-x) = \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial^\alpha \phi(x)(y-x)^\alpha)$$

$$= \sum_{|\alpha| \le m} a_\alpha \langle \partial^\alpha \delta_x, \phi \rangle \qquad \text{(definition of derivative of distribution.)}$$

We show now that action of u on the remainder ϕ_2 vanishes. By Theorem 2.3.3, there exists a smooth function $\chi \in C^\infty(\mathbb{R}^n)$ such that

$$\chi(y) = \begin{cases} 1 & |y| \le \frac{1}{2} \\ 0 & |y| \ge 1\,. \end{cases}$$

Define $\chi_\epsilon(y) := \chi(\epsilon^{-1}(y - x))$ to obtain:

$$\chi_\epsilon(y) = \begin{cases} 1 & \text{on } B(x, \epsilon/2) \\ 0 & \text{outside of } B(x, \epsilon) \end{cases} \qquad \text{and} \quad |\partial^\alpha \chi_\epsilon| \le C(\alpha) \epsilon^{-|\alpha|}\,.$$

Since $(1 - \chi_\epsilon)\phi_2 = 0$ in $B(x, \epsilon/2)$, and $\text{supp}\, u \subset \{x\}$, u must vanish on $(1 - \chi_\epsilon)\phi_2$ and, therefore,

$$\langle u, \phi_2 \rangle = \langle u, \chi_\epsilon \phi_2 + (1 - \phi_\epsilon)\phi_2 \rangle = \langle u, \chi_\epsilon \phi_2 \rangle\,.$$

By Exercise 2.3.2 and Exercise 2.4.7,

$$|\partial^\alpha (\chi_\epsilon \phi_2)| \le \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} \underbrace{|\partial^\gamma \phi_2|}_{\le C\epsilon^{m+1-|\gamma|}} \underbrace{|\partial^{\alpha-\gamma} \phi_\epsilon|}_{\le C\epsilon^{-|\alpha-\gamma|}} \le C\epsilon\,.$$

Consequently,

$$|\langle u, \phi_2 \rangle| \le C \sum_{|\alpha| \le m} \sup_K |\partial^\alpha (\chi_\epsilon \phi_2)| \le C\epsilon \to 0 \quad \text{as } \epsilon \to 0\,.$$

∎

Can you explain why index m in Theorem 2.4.3 must be finite ?

Exercises

Exercise 2.4.1 Existence of C^∞ functions with compact support. Consider the function:

$$u(x) := \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \le 0 \end{cases}$$

Prove that $\lim_{x \to 0_+} u^{(j)}(x) = 0$ and conclude that $u(x)$ is a C^∞ function. Use the function to construct a C^∞ function ψ with support equal to $[-1, 1]$ and different from zero in $(-1, 1)$.

Exercise 2.4.2 Prove equivalence of conditions (2.8).

Exercise 2.4.3 Consider a sequence of compact sets K_j as in the proof of Corollary 2.3.1. Choose $\chi_j \in C_0^\infty(\Omega)$ such that $\chi_j|_{K_j} = 1$. Let $\phi \in \mathcal{E}(\Omega)$. Prove that

$$\chi_j\phi \to \phi \quad \text{in } \mathcal{E}(\Omega)\,.$$

Consequently, space $\mathcal{D}(\Omega)$ is *dense* in space $\mathcal{E}(\Omega)$.

Exercise 2.4.4 Prove that

$$\phi_j \to 0 \text{ in } \mathcal{D}(\Omega) \quad \Rightarrow \quad \phi_j \to 0 \text{ in } \mathcal{E}(\Omega)\,.$$

Exercise 2.4.5 Let $u \in L^1_{loc}(\Omega)$. Prove that

$$\text{ess supp } u = \text{supp } R_u$$

where R_u is the regular distribution generated by function u.

Exercise 2.4.6 Continuity in $\mathcal{D}'(\Omega)$. Let u be a linear functional defined on $\mathcal{D}(\Omega)$. Show that u is sequentially continuous (and, therefore, continuous [20], p. 414) iff for any compact set $K \subset \Omega$, there exists an $m \in \mathbb{N}$ such that

$$|\langle u, \phi\rangle| \le C(m,K) \sum_{|\alpha|\le m} \sup_K |\partial^\alpha \phi|\,, \quad \forall\, \phi \in \mathcal{D}(\Omega) \text{ such that } \text{supp } \phi \subset K\,.$$

Hint: Use proof by contradiction. The condition appears naturally when you study the locally convex topological inductive limit [20], p. 414.

Exercise 2.4.7 Prove the estimate for function ϕ_2 from the proof of Theorem 2.4.3:

$$|\partial^\beta \phi_2| \le C(\alpha)\epsilon^{m+1-|\beta|} \quad \text{in } B(x,\epsilon)\,.$$

Exercise 2.4.8 Prove that the boundary integral in formula (2.9) defines an irregular distribution.

Exercise 2.4.9 Revisit Exercise 2.2.3 and prove the identity

$$\text{supp } u * v \subset \text{supp } u + \text{supp } v$$

for any $u, v \in L^1(\mathbb{R}^n)$ (not necessarily continuous). Symbol supp above denotes the *essential support* of an L^1-function. As in Exercise 2.2.3, we assume that u or v has a compact support.

Exercise 2.4.10 Let $\Omega = (0,1)$ and $u \in L^2(\Omega)$. Prove that $u'' \in L^2(\Omega)$ implies $u' \in L^2(\Omega)$ as well. All derivatives are understood in the sense of distributions.

Exercise 2.4.11 Prove that map (2.10) is continuous.

2.5 Fourier Transform

L-periodic functions. A measurable function $u : \mathbb{R}^n \to \mathbb{C}$ is L-periodic if

$$u(x + kL) = u(x) \qquad x \in \mathbb{R}^n,\, k \in \mathbb{Z}^n\,.$$

Restricting ourselves to functions that are L^2-integrable on cube $(0, L)^n$, we can equip the space with the inner product

$$(u, v) = \int_{(0,L)^n} u(x)\overline{v(x)}\, dx$$

to obtain a Hilbert space $L^2_{per}(\mathbb{R}^n)$. One can show then (not a cheap result...) that Laplace operator is a well-defined, closed and self-adjoint operator from (a dense subspace of) $L^2_{per}(\mathbb{R}^n)$ into itself. Spectral Theory for Self-Adjoint Operators [20], Section 6.11, shows that spectrum of the Laplacian consists of real non-negative eigenvalues only. Elementary separation of variables shows then that the eigenvectors are given by

$$\phi_k(x) = L^{-n/2} e^{i2\pi \frac{k\cdot x}{L}}$$

with $k \in \mathbb{Z}^n$, and the corresponding eigenvalues

$$\lambda_k = \left(\frac{2\pi|k|}{L}\right)^2 .$$

Note that the eigenvectors have been normalized to form an orthonormal system. It follows from the Spectral Theorem that eigenvectors ϕ from a complete orthonormal system, i.e.,

$$u(x) = \sum_{k\in\mathbb{Z}^n} (u, \phi_k)_{L^2_{per}(\mathbb{R}^n)} \phi_k(x) = \frac{1}{L^n} \sum_{k\in\mathbb{Z}^n} \hat{u}_L(\frac{k}{L}) e^{i2\pi \frac{k\cdot x}{L}}$$

where

$$\hat{u}_L(\xi) = \int_{(0,L)^n} u(x) e^{-i2\pi\xi\cdot x}\, dx\,.$$

Let $u \in L^2(\mathbb{R}^n)$ with a compact (essential) support in some $(-L/2, L/2)$. Consider its L-periodic extension and the corresponding frequency content illustrated in Fig. 2.2 for the one-dimensional case. Elementary calculations show that if we consider the original function with compact support in interval $(-L, L)$ and only then consider its $2L$-periodic extension (i.e. double the value of period L), the corresponding representation will consist of old frequencies (and identical values for them) and new frequencies in between the old ones. If we continue the process, we expect the frequencies to fill the entire real line. This is exactly the intuition behind the definition of the Fourier transform.

Classical Fourier transform. Let $u \in L^1(\mathbb{R}^n)$. We define its Fourier transform $\mathcal{F}u = \hat{u}$ by

$$(\mathcal{F}u)(\xi) = \hat{u}(\xi) := \int_{\mathbb{R}^n} u(x) e^{-i2\pi\xi x}\, dx\,.$$

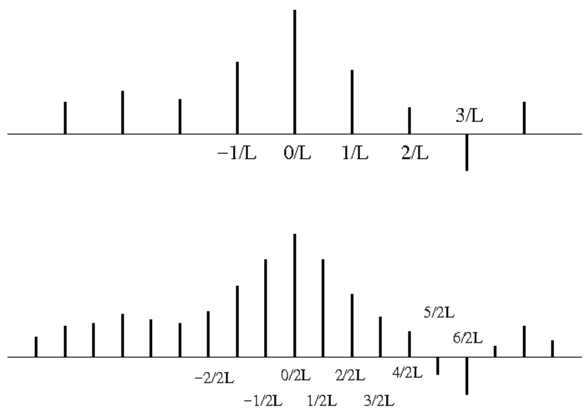

Figure 2.2
Change of frequency spectrum from L to $2L$.

Note that we use a simplified notation, ξx stands for the dot product $\xi \cdot x$. The formal (at this point) L^2-adjoint of the Fourier transform is equal to:

$$(\mathcal{F}^* u)(\xi) = \hat{u}(\xi) := \int_{\mathbb{R}^n} u(x) e^{i2\pi\xi x}\, dx\,.$$

We expect operator $\mathcal{F}$ to be invertible with the inverse equal to its adjoint, $\mathcal{F}^{-1} = \mathcal{F}^*$. Hölder inequality implies that the Fourier transform is well-defined. We have the following classical result.

THEOREM 2.5.1

Let $u, \hat{u} \in L^1(\mathbb{R}^n)$ and u is continuous at a point x. Then

$$u(x) = (\mathcal{F}^* \hat{u})(x)\,.$$

PROOF

Step 1: Gaussian is a fixed point for the Fourier transform. Let $\psi(x) = e^{-\pi|x|^2}$. One can prove, see Exercise 2.5.1, that

$$\mathcal{F}\psi = \psi\,.$$

The same property is shared by the adjoint. Indeed,

$$(\mathcal{F}^*\psi)(x) = (\mathcal{F}\psi)(-x) = \psi(-x) = \psi(x)\,.$$

Step 2: Consider now the scaled Gaussian:

$$\psi_\epsilon(x) := \epsilon^{-n}\psi(\epsilon^{-1}x)\,.$$

By Exercise 2.5.2,

$$\mathcal{F}_{x\to\xi}(\epsilon^{-n}\psi(\epsilon^{-1}x))(\xi) = (\mathcal{F}_{x\to\xi}\psi)(\epsilon\xi) = \psi(\epsilon\xi)$$

and,

$$(\mathcal{F}^*_{\xi\to x}\widehat{\psi_\epsilon})(x) = \epsilon^{-n}\,\underbrace{\mathcal{F}^*_{\xi\to x}\hat{\psi}}_{=\mathcal{F}^*\psi=\psi}(\epsilon^{-1}x) = \psi_\epsilon(x)\,.$$

In other words, the inversion formula holds for the scaled Gaussian.

Step 3: By Step 2 result, we have,

$$\begin{aligned}\int_{\mathbb{R}^n}\hat{u}(\xi)\widehat{\psi_\epsilon}(\xi)e^{i2\pi\xi x}\,d\xi &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}e^{-i2\pi\xi y}u(y)\,dy\right)\widehat{\psi_\epsilon}(\xi)\,e^{i2\pi\xi x}\,d\xi\\ &= \int_{\mathbb{R}^n}u(y)\int_{\mathbb{R}^n}e^{i2\pi\xi(x-y)}\widehat{\psi_\epsilon}(\xi)\,d\xi\,dy\\ &= \int_{\mathbb{R}^n}u(y)\psi_\epsilon(x-y)\,dy\\ &= (\psi_\epsilon * u)(x)\,.\end{aligned}$$

For every ξ,

$$\widehat{\psi_\epsilon}(\xi) = \hat{\psi}(\epsilon\xi) = \psi(\epsilon\xi) \to 1 \quad \text{as } \epsilon\to 0\,.$$

Thus, by the Lebesgue Theorem (show the dominating function),

$$\int_{\mathbb{R}^n}\hat{u}(\xi)\widehat{\psi_\epsilon}(\xi)e^{i2\pi\xi x}\,d\xi \to \int_{\mathbb{R}^n}\hat{u}(\xi)e^{i2\pi\xi x}\,d\xi\,.$$

In other words,

$$(\mathcal{F}^*\hat{u})(x) = \lim_{\epsilon\to 0}(\psi_\epsilon * u)(x)$$

and it remains to show that the limit on the right-hand side equals $u(x)$ at points of continuity of u.

Towards this goal, assume that u is continuous at x and pick an arbitrary $\epsilon_0 > 0$. There exists then a $\delta_0 > 0$ such that

$$|u(x-y) - u(x)| < \frac{\epsilon_0}{2} \quad \text{for } |y| < \delta_0\,.$$

Consequently,

$$\begin{aligned}|(\psi_\epsilon * u)(x) - u(x)| &= |\int_{\mathbb{R}^n}[u(x-y)-u(x)]\,\psi_\epsilon(y)\,dy|\\ &\le \int_{|y|<\delta_0}|u(x-y)-u(x)|\,\psi_\epsilon(y)\,dy + \int_{|y|\ge\delta_0}|u(x-y)-u(x)|\,\psi_\epsilon(y)\,dy\\ &\le \frac{\epsilon_0}{2}\underbrace{\int_{\mathbb{R}^n}\psi_\epsilon(y)\,dy}_{=1} + \left(\int_{\mathbb{R}^n}|u(x-y)-u(x)|\,dy\right)\sup_{|y|\ge\delta_0}\psi_\epsilon(y)\\ &\le \frac{\epsilon_0}{2} + 2\|u\|_{L^1}\epsilon^{-n}e^{-\pi(\delta_0/\epsilon)^2} \quad \forall\epsilon < \epsilon_0\,.\end{aligned}$$

As "exponential takes over any polynomial", term

$$\epsilon^{-n}e^{-\pi(\delta_0/\epsilon)^2}$$

converges to zero as $\epsilon \to 0$ so, for sufficiently small ϵ, the second term above is bounded by $\epsilon_0/2$ as well. ∎

COROLLARY 2.5.1

Let $\check{u}(x) := u(-x)$. Then

$$(\mathcal{F}(\mathcal{F}^* u))(x) = (\mathcal{F}^*(\mathcal{F}\check{u}))(-x) = \check{u}(-x) = u(x)\,.$$

In other words, we have $\mathcal{F}\mathcal{F}^* = \mathrm{id}$ at points of continuity of u as well.

Fourier transform has a regularizing effect on the function being transformed.

LEMMA 2.5.1

Let $u \in L^1(\mathbb{R}^n)$. Then its Fourier transform $\hat{u}$ is bounded and uniformly continuous over $\mathbb{R}^n$.

PROOF We have,

$$\begin{aligned} |\hat{u}(\xi+\eta) - \hat{u}(\xi)| &= \left| \int_{\mathbb{R}^n} u(x) e^{-i2\pi\xi x}(e^{-i2\pi\eta x} - 1)\, dx \right| \\ &\le \int_{\mathbb{R}^n} |u(x)|\, |e^{-i2\pi\eta x} - 1|\, dx \\ &\le 2\int_{\mathbb{R}^n - B_R} |u(x)|\, dx + \int_{B_R} |u(x)|\, \underbrace{|e^{-i2\pi\eta x} - 1|}_{\le 2\pi|\eta|\,|x|}\, dx \\ &\le 2\int_{\mathbb{R}^n - B_R} |u(x)|\, dx + 2\pi|\eta| R \int_{B_R} |u(x)|\, dx\,. \end{aligned}$$

Given $\epsilon > 0$, we choose sufficiently large R to bound the first term by $\epsilon/2$, and then we choose sufficiently small η to make the second term bounded by $\epsilon/2$ as well. Finally,

$$|\hat{u}(x)| \le \left| \int_{\mathbb{R}^n} e^{-i2\pi\xi x} u(x)\, dx \right| \le \int_{\mathbb{R}^n} |u(x)|\, dx\,,$$

for any $x \in \mathbb{R}^n$. ∎

Rapidly decreasing test functions. We introduce one more space of test functions defined on $\mathbb{R}^n$,

$$\mathcal{S}(\mathbb{R}^n) := \{\phi \in C^\infty(\mathbb{R}^n) \,:\, \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \phi(x)| < \infty \quad \forall \alpha, \beta\}$$

The countable set of seminorms

$$p_{\alpha,\beta}(\phi) := \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \phi(x)|$$

generates a first countable locally convex topological vector space topology so the continuity of any functional defined on $\mathcal{S}(\mathbb{R}^n)$ is equivalent to its sequential continuity. It follows that

$$\phi_j \to 0 \quad \Leftrightarrow \quad p_{\alpha,\beta}(\phi_j) = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \phi_j(x)| \to 0 \quad \forall \alpha,\beta\,. \tag{2.11}$$

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. Elementary calculations show that

$$\mathcal{F}_{x\to\xi}\left(\partial^\alpha \phi(x)\right)(\xi) = (i2\pi\xi)^\alpha \hat{\phi}(\xi)\,. \tag{2.12}$$

This in turn implies that

$$\mathcal{F}_{x\to\xi}\left((-i2\pi x)^\alpha \phi(x)\right) = \partial^\alpha \hat{\phi}(\xi)\,. \tag{2.13}$$

The classical Fourier transform restricted to space $\mathcal{S}(\mathbb{R}^n)$,

$$\mathcal{F} \,:\, \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)\,,$$

is well-defined, i.e., Fourier transform of a rapidly decaying function is also a rapidly decaying function, and continuous, see Exercise 2.5.3. Theorem 2.5.1 implies that $\mathcal{F}^{-1} = \mathcal{F}^*$.

As the following inclusions are continuous with dense images, see Exercise 2.5.4,

$$\mathcal{D}(\mathbb{R}^n) \quad \overset{d}{\hookrightarrow} \quad \mathcal{S}(\mathbb{R}^n) \quad \overset{d}{\hookrightarrow} \quad \mathcal{E}(\mathbb{R}^n)\,,$$

we can immediately conclude the corresponding embeddings for the dual spaces,

$$\mathcal{E}'(\mathbb{R}^n) \quad \hookrightarrow \quad \mathcal{S}'(\mathbb{R}^n) \quad \hookrightarrow \quad \mathcal{D}'(\mathbb{R}^n)\,.$$

Elements of dual $\mathcal{S}'(\mathbb{R}^n)$ are called *tempered distributions*. The following Proposition provides a sufficient condition for a regular distribution to be tempered. Let $u : \mathbb{R}^n \to \mathbb{C}$ be a function such that $u(x) = O(|x|^k)$, $|x| \to \infty$, for some integer k, i.e., there exists a constant $C > 0$ such that

$$|u(x)| \le C|x|^k \quad \text{for sufficiently large } |x|\,.$$

We say that function u is *slowly growing*.

PROPOSITION 2.5.1

Let $u \in L^1_{loc}(\mathbb{R}^n)$ be a slowly growing function. Then $R_u \in \mathcal{S}'(\Omega)$.

PROOF follows directly from the definition of topology in $\mathcal{S}(\mathbb{R}^n)$, comp. also [20], Exercise 5.2.6. ∎

Fourier transform of tempered distributions. Let $u \in L^1(\mathbb{R}^n)$. Definitions and Fubini's Theorem imply (check it) that

$$\int_{\mathbb{R}^n} \hat{u}\phi = \int_{\mathbb{R}^n} u\hat{\phi}\,, \quad \phi \in \mathcal{S}(\Omega)\,.$$

This motivates the definition of Fourier transform for tempered distributions. Let $u \in \mathcal{S}'(\mathbb{R}^n)$.

$$\langle \mathcal{F}u, \phi \rangle := \langle u, \mathcal{F}\phi \rangle\,, \quad \phi \in \mathcal{S}(\Omega)\,.$$

Note that the continuity of $\mathcal{F}$ on $\mathcal{S}(\Omega)$ implies that $\mathcal{F}(u)$ is well-defined, i.e., it belongs to $\mathcal{S}'(\mathbb{R}^n)$. In the same way we extend the definition of $\mathcal{F}^*$ to the tempered distributions. Note that both transforms are continuous operators on $\mathcal{S}'(\mathbb{R}^n)$. Property (2.12) extends to tempered distributions. Let $u \in \mathcal{S}'(\mathbb{R}^n)$. The $\partial^\alpha \in \mathcal{S}'(\mathbb{R}^n)$ as well, and

$$\begin{aligned}
\langle \mathcal{F}(\partial^\alpha u), \phi \rangle\ & \langle \partial^\alpha u, \mathcal{F}\phi \rangle && \text{(definition of Fourier transform for tempered distributions)} \\
& (-1)^{|\alpha|}\langle u, \partial^\alpha \hat{\phi} \rangle && \text{(definition of distributional derivative)} \\
& (-1)^{|\alpha|}\langle u, \mathcal{F}_{x\to\xi}((-i2x)^\alpha \phi(x))\rangle && \text{(Property (2.13))} \\
& \langle \mathcal{F}u, (i2\pi x)^\alpha \phi \rangle && \text{(definition of Fourier transform for tempered distributions)} \\
& \langle (i2\pi x)^\alpha \mathcal{F}u, \phi \rangle && \text{(definition of product of a } C^\infty \text{ function with a distribution)}\,,
\end{aligned}$$

i.e., $\mathcal{F}(\partial^\alpha u) = (i2\pi x)^\alpha \mathcal{F}u$. Additionally,

$$\langle \mathcal{F}^*\mathcal{F}u, \phi \rangle = \langle \mathcal{F}u, \mathcal{F}\phi \rangle = \langle u, \mathcal{F}^*\mathcal{F}\phi \rangle = \langle u, \phi \rangle, \qquad \phi \in \mathcal{S}(\mathbb{R}^n)\,,$$

i.e., $\mathcal{F}^*\mathcal{F} = I$. Similarly, $\mathcal{F}\mathcal{F}^* = I$ and, therefore, $\mathcal{F}^{-1} = \mathcal{F}^*$ and $(\mathcal{F}^*)^{-1} = \mathcal{F}$ on $\mathcal{S}'(\mathbb{R}^n)$. This and the property above imply that property (2.13) extends to tempered distributions as well.

Example 2.5.1

Fourier transform of Dirac's delta equals unity. Indeed,

$$\langle \mathcal{F}\delta, \phi \rangle = \langle \delta, \hat{\phi} \rangle = \hat{\phi}(0) = \int_{\mathbb{R}^n} e^{-i2\pi 0\xi}\phi(\xi)\,d\xi = \int_{\mathbb{R}^n} \phi(\xi)\,d\xi = \langle 1, \phi \rangle,$$

i.e., $\mathcal{F}\delta = 1$. Similarly, $\mathcal{F}^*\delta = 1$ as well and, therefore, we know immediately that, conversely, $\mathcal{F}1 = \delta$. It is more difficult to derive the last formula directly, comp. Exercise 2.5.7. □

THEOREM 2.5.2 (Plancherel)

Fourier transform is an isometry from $L^2(\mathbb{R}^n)$ into itself, i.e.,

$$(\mathcal{F}u, \mathcal{F}v)_{L^2} = (u, v)_{L^2}\,, \qquad u, v \in L^2(\mathbb{R}^n)\,.$$

Additionally, $\mathcal{F}^*$ represents the L^2-adjoint of $\mathcal{F}$ and $\mathcal{F}^{-1} = \mathcal{F}^*$ which proves that $\mathcal{F}$ is a surjection.*

PROOF Let $u, v \in \mathcal{S}(\mathbb{R}^n)$. By Theorem 2.5.1,

$$(\mathcal{F}u, \mathcal{F}v) = (\underbrace{\mathcal{F}^*\mathcal{F}u}_{=u}, v) = (u, v)\,,$$

*Not longer just formal adjoint.

so $\mathcal{F}$ is an isometry from $\mathcal{S}(\mathbb{R}^n)$ into itself. As isometries are automatically continuous and space $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ (explain, why?) operator $\mathcal{F}$ can be extended ***in a unique way*** to the L^2-space, comp. [20], Exercise 5.18.1. Let $u \in L^2(\mathbb{R}^n)$ and $v = \mathcal{F}u \in L^2(\mathbb{R}^n)$ be the corresponding value of the (L^2 extended) Fourier transform. As $L^2(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^n)$ and L^2-functions are slowly growing (explain, why?), we have the corresponding regular distribution R_v and,

$$\begin{aligned}
\langle \mathcal{F}R_u, \phi\rangle &= \int_{\mathbb{R}^n} u\,\mathcal{F}\phi && \text{(definition of Fourier transform for tempered distributions)}\\
&= \int_{\mathbb{R}^n} \underbrace{\mathcal{F}u}_{=v}\,\phi && \text{(property of classical Fourier transform + density argument)}\\
&= \langle R_v, \phi\rangle, \quad \forall\phi \in \mathcal{D}(\mathbb{R}^n)\,,
\end{aligned}$$

i.e., $\mathcal{F}R_u = R_v$. In other words, the L^2 extension of classical Fourier transform coincides with the distributional generalization. Same reasoning applies to $\mathcal{F}^*$. Now, by the density argument, the identity

$$(\mathcal{F}u, v) = (u, \mathcal{F}^*v)$$

extends from $u, v \in \mathcal{S}(\mathbb{R}^n)$ to $u, v \in L^2(\mathbb{R}^n)$ which "upgrades" $\mathcal{F}^*$ from the formal to the actual L^2-adjoint. Finally, replacing v above with $\mathcal{F}v$, we obtain that $\mathcal{F}^*\mathcal{F} = \mathrm{id}$. A similar argument with u shows that $\mathcal{F}\mathcal{F}^* = \mathrm{id}$ as well. Consequently, $\mathcal{F}^{-1} = \mathcal{F}^*$. ∎

Fourier transform of convolutions. Let $u, v \in L^1(\mathbb{R}^n)$. We have:

$$\begin{aligned}
\mathcal{F}_{x\to\xi}((u*v)(x)) &= \int_{\mathbb{R}^n} e^{-i2\pi\xi x} \int_{\mathbb{R}^n} u(x-y)v(y)\,dy\,dx\\
&= \int_{\mathbb{R}^n} \underbrace{\left(\int_{\mathbb{R}^n} u(x-y)\,dx\right) e^{-i2\pi\xi y}}_{\hat{u}(\xi)} v(y)\,dy\\
&= \hat{u}(\xi)\hat{v}(\xi)\,,
\end{aligned}$$

i.e., Fourier transform sets convolutions into products.

The result generalizes to distributions. First of all, the notion of tensor product of a distribution with a test function generalizes to tempered distributions. Let $u \in \mathcal{S}'(\mathbb{R}^n)$, $\psi \in \mathcal{D}(\mathbb{R}^n)$, and $\phi \in \mathcal{S}(\mathbb{R}^n)$. We define the tensor product $u * \psi$ by:

$$\langle u*\psi, \phi\rangle := \langle u, \check{\psi}*\phi\rangle\,.$$

where $\check{\psi}(x) = \psi(-x)$. As before, the fact that the operation,

$$\phi \ni \mathcal{S}(\mathbb{R}^n) \to \check{\psi}*\phi \in \mathcal{S}(\mathbb{R}^n) \tag{2.14}$$

is well-defined and continuous (Exercise 2.5.9) implies that the tensor product $u*\psi$ is a well-defined tempered distribution.

We are ready to compute now Fourier transform of convolution $u * \psi$. We have,

$$\begin{aligned}\langle \mathcal{F}(u*\psi), \phi\rangle &= \langle u*\psi, \mathcal{F}\phi\rangle = \langle u, \check{\psi} * \mathcal{F}\phi\rangle \\ &= \langle u, \mathcal{F}(\mathcal{F}\psi\, \phi)\rangle = \langle \mathcal{F}u, \mathcal{F}\psi\, \phi\rangle \\ &= \langle \mathcal{F}\psi\, \mathcal{F}u, \phi\rangle\end{aligned}$$

as,

$$\mathcal{F}^*(\check{\psi} * \mathcal{F}\phi) = \mathcal{F}(\psi)\,\phi \quad \Rightarrow \quad \check{\psi} * \mathcal{F}\phi = \mathcal{F}(\mathcal{F}\psi\,\phi)\,.$$

Thus $\mathcal{F}(u*\psi) = \mathcal{F}u \cdot \mathcal{F}\psi$.

Exercises

Exercise 2.5.1 ([18], Exercise 3.11) Prove that

$$\mathcal{F}_{x\to\xi}\left(e^{-\pi|x|^2}\right) = \prod_{j-1}^{n} \int_{-\infty}^{\infty} e^{-2\pi\xi_j x_j - \pi x_j^2}\, dx_j = e^{-\pi|\xi|^2}\,.$$

Hint: Method I: Use contour integration to prove that

$$\int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2}\, dx = 1, \quad \xi \in \mathbb{R}$$

and the fact that

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1\,,$$

see, e.g., [15], p.57.

Hint: Method II: Consider 1D case and prove that both Gaussian $u(x) = e^{\pi x^2}$ and its Fourier transform satisfy the differential equation:

$$u' + 2\pi x u = 0\,.$$

Exercise 2.5.2 ([18], Exercise 3.12) Let $\psi \in L^1(\mathbb{R}^n)$. Prove the scaling properties:

$$\mathcal{F}_{x\to\xi}(\epsilon^{-n}\psi(\epsilon^{-1}x))(\xi) = (\mathcal{F}_{x\to\xi}\psi)(\epsilon\xi)$$

and,

$$\mathcal{F}^*_{\xi\to x}(\psi(\epsilon\xi))(x) = \epsilon^{-n}(\mathcal{F}^*_{\xi\to x}\psi)(\epsilon^{-1}x)\,.$$

Exercise 2.5.3 Show that the classical Fourier transform restricted to the space of rapidly decaying test functions,

$$\mathcal{F} \,:\, \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)\,,$$

is well-defined and continuous.

Exercise 2.5.4 Show that the corresponding inclusions are continuous with dense images.

$$\mathcal{D}(\mathbb{R}^n) \quad \overset{d}{\hookrightarrow} \quad \mathcal{S}(\mathbb{R}^n) \quad \overset{d}{\hookrightarrow} \quad \mathcal{E}(\mathbb{R}^n)\,.$$

Hint: Use external approximation χ_n^ϵ of closed ball $\bar{B}(0,n)$ indicator function from Theorem 2.3.3.

Exercise 2.5.5 If you do not like the extension argument used in the text to define the Fourier transform for L^2-functions, here is another way to get there. Let $u \in L^2(\mathbb{R}^n)$. Take $N > 0$ and define

$$u_N(x) := \begin{cases} u(x) & \text{for } |x| < N \\ 0 & \text{otherwise} \end{cases}$$

Explain why $u_N \in L^1(\mathbb{R}^n)$, and define:

$$\underbrace{\mathcal{F}u}_{\text{new}} := \lim_{N\to\infty} \underbrace{\mathcal{F}u_N}_{\text{classical}}$$

where the limit is understood in the L^2 sense. Prove that the limit exists and show that the new definition delivers the same result as the two definitions discussed in the text.

Exercise 2.5.6 Prove the *Riemann-Lebesgue Lemma: Let* $u \in L^1(\mathbb{R}^n)$*. Then* $\hat{u}(\xi) \to 0$ *for* $|\xi| \to \infty$.

In other words:

$$\int_{\mathbb{R}^n} e^{-i2\pi\xi x} u(x)\,dx \to 0 \quad \text{as } |\xi| \to \infty\,.$$

Hint: Prove the result first for $u \in C_0^\infty(\mathbb{R}^n)$ and then use the density of $C_0^\infty(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$.

Conclude that also,

$$\int_{\mathbb{R}^n} \sin(2\pi\xi x)u(x)\,dx \to 0, \quad \int_{\mathbb{R}^n} \cos(2\pi\xi x)u(x)\,dx \to 0, \quad \text{as } |\xi| \to \infty\,.$$

Exercise 2.5.7 Compute directly, using the definition of Fourier transform for tempered distributions only, Fourier transform of unity function. Note that

$$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2} \tag{2.15}$$

where the integral above is *not* a Lebesgue integral but it is a singular integral defined by the limit:

$$\int_0^\infty \frac{\sin t}{t}\,dt := \lim_{b\to\infty} \int_0^b \frac{\sin t}{t}\,dt\,.$$

Exercise 2.5.8 Compute Fourier transform of Heaviside function.

Exercise 2.5.9 Prove that operation (2.14) is well-defined and continuous.

2.6 Tensor Product of Distributions

Tensor product of test functions. Let $\Omega = \Omega_1 \times \Omega_2$ where $\Omega_1 \subset \mathbb{R}^{n_1}$ and $\Omega_2 \subset \mathbb{R}^{n_2}$ are open sets with $n_1 + n_2 = n$. Let $\phi_i \in \mathcal{D}(\Omega_i),\ i = 1, 2$ be arbitrary test functions. *Tensor product of test functions* is defined as

$$(\phi_1 \otimes \phi_2)(x_1, x_2) := \phi_1(x_1)\, \phi_2(x_2)\,, \quad x = (x_1, x_2) \in \Omega_1 \times \Omega_2\,. \tag{2.16}$$

Obviously, function $\phi_1 \otimes \phi_2 \in C_0^\infty(\Omega)$.

LEMMA 2.6.1

Finite sums of tensor products of test functions,

$$\sum_k \phi_1^k \otimes \phi_2^k\,, \quad \phi_i \in \mathcal{D}(\Omega_i),\, i = 1, 2\,,$$

form a dense subset of $\mathcal{D}(\Omega)$.

PROOF First of all, note that the number of terms in the sums above, although always finite, can be arbitrary large. The result is a direct consequence of a multidimensional version of the celebrated Weierstrass Theorem for approximating smooth functions with polynomials on compact sets. Let $\phi \in \mathcal{D}(\Omega)$ be an arbitrary test functions and $K_i \subset \Omega_i$ compacts sets such that supp $\phi \subset K := K_1 \times K_2$. Let $P_j(x_1, x_2)$ be a sequence of polynomials converging uniformly to function ϕ along with all its derivatives. Choose two arbitrary test functions $\chi_i \in \mathcal{D}(\Omega_i)$ such that $\chi_i = 1$ in K_i, and consider a sequence of functions

$$P_j(x_1, x_2)\chi_1(x_1)\chi_2(x_2)$$

As polynomials are finite sums of monomials $x_1^{\alpha_1} x_2^{\alpha_2}$, functions above are finite sums of tensor products of test functions $x_i^{\alpha_i}\chi_i(x_i)$. Uniform convergence of polynomials to function ϕ implies convergence of test functions above to ϕ in $\mathcal{D}(\Omega)$. ∎

Take now an arbitrary test function $\phi \in \mathcal{D}(\Omega)$, distribution $u_2 \in \mathcal{D}'(\Omega_2)$ and define:

$$\phi_1(x_1) = \langle u_2, \phi(x_1, \cdot)\rangle \tag{2.17}$$

LEMMA 2.6.2

Function $\phi_1 \in \mathcal{D}(\Omega_1)$. *Moreover, if* $\mathcal{D}(\Omega) \ni \phi^j \to 0$ *in* $\mathcal{D}(\Omega)$ *then the corresponding functions* ϕ_1^j *converge to zero in* $\mathcal{D}(\Omega_1)$. *The same conclusions hold if we replace test functions* $\mathcal{D}(\Omega)$ *with rapidly decaying test functions* $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ *and distributions* $\mathcal{D}'(\Omega_2)$ *with tempered distributions* $\mathcal{S}'(\mathbb{R}^m)$.

PROOF The first part follows immediately from Exercise 2.6.2. To prove the result for rapidly decaying test functions, recall that $\mathcal{S}(\mathbb{R}^m)$ is a locally convex topological vector space (l.c.t.v.s) with the topology generated by the family of seminorms,

$$p_{\alpha,\beta}(u) = \sup_{\mathbb{R}^m} |x^\alpha \partial^\beta u| \, .$$

Moreover, a linear functional u defined on the l.c.t.v.s is continuous iff there exists a finite subset I of indices α, β, and a constant C such that

$$|\langle u, \phi \rangle| \le C \sum_{(\alpha,\beta)\in I} p_{\alpha,\beta}(u)$$

(see [20], Exercise 5.2.6). Let γ be an arbitrary multiindex. By Exercise **??**, we can migrate derivative $\partial^\gamma_{x_1}$ under the distribution, i.e.

$$\partial^\gamma_{x_1} \langle u, \phi(x_1, \cdot) \rangle = \langle u, \partial^\gamma_{x_1} \phi(x_1, \cdot) \rangle \, .$$

Using the continuity criterion for u we have, for every multiindex δ,

$$\begin{aligned} \sup_{x_1} |x_1^\delta \partial^\gamma_{x_1} \langle u, \phi(x_1, \cdot)\rangle| &= \sup_{x_1} |x_1^\delta \langle u, \partial^\gamma_{x_1} \phi(x_1, \cdot)\rangle| \\ &\le \sup_{x_1} |x_1^\delta| \; C \sum_{(\alpha,\beta)\in I} \sup_{x_2} |x_2^\alpha \partial_y^\gamma \phi(x_1, x_2)| \end{aligned}$$

which remains bounded by the assumption that $\phi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$. The continuity property follows. ∎

Tensor product of distributions. Let $u_i \in \mathcal{D}'(\Omega_i)$, $i = 1, 2$. Let $\phi \in \mathcal{D}(\Omega)$ and ϕ_1 be defined by (2.17). *Tensor product of distributions* u_i is defined as

$$\langle u_1 \otimes u_2, \phi \rangle := \langle u_1, \phi_1 \rangle \, . \tag{2.18}$$

First of all, Lemma 2.6.2 implies that the tensor product $u_1 \otimes u_2$ is well defined. Indeed,

$$\phi^j \to 0 \text{ in } \mathcal{D}(\Omega) \quad \Rightarrow \quad \phi_1^j \to 0 \text{ in } \mathcal{D}(\Omega_1) \quad \Rightarrow \quad \langle u_1, \phi_1^j \rangle \to 0 \, .$$

i.e., $u_1 \otimes u_2$ is continuous. In the same way, tensor product of tempered distributions is a well-defined tempered distribution as well.

Secondly, notice that the density result from Lemma 2.6.1 implies that it is sufficient to define action of $u_1 \otimes u_2$ on tensor products of test functions for which definition (2.18) reduces to:

$$\langle u_1 \otimes u_2, \phi_1 \otimes \phi_2 \rangle = \langle u_1, \phi_1 \rangle \, \langle u_2, \phi_2 \rangle \, .$$

The same density result implies that the tensor product of distributions is commuttative,

$$u_1 \otimes u_2 = u_2 \otimes u_1 \, .$$

For regular distributions $u_i \in L^1_{loc}(\Omega_i)$, tensor product reduces to the iterated integral,

$$\langle u_1 \otimes u_2, \phi \rangle = \int_{\Omega_1} u_1(x_1) \int_{\Omega_2} u_2(x_2)\phi(x_1, x_2)\, dx_1\, dx_2$$

and defines a regular distribution generated by tensor product of functions u_1, u_2. For more examples of tensor products of distributions, see [21] and Section 3.5. For instance, we have trivially:

$$\delta_{x_1} \otimes \delta_{x_2} = \delta_{(x_1,x_2)}\,.$$

LEMMA 2.6.3

Let $\phi \in \mathcal{D}(\Omega_1 \times \Omega_2)$ (or $\mathcal{S}(\Omega_1 \times \Omega_2)$). Let $u \in \mathcal{D}'(\Omega_2)$ (or $\mathcal{S}'(\Omega_1)$) and $G \subset \Omega_1$. Then

$$\int_G \langle u, \phi(x,\cdot)\rangle\, dx = \langle u, \int_G \phi(x,\cdot)\, dx\rangle\,,$$

i.e., action of u and integration in x commute with each other.

PROOF The result is a consequence of the commuttativity of tensor product of distributions. Let $\chi_G \in L^1_{loc}(\Omega_1)$ be the indicator function of set G. Then

$$\langle \chi_G \otimes u, \phi\rangle := \int_{\Omega_1} \chi_G(x)\langle u, \phi(x,\cdot)\rangle\, dx = \int_G \langle u, \phi(x,\cdot)\rangle\, dx$$

and,

$$\langle u \otimes \chi_G, \phi\rangle := \langle u, \int_{\Omega_1} \chi_G(x)\phi(x,\cdot)\, dx\rangle = \langle u, \int_G \phi(x,\cdot)\, dx\rangle\,.$$

∎

As we have seen from Lemma 2.5.1, Fourier transform of an L^1-function is a continuous function. Identity:

$$\mathcal{F}_{x\to\xi}((-i2\pi x)^\alpha u(x))(\xi) = \partial^\alpha \hat{u}(\xi)$$

implies that, if additionally, u has a compact support (and, therefore, every $(-i2\pi x)^\alpha u(x)$ is an L^1-function as well), its Fourier transform is a C^∞ function. It turns out that the observation is true for *any distribution with compact support*.

LEMMA 2.6.4

Let $u \in \mathcal{D}'(\mathbb{R}^n)$ with a compact support. Then $\hat{u}$ is a regular distribution generated by a C^∞-function, denoted with the same symbol, and defined by:

$$\hat{u}(\omega) := \langle u, \chi(\cdot)e^{-i2\pi\omega\cdot}\rangle \tag{2.19}$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$, equal one in the support of u.

PROOF First of all, $\hat{u}(\omega)$ is well-defined, i.e., its value is independent of choice of function χ, comp. Exercise 2.6.1. We have now,

$$\begin{aligned}\langle \hat{u}, \phi \rangle = \langle u, \hat{\phi} \rangle &= \langle u, \int_{\mathbb{R}^n} \phi(x)\chi(x)e^{-i2\pi x\cdot}\, dx \rangle \\ &= \int_{\mathbb{R}^n} \underbrace{\langle u, \chi(x)e^{-i2\pi x\cdot} \rangle}_{=:\hat{u}(x)} \phi(x)\, dx \qquad \text{(Lemma 2.6.3)}\,.\end{aligned}$$

Function $\hat{u}(x)$ is a C^∞-function, comp. Exercise 2.6.2. ∎

Exercises

Exercise 2.6.1 Prove that value of (2.19) is independent of the choice of function χ.

Exercise 2.6.2 Let $B = B(y_0, \epsilon) \subset \mathbb{R}^m$. Consider a function

$$\mathbb{R}^n \times B \ni (x, y) \to \phi(x, y) \in \mathbb{C}$$

with the following properties:

(i) there exists a compact set $K \subset \mathbb{R}^n$ such that

$$\text{supp}\, \phi(\cdot, y) \subset K \quad \forall y \in B\,,$$

(ii) for every multiindex α, derivative $\partial_x^\alpha \phi(x, y)$ exists for all (x, y) in the domain of ϕ and,

$$\partial_x^\alpha \phi(x, \cdot) \in C^1(B) \quad \forall x \in \mathbb{R}^n\,.$$

Prove that, for arbitrary $u \in \mathcal{D}'(\mathbb{R}^n)$, function

$$f(y) := \langle u, \phi(\cdot, y) \rangle, \quad y \in B$$

is differentiable at y_0, and

$$\frac{\partial f}{\partial y_j} = \langle u, \frac{\partial \phi}{\partial y_j}(\cdot, y_0) \rangle\,.$$

Consult [21], Theorem 2.7.2, if necessary.

Exercise 2.6.3 This problem is a slight variation of Exercise 2.6.2. Consider a function $\phi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$. Prove that, for any tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$,

$$\frac{\partial}{\partial y_j} \langle u, \phi(\cdot, y) \rangle = \langle u, \frac{\partial \phi}{\partial y_j}(\cdot, y) \rangle\,.$$

3

Sobolev Spaces

3.1 Sobolev Spaces H^s

Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain (= open and connected set). Let k be a natural number and $p \in [1, \infty]$. The classical Sobolev space is defined as:

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \, : \, \partial^\alpha u \in L^p(\Omega), \quad |\alpha| \leq k\} \, ,$$

with the norm:

$$\|u\|^p_{W^{k,p}(\Omega)} := \sum_{l=0}^{k} \sum_{|\alpha|=l} \frac{l!}{\alpha!} \, \|\partial^\alpha u\|^p_{L^p(\Omega)} \, ,$$

and the corresponding *seminorm* of order k,

$$|u|^p_{W^{k,p}(\Omega)} := \sum_{|\alpha|=k} \frac{k!}{\alpha!} \, \|\partial^\alpha u\|^2_{L^p(\Omega)} \, ,$$

compare formulas for the differentials. The derivatives in the definition above are understood in the sense of distributions. This is a delicate and crucial point. We begin with an L^p-function u and consider the corresponding regular distribution R_u. We can differentiate R_u in the sense of distributions as many times as we wish. When we request derivative

$$\partial^\alpha u := \partial^\alpha R_u$$

to be an L^p-function as well, we request first of all that distribution $\partial^\alpha R_u$ *is regular*, i.e., it is generated by an L^1_{loc}-function, denoted with the same symbol $\partial^\alpha u$ and, additionally, this function is L^p-integrable. Thus, function from Example 2.4.1 will not live in Sobolev space $W^{1,p}(a,b)$ *unless it is globally continuous*. Otherwise, the distributional derivative includes the Dirac's delta which is *not* a regular distribution. The same comment applies to Example 2.9.

Alternatively, we can say that function $v = \partial^\alpha u$ in the sense of distributions iff

$$\int_\Omega v \, \phi = (-1)^{|\alpha|} \int_\Omega u \, \partial^\alpha \phi \qquad \forall \phi \in \mathcal{D}(\Omega) \, .$$

Take a second to realize why the two reasonings are equivalent. The Sobolev space is a Banach space, i.e., it is complete. Indeed, let u_n be a Cauchy sequence in $W^{k,p}(\Omega)$. Definition of the norm implies that $\partial^\alpha u_n$ is Cauchy in $L^p(\Omega)$ for every $|\alpha| \leq k$. As $L^p(\Omega)$ is complete, $\partial^\alpha u_n$ must converge to some $v_\alpha \in L^p(\Omega)$. We need only to show that $\partial^\alpha u = v_\alpha$ in the sense of distributions. By definition,

$$\int_\Omega \partial^\alpha u_n \, \phi = (-1)^{|\alpha|} \int_\Omega u_n \, \partial^\alpha \phi \qquad \forall \phi \in \mathcal{D}(\Omega) \, .$$

Passing to the limit with $n \to \infty$ and utilizing the continuity of both sides, we get:

$$\int_\Omega v_\alpha \, \phi = (-1)^{|\alpha|} \int_\Omega u \, \partial^\alpha \phi \qquad \forall \phi \in \mathcal{D}(\Omega)$$

which proves exactly what we wanted.

In the case of $p = 2$ we have a Hilbert space with the norm derived from the inner product:

$$(u, v)_{W^k(\Omega)} := \sum_{l=0}^{k} \sum_{|\alpha|=l} \frac{l!}{\alpha!} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)} \, .$$

For Lipschitz domains (to be defined), the $W^k(\Omega) := W^{k,2}(\Omega)$ space coincides with space $H^k(\Omega)$ (with equivalent but not equal norms) that will be defined momentarily. For that reason, very often in the literature, symbols $H^k(\Omega)$ and $W^k(\Omega)$ are used interchangeably. In these notes, most of the time, we will restrict ourselves to the case $p = 2$ only. Recall examples of variational formulations using spaces $H^1(\Omega)$ or $H^2(\Omega)$.

Fractional Sobolev spaces on $\mathbb{R}^n$. We would like now to extend the definition of (Hilbert) Sobolev space $W^k(\Omega)$ to an arbitrary real value of k. The need for such an extension can be motivated in many ways. First of all, Sobolev spaces are not only used as energy spaces for various formulations but also for accessing regularity of solutions. Many solutions live in a fractional Sobolev space. Recall that the h-convergence rate r equals the minimum of polynomial order p and the regularity index s, $r = \min\{p, s\}$. Without means for using real values of s, we cannot estimate precisely the rate. More recently, fractional Sobolev spaces have also been identified as energy spaces for non-local formulations (peridynamics).

We will begin with the case of $\Omega = \mathbb{R}^n$. Recalling the action of Fourier transform on derivatives,

$$\widehat{\partial^\alpha u}(\xi) = (i 2\pi \xi)^\alpha \hat{u}(\xi) \, ,$$

we can represent the Sobolev norm in the frequency domain as:

$$\|u\|^2_{W^k(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left(\sum_{l=0}^{k} \sum_{|\alpha|=l} \frac{k!}{\alpha!} \, (2\pi\xi)^{2\alpha} \right) |\hat{u}(\xi)|^2 \, d\xi$$

The weight in the integral is equivalent to the *Bessel kernel*,

$$\sum_{l=0}^{k} \sum_{|\alpha|=l} \frac{l!}{\alpha!} \, (2\pi)^l \xi^{2\alpha} = \sum_{l=0}^{k} (2\pi)^l \sum_{|\alpha|=l} \frac{l!}{\alpha!} \, \xi^{2\alpha} \sim (1 + |\xi|^2)^k \, .$$

comp. Exercise 3.1.1.

This leads to the natural and elegant definition of fractional Sobolev spaces on $\mathbb{R}^n$ for any $s \in \mathbb{R}$:

$$H^s(\mathbb{R}^n) := \{ u \in \mathcal{S}'(\mathbb{R}^n) \, : \, (1 + |\xi|^2)^{s/2} \hat{u}(\xi) \in L^2(\mathbb{R}^n) \}$$

with the inner product given by the weighted L^2-product in the frequency domain:

$$(u,v)_{H^s} := \int_{\mathbb{R}^n} (1+|\xi|^2)^s \hat{u}(\xi)\overline{\hat{v}(\xi)}\, d\xi \,.$$

By construction, spaces $W^k(\mathbb{R}^n)$ and $H^k(\mathbb{R}^n)$ are equal, with equivalent norms.

Bessel potential of order $s \in \mathbb{R}$ is defined by:

$$(J^s u)(x) := \int_{\mathbb{R}^n} (1+|\xi|^2)^{s/2}\, \hat{u}(\xi)\, e^{i2\pi x\xi}\, d\xi = \mathcal{F}^*_{\xi\to x}((1+|\xi|^2)^{s/2}\hat{u}(\xi))(x), \quad x \in \mathbb{R}^n \,,$$

or, using argument-less notation,

$$J^s u = \mathcal{F}^*_{\xi\to x}((1+|\xi|^2)^{s/2}\mathcal{F}_{x\to\xi} u)\,.$$

Bessel potential is thus a composition of three operators: Fourier transform, multiplication with the weight $(1+|\xi|^2)^{s/2}$, and the adjoint (inverse) Fourier transform. As all three operations are continuous on $\mathcal{S}(\mathbb{R}^n)$, Bessel potential is a continuous linear map from $\mathcal{S}(\mathbb{R}^n)$ into itself. It follows immediately from the definition that

$$\mathcal{F}_{x\to\xi}((J^s u)(x))(\xi) = (1+|\xi|^2)^{s/2}\hat{u}(\xi)\,.$$

We also easily have:

$$J^{s+r} = J^s J^r, \quad J^0 = I \qquad \Rightarrow \qquad (J^s)^{-1} = J^{-s}\,,$$

i.e., J^s is actually an isomorphism from $\mathcal{S}(\mathbb{R}^n)$ onto itself, with the inverse equal to J^{-s}. We also have:

$$\langle J^s u, v\rangle = \langle u, J^s v\rangle \qquad \text{or} \qquad (J^s u, v) = (u, J^s v), \qquad u,v \in \mathcal{S}(\mathbb{R}^n)$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing, and

$$(u,v) := \langle u, \overline{v}\rangle, \quad v \in V, u \in V'\,,$$

with V denoting any topological vector space. For $V = L^2(\Omega)$, $(\cdot,\cdot)$ coincides with the L^2 inner product which justifies the notation. The last properties motivate definition of the Bessel potential for tempered distributions:

$$\langle J^s u, \phi\rangle := \langle u, J^s\phi\rangle \quad \text{or} \quad (J^s u, \phi) := (u, J^s\phi), \quad \phi \in \mathcal{S}(\mathbb{R}^n)\,.$$

It follows that (the extended potential) J^s is a well defined continuous map from $\mathcal{S}'(\mathbb{R}^n)$ into itself with the same properties as the original potential J^s.

The fractional Sobolev space can now be characterized as the inverse image of L^2-space through the Bessel potential,

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \,:\, J^s u \in L^2(\mathbb{R}^n)\}\,.$$

By construction, $J^s : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a unitary isomorphism. This implies immediately the following properties (see Exercises 3.1.2 and 3.1.3).

- $H^s(\mathbb{R}^n)$ is separable.
- $\mathcal{D}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.
- The fractional Sobolev spaces form a scale:

$$H^t(\mathbb{R}^n) \overset{d}{\hookrightarrow} H^s(\mathbb{R}^n) \qquad s < t\,.$$

- Topological dual of $H^s(\mathbb{R}^n)$ equals $H^{-s}(\mathbb{R}^n)$ with *equal norms*:

$$\|u\|_{H^{-s}} = \sup_v \frac{|\langle u, v\rangle|}{\|v\|_{H^s}}\,.$$

Fractional Sobolev spaces on an arbitrary domain. Let $\Omega \subset \mathbb{R}^n$ be now an arbitrary domain. The Sobolev space $H^s(\Omega)$ consists of restrictions from $H^s(\mathbb{R}^n)$ to Ω, and it is equipped with the *minimum energy extension norm*:

$$H^s(\Omega) := \{u \in \mathcal{D}'(\Omega) \, : \, \exists U \in H^s(\mathbb{R}^n) \, : \, u = U|_\Omega\}$$

$$\|u\|_{H^s(\Omega)} := \inf_{\substack{U \in H^s(\mathbb{R}^n) \\ U|_\Omega = u}} \|U\|_{H^s(\mathbb{R}^n)}\,.$$

We will relate now the definition with standard constructions in Functional Analysis to prove, among other things, that $H^s(\Omega)$ is a Hilbert space, i.e., it is complete, and the minimum extension norm is generated by an inner product.

Let F be a *closed* subset of $\mathbb{R}^n$. We define:

$$H^s_F(\mathbb{R}^n) := \{u \in H^s(\mathbb{R}^n) \, : \, \text{supp}\, u \subset F\}.$$

It follows immediately (show it) that $H^s_F(\mathbb{R}^n)$ is a *closed subspace* of $H^s(\mathbb{R}^n)$. Consequently, the quotient space $H^s(\mathbb{R}^n)/H^s_{\Omega'}(\mathbb{R}^n)$ is well defined, see [20], Lemma 5.17.1, and embedding

$$\iota \, : \, H^s(\Omega) \ni u \to [U] = U + H^s_{\Omega'}(\mathbb{R}^n) \in H^s(\mathbb{R}^n)/H^s_{\Omega'}(\mathbb{R}^n), \quad \text{where } U \in H^s(\mathbb{R}^n),\, U|_\Omega = u\,,$$

is a well defined isometric isomorphism, comp. Exercise 3.1.4. This is a standard reasoning for Banach spaces. For Hilbert spaces we have the Orthogonal Decomposition Theorem, see [20], Theorem 6.2.1,

$$H^s(\mathbb{R}^n) = H^s_{\Omega'}(\mathbb{R}^n) \overset{\perp}{\oplus} (H^s_{\Omega'}(\mathbb{R}^n))^\perp\,,$$

and space $H^s(\Omega)$ is isometrically isomorphic with the orthogonal complement. This shows immediately that space $H^s(\Omega)$ is a Hilbert space. The Orthogonal Decomposition Theorem helps also to identify the inner product in $H^s(\Omega)$, although in somehow abstract way. Let $P, Q = I - P$ be the orthogonal projections of $H^s(\mathbb{R}^n)$ onto $H^s_{\Omega'}(\mathbb{R}^n)$ and its orthogonal complement. The inner product in space $H^s(\Omega)$ is equal to:

$$(u, v)_{H^s(\Omega)} = (QU, QV)_{H^s(\mathbb{R}^n)} \qquad \text{where } U|_\Omega = u,\, V|_\Omega = v\,.$$

Once we understand the Functional Analysis picture, a number of immediate observations follows.

- Space $H^s(\Omega)$ is separable.
- Restriction operator:
$$H^s(\mathbb{R}^n) \ni U \to U|_\Omega \in H^s(\Omega)$$
is continuous with norm equal one.
- Subspace:
$$C_0^\infty(\overline{\Omega}) := \{U|_\Omega \,:\, U \in C_0^\infty(\mathbb{R}^n)\}$$
is dense in $H^s(\Omega)$ (consequence of density of $C_0^\infty(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$ and continuity of the restriction operator).

Topological dual of $H^s(\Omega)$. For $\Omega = \mathbb{R}^n$, the dual of space H^s is simply space H^{-s}. For general domains Ω, the story is much more technical. Let $s \in \mathbb{R}$. We start with the definition of a new space:

$$\tilde{H}^s_\Omega(\mathbb{R}^n) := \overline{C_0^\infty(\Omega)}^{H^s(\mathbb{R}^n)} \subset H^s_{\overline{\Omega}}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)\,.$$

By $C_0^\infty(\Omega)$ we really understand zero extensions of such test functions in Ω to the whole $\mathbb{R}^n$.

THEOREM 3.1.1

We have,

$$\tilde{H}^{-s}_\Omega(\mathbb{R}^n) \hookrightarrow (H^s(\Omega))' \hookrightarrow H^{-s}_{\overline{\Omega}}(\mathbb{R}^n)\,.$$

More precisely, the topological dual of space $H^s(\Omega)$ is isometrically isomorphic with a subspace of $H^{-s}_{\overline{\Omega}}(\mathbb{R}^n)$ containing $\tilde{H}^s_\Omega(\mathbb{R}^n)$.

We will prove shortly that if Ω is a C^0-domain (to be defined) then

$$\tilde{H}^{-s}_\Omega(\mathbb{R}^n) = H^{-s}_{\overline{\Omega}}(\mathbb{R}^n)\,.$$

Consequently, for C^0-domains, the dual of $H^s(\Omega)$ can be identified with either of the two spaces.

PROOF of Theorem 3.1.1. Let $U \in \tilde{H}^{-s}_\Omega(\mathbb{R}^n)$. Take $v \in H^s(\Omega)$ and an arbitrary extension $V \in H^s(\mathbb{R}^n)$, $V|_\Omega = v$. Product

$$(U,V)_{\mathbb{R}^n} := \begin{cases} \langle U, \overline{V}\rangle \text{ for } s \geq 0 \\ \overline{\langle V, \overline{U}\rangle} \text{ for } s \leq 0 \end{cases}$$

depends only upon v. Indeed, let $V|_\Omega = 0$, $U = \lim_{k\to\infty} U_k$, $U_k \in C_0^\infty(\Omega)$ where the convergence is understood in the $H^{-s}(\mathbb{R}^n)$ norm. Passing to the limit in

$$(U_k, V) = 0\,,$$

we obtain $(U,V)=0$ as well. Define thus

$$(U,v) := (U,V)_{\mathbb{R}^n} \quad \text{where } V|_\Omega = v,\ V \in H^s(\mathbb{R}^n)\,.$$

We have,

$$|(U,v)| = |(U,V)_{\mathbb{R}^n}| \le \|U\|_{H^{-s}(\mathbb{R}^n)}\,\|V\|_{H^s(\mathbb{R}^n)}\,.$$

Taking infimum wrt V, we obtain,

$$|(U,v)| \le \|U\|_{H^{-s}(\mathbb{R}^n)}\,\|v\|_{H^s(\Omega)}$$

which proves that

$$(U,\cdot) \in (H^s(\Omega))' \quad \text{and} \quad \|(U,\cdot)\|_{(H^s(\Omega))'} \le \|U\|_{H^{-s}(\mathbb{R}^n)}\,.$$

To prove the second embedding, consider an arbitrary $l \in (H^s(\Omega))'$, and define the corresponding functional on $H^s(\mathbb{R}^n)$:

$$\{H^s(\mathbb{R}^n) \ni V \to l(V|_\Omega) \in \mathbb{C}\} \in (H^s(\mathbb{R}^n))'\,.$$

As $H^{-s}(\mathbb{R}^n)$isdual to $H^s(\mathbb{R}^n)$, there exists an $U \in H^{-s}(\mathbb{R}^n)$ such that

$$(U,V) = l(V|_\Omega), \quad V \in H^s(\mathbb{R}^n)$$

and,

$$\|U\|_{H^{-s}(\mathbb{R}^n)} = \sup_{V \in H^s(\mathbb{R}^n)} \frac{|l(V|_\Omega)|}{\|V\|_{H^s(\mathbb{R}^n)}}\,.$$

Take now any $V \in C_0^\infty(\mathbb{R}^n - \overline{\Omega})$. Then $V|_\Omega = 0$ which implies that

$$(U,V)_{\mathbb{R}^n} = l(V|_\Omega) = 0\,.$$

This proves that supp $U \subset \overline{\Omega}$, i.e., $U \in H^{-s}_{\overline{\Omega}}$. ∎

Comment: We will show that, for C^0-domains and $s \ge -\frac{1}{2}$, we can identify space $\tilde{H}^s_\Omega(\mathbb{R}^n) = H^s_{\overline{\Omega}}(\mathbb{R}^n)$ of distributions defined on the whole space $\mathbb{R}^n$, with a subspace of distributions defined on Ω, denoted $\tilde{H}^s(\Omega)$. Consequently, for range $s \in [-\frac{1}{2}, \frac{1}{2}]$, spaces $H^s(\Omega)$ and $\tilde{H}^{-s}(\Omega)$ are dual to each other.

Exercises

Exercise 3.1.1 Prove that

$$\sum_{l=0}^{k} (2\pi)^l \sum_{|\alpha|=l} \frac{l!}{\alpha!}\,\xi^{2\alpha} \le C(k)(1+|\xi|^2)^k \qquad \text{and} \qquad (1+|\xi|^2)^k \le \sum_{l=0}^{k} (2\pi)^l \sum_{|\alpha|=l} \frac{l!}{\alpha!}\,\xi^{2\alpha}\,,$$

for all $\xi \in \mathbb{R}^n$.

Exercise 3.1.2 Weighted L^2 spaces. Consider a weighted $L^2_w(\Omega)$ space with the inner product:

$$(u,v)_w = \int_\omega w u \overline{v}$$

where w is a measurable function almost everywhere positive in Ω. Argue why the completness of the standard L^2-space implies the completness of the weighted space. Prove that the following maps are unitary isomorphisms.

$$L^2_w \ni u \to w^{1/2} u \in L^2$$
$$L^2 \ni u \to w^{1/2} u \in L^2_{1/w}$$
$$L^2_{1/w} \ni u \to \{L^2_w \ni v \to \textstyle\int_\Omega u\overline{v} \in \mathbb{C}\} \in (L^2_w)' .$$

Exercise 3.1.3 Explain in detail why $\mathcal{D}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$, for any $s \in \mathbb{R}^n$. *Hint:* Use the fact that Bessel potential is a unitary isomorphism and it maps fast decaying test functions into itself. Recall also that $\mathcal{D}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$.

Exercise 3.1.4 Prove that the map

$$\iota \,:\, H^s(\Omega) \ni u \to [U] = U + H^s_{\Omega'}(\mathbb{R}^n) \in H^s(\mathbb{R}^n)/H^s_{\Omega'}(\mathbb{R}^n), \quad \text{where } U \in H^s(\mathbb{R}^n),\, U|_\Omega = u\,,$$

is a well defined isometric isomorphism.

3.2 Sobolev Spaces W^s

Slobodeckij seminorm. Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $\mu \in (0,1)$. The Slobodeckij seminorm is defined as:

$$|u|^2_{\mu,\Omega} := \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\mu}}\, dxdy\,. \tag{3.1}$$

For $\Omega = \mathbb{R}^n$, we will use a simplified notation $|u|_\mu := |u|_{\mu,\mathbb{R}^n}$. The Slobodeckij seminorm can then be expressed in terms of Fourier transform $\hat{u}$.

LEMMA 3.2.1

Let $\mu \in (0,1)$. We have:

$$|u|^2_\mu := |u|^2_{\mu,\mathbb{R}^n} = a_\mu \int_{\mathbb{R}^n} |\xi|^{2\mu}\, |\hat{u}(\xi)|^2 \, d\xi \tag{3.2}$$

where

$$a_\mu := \int_0^\infty t^{-2\mu-1} \int_{|\omega|=1} |e^{i2\pi\omega_1 t} - 1|^2 \, d\omega dt$$

with ω_1 being the first component of $\omega \in \mathbb{R}^n$.

PROOF Elementary calculations show that

$$(\mathcal{F}_{x\to\xi}u(x+h))(\xi) = e^{i2\pi\xi h}\left(\mathcal{F}_{x\to\xi}u(x)\right)(\xi)\,.$$

Consequently, for $(\delta_h u)(x) := u(x+h) - u(x)$,

$$(\mathcal{F}_{x\to\xi}\delta_h u)(\xi) = (e^{i2\pi\xi h} - 1)\hat{u}(\xi)\,.$$

We have now,

$$\begin{aligned}
|u|_\mu^2 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(y+h) - u(u)|^2}{|h|^{n+2\mu}}\,dhdy && \text{(change of variable } x = y+h)\\
&= \int_{\mathbb{R}^n} \frac{1}{|h|^{n+2\mu}} \int_{\mathbb{R}^n} |\delta_h u|^2\,dydh && \text{(Fubini)}\\
&= \int_{\mathbb{R}^n} \frac{1}{|h|^{n+2\mu}} \int_{\mathbb{R}^n} |e^{i2\pi\xi h} - 1|^2\,|\hat{u}(\xi)|^2\,d\xi dh && \text{(Plancherel Theorem)}\\
&= \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \underbrace{\left(\int_{\mathbb{R}^n} \frac{|e^{i2\pi\xi h} - 1|^2}{|h|^{n+2\mu}}\,dh\right)}_{\text{a weight}} d\xi && \text{(Fubini)}\,.
\end{aligned}$$

We now focus on computing the weight. First, we switch to (n-dimensional) spherical coordinates:

$$h = \rho\omega, \quad \rho = |h|, \quad \omega = \frac{h}{|h|} \in \text{ unit sphere } S, \quad dh = \rho^{n-1}\,d\rho\,d\omega\,,$$

to obtain

$$\int_{\mathbb{R}^n} \frac{|e^{i2\pi\xi h} - 1|^2}{|h|^{n+2\mu}}\,dh = \int_0^\infty \rho^{-2\mu} \int_{|\omega|=1} |e^{i2\pi\rho\xi\omega} - 1|^2\,d\omega\,d\rho\,.$$

After another change of variable:

$$t = \rho|\xi|, \quad \rho\xi\omega = \underbrace{\rho|\xi|}_{=t}\frac{\xi}{|\xi|}\omega\,,$$

the inner integral over the unit sphere transforms into:

$$\int_{|\omega|=1} |e^{i2\pi t\frac{\xi}{|\xi|}\omega} - 1|^2\,d\omega\,.$$

As the integral is invariant wrt rotations, we can rotate the system of coordinates for ω in such a way that the first coordinate alligns with vector $\frac{\xi}{|\xi|}$. In the new system of coordinates vector $\frac{\xi}{|\xi|} = (1, 0, \ldots, 0)$ so the whole integral becomes:

$$\int_{|\omega|=1} |e^{i2\pi t\omega_1} - 1|^2\,d\omega\,.$$

Consequently, we arrive at the formula:

$$\int_{-\infty}^{\infty} (t|\xi|^{-1})^{-1-2\mu}|\xi|^{-1} \int_{|\omega|=1} |e^{i2\pi t\omega_1} - 1|^2\,d\omega\,dt = |\xi|^{2\mu}\,a_\mu\,.$$

Note that $\int_{|\omega|=1} |e^{i2\pi\omega_1 t} - 1|^2\,d\omega$ is $O(t^2)$ for $t \to 0$ and $O(1)$ for $t \to \infty$, so the constant a_μ is finite but it blows up for both $\mu \to 0$ and $\mu \to 1$. ∎

Slobodeckij norm. Let $s = k + \mu$ where $k \in \mathbb{N}$, $\mu \in (0,1)$. We define the Slobodeckij norm as:

$$\|u\|^2_{W^s(\Omega)} := \|u\|^2_{W^k(\Omega)} + a_\mu^{-1} \sum_{|\alpha|=k} \frac{k!}{\alpha!} |D^\alpha u|^2_{\mu,\Omega} \tag{3.3}$$

Notice that we have rescaled the seminorm contributions with factor a_μ^{-1}.

For $\Omega = \mathbb{R}^n$, the Bessel norm and the Slobodeckij norm are equivalent. Indeed,

$$\begin{aligned} \|u\|^2_{H^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (1+|\xi|^2)^s \, |\hat{u}(\xi)|^2 \, d\xi \quad \text{and,} \\ \|u\|^2_{W^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left(\sum_{l=0}^{k} (2\pi)^l \sum_{|\alpha|=l} \frac{l!}{\alpha!} \, \xi^{2\alpha} + |\xi|^{2\mu} (2\pi)^k \sum_{|\alpha|=k} \frac{k!}{\alpha!} \, \xi^{2\alpha} \right) |\hat{u}(\xi)|^2 \, d\xi \end{aligned}$$

The kernels

$$(1+|\xi|^2)^s \quad \text{and} \quad \left(\sum_{l=0}^{k} (2\pi)^l \sum_{|\alpha|=l} \frac{l!}{\alpha!} \, \xi^{2\alpha} + |\xi|^{2\mu} (2\pi)^k \sum_{|\alpha|=k} \frac{k!}{\alpha!} \, \xi^{2\alpha} \right) \tag{3.4}$$

are equivalent, see Exercise 3.2.1, with the equivalence constants dependent upon s only. For instance, for $s = \mu \in (0,1)$, we have

$$(1+|\xi|^2)^\mu \quad \text{and} \quad (1+|\xi|^{2\mu}) \, .$$

Then

$$\begin{aligned} &\text{For } |\xi| \leq 1 \, : (1+|\xi|^2)^\mu \leq 2^\mu \leq 2 \leq 2(1+|\xi|^{2\mu}) \\ &\text{For } |\xi| \geq 1 \, : (1+|\xi|^2)^\mu \leq (2|\xi|^2)^\mu \leq 2^\mu |\xi|^{2\mu} \leq 2(1+|\xi|^{2\mu}) \end{aligned}$$

so,

$$(1+|\xi|^2)^\mu \leq 2(1+|\xi|^{2\mu})$$

Similarly,

$$\begin{aligned} &\text{For } |\xi| \leq 1 \, : 1+|\xi|^{2\mu} \leq 2 \leq 2(1+|\xi|^2)^\mu \\ &\text{For } |\xi| \geq 1 \, : (1+|\xi|^2)^\mu \leq (2|\xi|^2)^\mu \leq 2|\xi|^{2\mu} \leq 2(1+|\xi|^2)^\mu \end{aligned}$$

so,

$$(1+|\xi|^{2\mu}) \leq 2(1+|\xi|^2)^\mu$$

as well. We might say that choosing the Bessel kernel over the Slobodeckij kernel is a matter of an algebraic convenience or elegance only.

The equivalence of Bessel and Slobodeckij norms in $\mathbb{R}^n$ transfers to arbitrary domains under some additional assumptions. Let $\Omega \subset \mathbb{R}^n$. Recall that the norm in $H^s(\Omega)$ is defined through the minimum energy extension norm,

$$\|u\|^2_{H^s(\Omega)} = \min\{\|U\|^2_{H^s(\mathbb{R}^n)} \, : \, U|_\Omega = u\} \, .$$

The Slobodeckij norm (3.3) is always bounded by the minimum extension norm. Indeed, let $U \in H^s(\mathbb{R}^n)$ be the minimum energy extension of u to $\mathbb{R}^n$.

$$\begin{aligned} \|u\|_{W^s(\Omega)} &\leq \|U\|_{W^s(\mathbb{R}^n)} && \text{since } \Omega \subset \mathbb{R}^n \\ &\approx \|U\|_{H^s(\mathbb{R}^n)} && \text{equivalence of norms} \\ &= \|u\|_{H^s(\Omega)} && \text{minimum energy extension} \end{aligned}$$

The inverse inequality requires some assumptions on regularity of the domain.

THEOREM 3.2.1

Let $s \geq 0$. Assume there exists a continuous extension operator $E : W^s(\Omega) \to W^s(\mathbb{R}^n)$. Then spaces $W^s(\Omega)$ and $H^s(\Omega)$ coincide with each other with equivalent norms.

PROOF

$$\begin{aligned} \|u\|_{H^s(\Omega)} &\leq \|Eu\|_{H^s(\mathbb{R}^n)} && \text{(minimum energy extension argument)} \\ &\approx \|Eu\|_{W^s(\mathbb{R}^n)} && \text{(equivalence of norms)} \\ &\leq C\|u\|_{W^s(\Omega)} && \text{(continuity of extension operator } E) \,. \end{aligned}$$

∎

We will construct such extension operators for Lipschitz domains (to be defined). In the case of a general domain Ω, space $W^s(\Omega)$ may be larger, see Exercise 3.2.3.

The Sloboditskij norm is indispensable in proving many results for fractional Sobolev spaces. Here is one of them.

LEMMA 3.2.2

Let $\epsilon > 0$. Let $K_j = \overline{B}(0, j)$, and $\chi_j^\epsilon \in C_0^\infty(\mathbb{R}^n)$ be the corresponding C^∞ approximation of indicator function $\chi_j := \chi_{\overline{B}_j}$ from Theorem 2.3.3. Let $u \in H^s(\mathbb{R}^n)\,, s \in \mathbb{R}$. We have:

$$\chi_j^\epsilon u \to u \quad \text{in } H^s(\mathbb{R}^n)\,.$$

PROOF **Case:** $s = k \in \mathbb{N}$. We start with the L^2-estimate.

$$\int_{\mathbb{R}^n} |(1 - \chi_j^\epsilon)u(x)|^2 \leq \int_{\mathbb{R}^n - B_j} |u(x)|^2 \to 0 \quad \text{as } j \to \infty$$

since

$$\int_{\mathbb{R}^n} |u(x)|^2 \, dx < \infty\,.$$

The same reasoning applies to arbitrary derivative ∂^α. The formula from Exercise 2.3.2 extends to distributional derivatives (prove it...),

$$\partial^\alpha((1 - \chi_j^\epsilon)u) = \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \partial^\gamma (1 - \chi_j^\epsilon) \; \partial^{\alpha - \gamma} u\,.$$

Each of derivatives $\partial^\gamma(1 - \chi_j^\epsilon)$ is bounded (pointwise) with a bound independent of j (see Theorem 2.3.3) and each derivative $\partial^{\alpha-\gamma}u$ is L^2-integrable.

Notice that in process of proving the convergence, we have also proved the bound:

$$\|(1-\chi_j^\epsilon)u\|_{H^k(\mathbb{R}^n)} \le C\|u\|_{H^k(\mathbb{R}^n)}$$

with C depending upon k and ϵ, but independent of u. By the interpolation argument, the bound extends* to arbitrary $s \ge 0$. By the duality argument, the bound extends to negative s. Indeed,

$$\begin{aligned} |\langle(1-\chi_j^\epsilon)u,\phi\rangle| &= |\langle u,(1-\chi_j^\epsilon)\phi\rangle| \\ &\le \|u\|_{H^{-s}(\mathbb{R}^n)}\,\|(1-\chi_j^\epsilon)\phi\|_{H^s(\mathbb{R}^n)} \\ &\le \|u\|_{H^{-s}(\mathbb{R}^n)}\,C\|\phi\|_{H^s(\mathbb{R}^n)} \end{aligned}$$

which implies that

$$\|(1-\chi_j^\epsilon)u\|_{H^{-s}(\mathbb{R}^n)} \le C\|u\|_{H^{-s}(\mathbb{R}^n)}\,.$$

Case: Arbitrary $s \ge 0$. For the Sloboditskij seminorm things are a bit more technical but the idea is exactly the same. You have to mimic with differences what we have done above with derivatives. Let $\phi_j := 1-\chi_j^\epsilon$. We have a simple estimate

$$\begin{aligned} |\phi_j(x)u(x)-\phi_j(y)u(y)| &\le |\phi_j(x)|\,|u(x)-u(y)| + |\phi_j(x)-\phi_j(y)|\,|u(y)| \\ &\le |u(x)-u(y)| + |\phi_j(x)-\phi_j(y)|\,|u(y)|\,. \end{aligned}$$

Consequently, for $\mu \in (0,1)$,

$$\begin{aligned} \int_{\mathbb{R}^n\times\mathbb{R}^n - B_j\times B_j} \frac{|\phi_j(x)u(x)-\phi_j(y)u(y)|^2}{|x-y|^{n+2\mu}}\,dxdy &\le 2\int_{\mathbb{R}^n\times\mathbb{R}^n - B_j\times B_j} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\mu}}\,dxdy \\ &\quad +2\int_{\mathbb{R}^n\times\mathbb{R}^n - B_j\times B_j} \frac{|\phi_j(x)-\phi_j(y)|^2\,|u(y)|^2}{|x-y|^{n+2\mu}}\,dxdy \end{aligned}$$

The first integral goes to zero by the integrability argument. The second integral is bounded by:

$$\int_{\mathbb{R}^n} |u(y)|^2 \underbrace{\int_{\mathbb{R}^n} \frac{|\phi_j(y+h)-\phi_j(y)|^2}{|h|^{n+2\mu}}\,dh}_{=:w_j(y)}\,dy\,.$$

As $u \in L^2(\mathbb{R}^n)$, its is sufficient to show that $w_j(y)$ goes pointwise to zero. Fix thus $y \in \mathbb{R}^n$. For $j > |y|, \phi_j(y) = 0$, and the integral reduces to:

$$\int_{\mathbb{R}^n} \frac{|\phi_j(y+h)|^2}{|h|^{n+2\mu}}\,dh = \int_{\mathbb{R}^n} \frac{|\phi_j(z)|^2}{|z-y|^{n+2\mu}}\,dz \le \int_{|z|\ge j} \frac{1}{|z-y|^{n+2\mu}}\,dz \to 0 \quad \text{as } j\to\infty\,.$$

This finishes the proof for $s = \mu$. For $s = k+\mu$, the reasoning has to be applied to the k-th derivatives.

Case: $-s < 0$. Pick an arbitrary $\epsilon_0 > 0$. Let $u \in H^{-s}(\mathbb{R}^n)$ and $v \in C_0^\infty(\mathbb{R}^n)$. Let $\phi_j^\epsilon = 1-\psi_j^\epsilon$ as above. We have:

$$\begin{aligned} \|\phi_j^\epsilon u\|_{H^{-s}(\mathbb{R}^n)} &\le \|\phi_j^\epsilon(u-v)\|_{H^{-s}(\mathbb{R}^n)} + \|\phi_j^\epsilon v\|_{H^{-s}(\mathbb{R}^n)} \\ &\le C\|u-v\|_{H^{-s}(\mathbb{R}^n)} + \|\phi_j^\epsilon v\|_{L^2(\mathbb{R}^n)} \end{aligned}$$

*You can also deduce the bound from the Uniform Boundedness Theorem, see [20],Theorem 5.8.1.

By density of test functions in $H^{-s}(\mathbb{R}^n)$, we can select a $v \in C_0^\infty(\mathbb{R}^n)$ such that the first term is bounded by $\epsilon_0/2$. By the result for non-negative s, the second term is bounded by $\epsilon_0/2$ for sufficiently large j as well. ∎

REMARK 3.2.1

1. We proved that $\phi_j u \to 0$ in $H^s(\mathbb{R}^n)$ for any $u \in H^s(\mathbb{R}^n)$ and $\phi_j = 1 - \chi_j^\epsilon$. Upon examination of the proof, we can see that functions ϕ_j can be replaced with any other sequence of C^∞ functions with derivatives bounded uniformly in j, and vanishing on ball B_j.

2. Lemma 3.2.2 illustrates the fact that the definition of space $H^s(\mathbb{R}^n)$ involves a certain decay of functions at infinity. This is intuitively clear for positive values of s but less so for negative s. Note also that closed balls $\overline{B_j}$ can be replaced with any sequence of compact sets, K_j such that

$$K_1 \subset\subset K_2 \subset\subset \ldots K_j \subset\subset \qquad \bigcup_{j=1}^{\infty} K_j = \mathbb{R}^n \,.$$

∎

Exercises

Exercise 3.2.1 Prove that kernels (3.4) are equivalent with equivalence constants depending upon s only.

Exercise 3.2.2 Consider $\Omega = B(0, \frac{1}{2}) \subset \mathbb{R}^2$ and function

$$u = \ln|\ln r|$$

where r, θ are polar coordinates. Show that $u \in H^1(\Omega)$.

Exercise 3.2.3 Let Ω be the crack domain.

$$\Omega = (-1,1) \times (-1,1) - [0,1) \times \{0\}\,,$$

see Fig. 3.1. Prove that there exists no continuous extension operator from $W^1(\Omega)$ to $W^1(\mathbb{R}^2) \sim H^1(\mathbb{R}^2)$. Consequently, space $W^1(\Omega)$ is larger than space $H^1(\Omega)$.

3.3 Domain Regularity and Density Results

All results proved so far, hold for arbitrary domains $\Omega \subset \mathbb{R}^n$. In this section, we learn how to characterize regularity of a domain or, more precisely, its boundary $\Gamma = \partial\Omega := \overline{\Omega} - \Omega$.

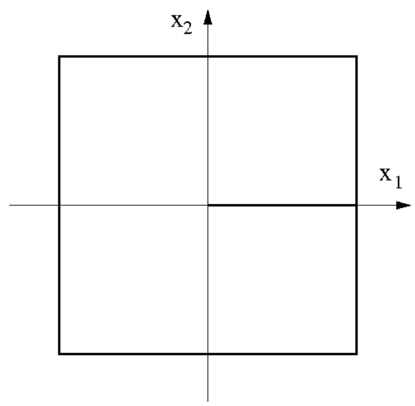

Figure 3.1
The crack domain.

Hypographs. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We will use the notation:

$$x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \sim \mathbb{R}^n \quad \text{where} \quad x' = (x_1, \ldots, x_{n-1}) \,.$$

Let $\zeta : \mathbb{R}^{n-1} \ni x' \to x_n = \zeta(x') \in \mathbb{R}^n$ be now a continuous function. By the *hypograph of function* ζ, the hypograph domain or, shortly, a C^0-hypograph, we mean the open set:

$$\{x \in \mathbb{R}^n \,:\, x_n < \zeta(x') \quad x' \in \mathbb{R}^{n-1}\} \,.$$

Explain why the set is open. If function ζ is Lipschitz, we talk about a *Lipschitz hypograph* or $C^{0,1}$ hypograph. If ζ is a $C^{k,1}$ function, $k = 1, \ldots, \infty$, we speak about a $C^{k,1}$ hypograph. Recall that notation $C^{k,\theta}$ is used for k times differentiable functions such that k-th derivatives are Hölder continuous with exponent $\theta \in (0, 1]$. For $\theta = 1$ we have Lipschitz continuous functions, hence the notation.

Finally, if function ζ is continuous and piecewise smooth, we will call it a *piecewise smooth hypograph*. More precisely, we say that a continuous function f defined on an open set G

$$f \,:\, G \to \mathbb{R} \,,$$

is piecewise smooth if G can be partitioned into a *finite* number of open sets G_j, $j = 1, \ldots, N$ such that

$$G_i \cap G_j = \emptyset \quad \text{for } i \neq j \quad , \quad \overline{G} = \bigcup_{j=1}^{N} \overline{G_j} \,,$$

and restriction $f|_{G_j} \in C^1(\overline{G_j})$, for each $j = 1, \ldots, N$.

C^k, $C^{k,1}$, Lipschitz and polyhedral domains. A domain $\Omega \subset \mathbb{R}^n$ is said to be a C^k or $C^{k,1}$ domain if its boundary Γ is compact and there exist open cubes $G_j = (a_1^j, b_1^j) \times \ldots \times (a_n^j, b_n^j)\,, j = 1, \ldots, J$, such that:

- $G_j, j = 1, \ldots, J$ is an open cover of Γ;

- for each $j = 1, \ldots, J$, cube G_j can by extended to the whole space with $\Omega \cap G_j$ extending to a C^k or $C^{k,1}$ hypograph. The local systems of coordinates for the cube may be obtained from the canonical coordinates in $\mathbb{R}^n$ by a rigid body motion.

The definition is illustrated in Fig. 3.2. A $C^{0,1}$ domain is called a *Lipschitz domain*. Finally, if each hypograph is piecewise smooth, we speak about a *polyhedral domain*.

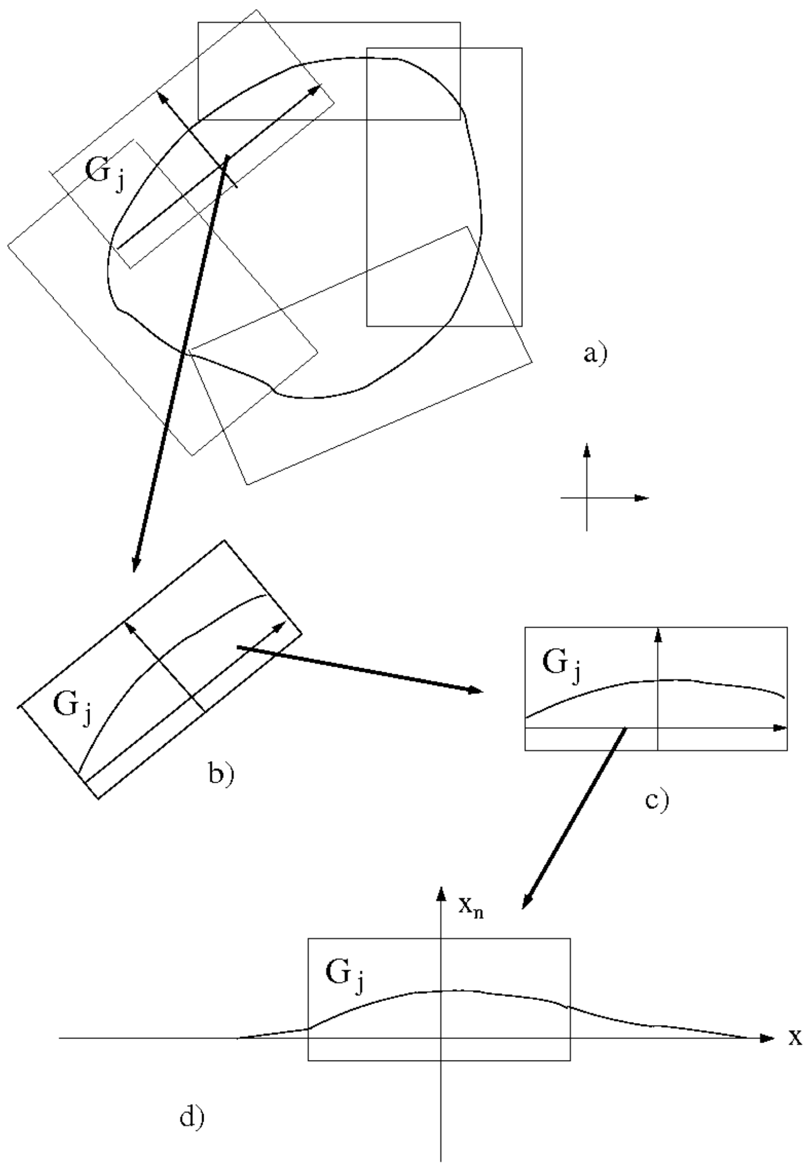

Figure 3.2

Regularity of a domain. a) Open cover of domain boundary Γ. b) A particular open set G_j covering a part of Γ. c) Same set G_j after rotation. d) Extension of $\Omega \cap G_j$ to a hypograph domain.

Partition of unity. A finite or infinite sequence of functions $\psi_j \in C^\infty(\mathbb{R}^n)$ is called a *a partition of unity for a set S* if

- $\psi_j \geq 0$;
- For each $x \in S$, there exists a neighborhood B such that only a finite number of functions ψ_j is non-zero in B;
- $\sum_j \psi_j(x) = 1 \quad x \in S$.

Note that, by the second assumption, the sum in the third condition is always finite.

LEMMA 3.3.1

Let S be an arbitrary set in $\mathbb{R}^n$ and $\mathcal{G}$ an arbitrary[†] *open cover for set S. There exists a partition of unity ϕ_j such that*

$$\forall\, j \quad \exists\, G \in \mathcal{G}\, :\, \text{supp}\, \phi_j \subset G\,.$$

PROOF Let H be the union of all sets G from family $\mathcal{G}$. Obviously, H is an open set. There exists[‡] a sequence of compact sets $K_i \subset H$ such that

$$K_i \subset \text{int} K_{i+1} \quad i = 1, 2, \ldots \quad \text{and} \quad \bigcup_1^\infty K_i = H\,.$$

Consider compact sets

$$F_1 = K_1, \quad F_j = K_j - \text{int} K_{j-1} \quad j = 2, 3, \ldots\,.$$

Note that $F_j \cap K_{j-2} = \emptyset$ for $j > 2$. For every $x \in F_j$ there exists an open set $G \in \mathcal{G}$ such that $x \in G$ and $x \in G - K_{j-2}$ for $j > 2$. Pick an open cube $G_{j,x}$ neighborhood of x, contained in G for $j = 1, 2$ and in $G - K_{j-2}$, for $j > 2$. The family of open cubes $G_{j,x}$, $x \in F_j$ constitutes an open cover for the compact set F_j and, therefore, we can always extract a finite number of those cubes that still cover set F_j. Notice that each cube $G_{j,x}$ may intersect only with a finite number of cubes corresponding to sets $F_1, \ldots, F_{j+1}$. Collect now all the cubes into one countable family of cubes $\{G_{j,x}\}$. For every cube $G_{j,x}$, pick a C^∞ function ψ_j with support in the cube (comp. Exercise 2.4.1) and different from zero at x. The construction implies that only a finite number of functions is different from zero at point x. Normalize,

$$\phi_j(x) = \frac{\psi_j(x)}{\sum_i \psi_i(x)}\,,$$

to obtain the partition of unity. ∎

[†] It need not be countable.
[‡] See. e.g., Lemma 5.3.1 in [20].

THEOREM 3.3.1

Let G_j be a countable open cover for a domain $\Omega \subset \mathbb{R}^n$. There exists a corresponding partition of unity ψ_j for domain Ω such that

$$\text{supp}\, \psi_j \subset G_j, \quad \forall j\,.$$

We will say that partition ψ_j is <u>*subordinate to cover*</u> *G_j.*

PROOF Let ϕ_i be a partition of unity from Lemma 3.3.1. For each j, define the index set:

$$I_j := \{ i \,:\, \text{supp}\, \phi_i \subset G_j, \quad i \notin \bigcup_{k=1}^{j-1} I_k \}$$

and set:

$$\psi_j := \sum_{i \in I_j} \phi_i\,.$$

∎

LEMMA 3.3.2

Let $s \in \mathbb{R}$ and $\epsilon > 0$. For every $u \in H^s(\mathbb{R}^n)$ there exists a function $v \in C_0^\infty(\mathbb{R}^n)$ such that

$$\|u - v\|_{H^s(\mathbb{R}^n)} < \epsilon \quad \text{and} \quad \text{supp}\, v \subset \{ x \in \mathbb{R}^n \,:\, d(x, \text{supp}\, u) < \epsilon \}\,.$$

PROOF Proof is a direct consequence of Lemma 3.2.2 and Exercise 3.3.1. ∎

LEMMA 3.3.3

Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open set, and $s \geq 0$. The set

$$W^s(\Omega) \cap C^\infty(\Omega)$$

is dense in $W^s(\Omega)$.

PROOF Consider open sets:

$$G_j := \{ x \in \Omega \,:\, d(x, \Gamma) < \frac{1}{j},\, |x| < j \}\,, \quad j = 1, 2, \ldots$$

and the corresponding partition of unity ψ_j subordinate to G_j. Let $u \in W^s(\Omega)$, $s \geq 0$. Take an arbitrarily small $\epsilon > 0$. We claim that functions $\widetilde{\psi_j u}$ where tilde denotes the zero extension, belong to $W^s(\mathbb{R}^n) = H^s(\mathbb{R}^n)$. We start with $s = 1$. Indeed, for $u \in W^1(\Omega), \psi \in \mathcal{D}(\Omega)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$,

$$\begin{aligned}
\langle \partial_j \widetilde{\psi u}, \phi \rangle &= -\langle \widetilde{\psi u}, \partial_j \phi \rangle && \text{(definition of distributional derivative)} \\
&= -\langle \tilde{\psi}\tilde{u}, \partial_j \phi \rangle && (\widetilde{\psi u} = \tilde{\psi}\tilde{u}) \\
&= -\langle \tilde{u}, \tilde{\psi}\partial_j \phi \rangle && \text{(definition of product of } C^\infty \text{ function with distribution)} \\
&= -\langle u, \partial_j(\tilde{\psi}\phi) - \partial_j \tilde{\psi}\phi \rangle && (\tilde{\psi}\partial_j \phi \text{ has a support in } \Omega) \\
&= \langle \partial_j u, \tilde{\psi}\phi \rangle + \langle u, \partial_j \tilde{\psi}\phi \rangle \\
&= \langle \tilde{\psi}\widetilde{\partial_j u}, \phi \rangle + \langle \partial_j \tilde{\psi} u, \phi \rangle \, ,
\end{aligned}$$

i.e.,

$$\partial_j(\widetilde{\psi u}) = \tilde{\psi}\, \widetilde{\partial_j u} + \widetilde{\partial_j \psi}\, \tilde{u} \in L^2(\mathbb{R}^n)$$

and, $\widetilde{\psi u} \in W^1(\Omega)$ as claimed. By induction, the result holds for any integer s. Consider now $\mu \in (0, 1)$. We have:

$$\psi(x)u(x) - \psi(y)u(y) = \psi(x)(u(x) - u(y)) + (\psi(x) - \psi(y))u(y)$$

and, therefore,

$$|\psi(x)u(x) - \psi(y)u(y)|^2 \leq 2|\psi(x)|^2\, |u(x) - u(y)|^2 + 2|\psi(x) - \psi(y)|^2\, |u(y)|^2 \, . \tag{3.5}$$

Consider an auxiliary 1D function:

$$\chi(t) := \psi(x + t(y - x)) \, .$$

By the Mean-Value Theorem,

$$\psi(y) - \psi(x) = \chi(1) - \chi(0) = \chi'(\xi) = \frac{\partial \psi}{\partial x_j}(x + \xi(y - x))(y_j - x_j) \, .$$

Thus, by the Weierstrass Theorem, there exists a constant $C > 0$ such that

$$|\psi(x)| \leq C \quad \text{and} \quad |\psi(x) - \psi(y)| \leq C|x - y|$$

for all x. The Slobodeckij integral corresponding to the first term in (3.5) is thus bounded by the Slobodeckij seminorm of u. For the second term we have:

$$\int_\Omega \int_\Omega \frac{|\psi(y) - \psi(x)|^2}{|x - y|^{n+2\mu}} |u(x)|^2 \, dx\, dy = \int_\Omega \int_\Omega \frac{|\psi(y) - \psi(x)|^2}{|x - y|^{n+2\mu}} |u(x)|^2 \, dy\, dx \leq C \int_\Omega |u(x)|^2 \, dx$$

since,

$$\int_{\mathbb{R}^n} \frac{|\psi(y) - \psi(x)|^2}{|x - y|^{n+2\mu}} \, dy \leq C \int_{|x-y| \leq 1} |y - x|^{-n-2\mu+2} \, dy + C \int_{|x-y|>1} |y - x|^{-n-2\mu} \, dy$$

and both integrals on the right-hand side are finite and independent of x.

By Lemma 3.3.2, there exist functions $v_j \in \mathcal{D}(\Omega)$ such that

$$\|\psi_j u - v_j\|_{W^s(\Omega)} \le \|\widetilde{\psi_j u} - \tilde{v}_j\|_{H^s(\mathbb{R}^n)} \le \frac{\epsilon}{2j} \quad \text{and} \quad \text{supp } v_j \subset \frac{1}{j} - \text{neighborhood of supp } \tilde{\psi_j u}\,. \quad (3.6)$$

Define now $v(x) = \sum_{j=1}^{\infty} v_j$. We claim that the sum is locally finite and, therefore, we can conclude that $v \in C^\infty(\Omega)$. Indeed, by the definition of partition of unity, for any $x \in \Omega$, there exists a neighborhood $B(x, \delta_x)$ and a constant N_x such that

$$\sum_{j=1}^{N_x} \psi_j = 1 \quad \text{in } B(x, \delta_x)\,,$$

i.e., $\psi_j = 0$ in $B(x, \delta_x)$ for $j > N_x$. Condition (3.6) on support of v_j implies that, for $\frac{1}{j} < \frac{\delta_x}{2}$, $v_j = 0$ in $B(x, \frac{\delta_x}{2})$. Finally,

$$\|u - v\|_{W^s(\Omega)} \le \sum_{j=1}^{\infty} \|\psi_j u - v_j\|_{W^s(\Omega)} < \epsilon\,.$$

∎

THEOREM 3.3.2 (Theorem 3.29 in [18])

Let Ω be a C^0-domain in $\mathbb{R}^n$. Then

(i) $C_0^\infty(\overline{\Omega})$ is dense in $W^s(\Omega)$, for all $s \ge 0$,

(ii) $C_0^\infty(\Omega)$ is dense in $H^s_{\overline{\Omega}}(\mathbb{R}^n)$, for every $s \in \mathbb{R}$. Consequently,

$$\tilde{H}^s_\Omega(\mathbb{R}^n) = H^s_{\overline{\Omega}}(\mathbb{R}^n) \quad s \in \mathbb{R}\,.$$

PROOF (i) *Case:* Ω is a C^0-hypograph.

We may assume (compactness of Γ) that function ζ defining the hypograph has a compact support.

Let $u \in W^s(\Omega)$, $s \ge 0$. As $C^\infty(\Omega) \cap W^s(\Omega)$ is dense in $W^s(\Omega)$ (Lemma 3.3.3), we may assume additionally that $u \in C^\infty(\Omega)$. Let $\epsilon > 0$. For each $\delta > 0$, define the "shifted function":

$$u_\delta(x) := u(x', x_n - \delta), \quad x \in \Omega_\delta := \{x \in \mathbb{R}^n \,:\, x_n < \zeta(x') + \delta\}\,.$$

Differentiation and shifting commute,

$$\partial^\alpha u_\delta = (\partial^\alpha u)_\delta\,, \quad (3.7)$$

i.e. derivative of the shifted function equals the shifted derivative. Recalling the results on p-modulus of continuity for L^p functions, we obtain,

$$u_\delta|_\Omega \to u \text{ in } W^s(\Omega) \text{ as } \delta \to 0\,.$$

Note that the for the Sloboditskij seminorm, the reasoning must be applied to the L^2 function

$$v(x,y) := \frac{w(x) - w(y)}{|x-y|^{\frac{n}{2}+\mu}}, \quad (x,y) \in \Omega \times \Omega$$

where w stands for the highest involved derivatives of u. The commuttativity property (3.7) is the key to the result.

Choose now such a δ that

$$\|u - u_\delta|_\Omega\|_{W^s(\Omega)} < \frac{\epsilon}{2}\,.$$

Next, choose a cutoff function $\chi \in C^\infty(\mathbb{R}^n)$ such that,

$$\chi = \begin{cases} 1 \text{ on } \Omega \\ 0 \text{ on } \mathbb{R}^n - \Omega_{\delta/2} \end{cases}$$

We have,

$$\chi u_\delta \in W^s(\mathbb{R}^n) = H^s(\mathbb{R}^n) = \overline{C_0^\infty(\mathbb{R}^n)}\,.$$

Consequently, there exists a test function $V \in C_0^\infty(\mathbb{R}^n)$ such that

$$\|\chi u_\delta - V\|_{W^s(\mathbb{R}^n)} < \frac{\epsilon}{2}\,.$$

Set $v = V|_\Omega$, and use triangle inequality to finish the argument.

$$\begin{aligned} \|u - v\|_{W^s(\Omega)} &= \|u - u_\delta|_\Omega + (\chi u_\delta - V)|_\Omega\|_{W^s(\Omega)} \\ &\leq \|u - u_\delta|_\Omega\|_{W^s(\Omega)} + \|\chi u_\delta - V\|_{W^s(\mathbb{R}^n)} < \epsilon\,. \end{aligned}$$

(i) ***Case:*** Ω is a C^0-domain. Let G_j, $j = 1, \ldots, J$ be an open cover for $\Gamma = \partial\Omega$ from the definition of Lipschitz domain. Define,

$$G_0^\delta = \{x \in \Omega \,:\, d(x,\Gamma) > \delta\}\,.$$

Claim: there exists a $\delta > 0$ such that

$$\overline{\Omega} \subset G_0^\delta \cup \bigcup_{j=1}^J G_j\,.$$

Indeed, assume to the contrary that there exists a sequence $x_n \in \overline{\Omega}$ such that

$$d(x_n,\Gamma) \leq \frac{1}{n} \quad \text{and} \quad x_n \notin \bigcup_{j=1}^J G_j\,.$$

Let $y_n \in \Gamma$ be the corresponding sequence of points on Γ that realize the distance, i.e.

$$d(x_n,\Gamma) = d(x_n,y_n)\,.$$

Compactness of Γ implies that there exists a subsequence, denoted with same symbol y_n that converges to a point $y \in \Gamma$. Consequently, x_n converges to y as well. But y must belong to an open set G_j from the cover. Consequently, for sufficiently large n, $x_n \in G_j$ as well, a contradiction.

Let now $u \in W^s(\Omega)$ and $\epsilon > 0$ be an arbitrary constant. Let $\{\psi_j\}_0^J$ be a partition of unity subordinate to cover $G_0 := G_0^\delta, G_1, \ldots, G_J$ of domain Ω. By the result above for the hypograph, for each product $\psi_j u$, there exists a function $v_j \in C^\infty(\overline{G_j})$ such that

$$\|\psi_j u - v_j\|_{W^s(G_j \cap \Omega)} < \frac{\epsilon}{J+1}, \qquad j = 1, \ldots, J.$$

By Lemma 3.3.2 and equivalence of Bessel and Sloboditskij norms in $\mathbb{R}^n$, there exists also a function $v_0 \in C_0^\infty(\Omega)$ such that the same estimate holds:

$$\|\psi_0 u - v_0\|_{W^s(\Omega)} < \frac{\epsilon}{J+1}.$$

Function $v := \sum_{j=0}^J v_j \in C^\infty(\mathbb{R}^n)$, and

$$\|u - v|_\Omega\|_{W^s(\Omega)} = \|\sum_{j=0}^J (\psi_j u - v_j)\|_{W^s(\Omega)} \leq \|\psi_0 u - v_0\|_{W^s(\Omega)} + \sum_{j=1}^J \|\psi_j u - v_j\|_{W^s(G_j)} < \epsilon.$$

(ii) ***Case:*** Ω is a C^0-hypograph. Take an arbitrary $u \in H^s_{\overline{\Omega}}(\mathbb{R}^n)$, $s \in \mathbb{R}$, and $\epsilon > 0$. For $\delta > 0$, define a shifted[§] function,

$$u_\delta(x) := u(x', x_n + \delta), \quad u_\delta \in H^s(\mathbb{R}^n), \quad \text{supp}\, u_\delta \subset \{x \in \mathbb{R}^n : x_n \leq \zeta(x') - \delta\}.$$

We have,

$$u_\delta \to u \text{ in } H^s(\mathbb{R}^n) \text{ as } \delta \to 0.$$

Indeed, for any h,

$$\|u(\cdot + h) - u(\cdot)\|^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |e^{i2\pi h\xi} - 1|^2 \, |\hat{u}(\xi)|^2 \, d\xi.$$

and the result is implied by the Lebesgue Dominated Convergence Theorem. Finish the proof by applying Lemma 3.3.2 to function u_δ.

Contrary to the reasoning in the first part of this proof, this is an easy argument.

(ii) ***Case:*** Ω is a C^0-domain. The proof is fully analogous to the one for (i).

▮

REMARK 3.3.1 Let $s \in \mathbb{R}$. Theorem 3.3.2(ii) and Theorem 3.1.1 imply now that

$$(H^s(\Omega))' = \tilde{H}^{-s}_\Omega(\mathbb{R}^n) = H^{-s}_{\overline{\Omega}}(\mathbb{R}^n),$$

with equivalent norms. By reflexivity of Hilbert spaces, conversely,

$$(\tilde{H}^s_\Omega(\mathbb{R}^n))' = H^{-s}(\Omega),$$

[§]Inside of Ω this time.

with equivalent norms, as well. ∎

Exercises

Exercise 3.3.1 ([18], Exercise 3.17) Let ψ and ψ_ϵ be functions like in (2.7).

(i) Let $u \in \mathcal{D}'(\mathbb{R}^n)$. Show that convolution $u_\epsilon := \psi_\epsilon * u$ lives in $\mathcal{E}(\mathbb{R}^n)$ and

$$\text{supp}\, u_\epsilon \subset \{x \in \mathbb{R}^n \,:\, d(x, \text{supp}\, u) \le \epsilon\}\,.$$

(ii) Let $u \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$. Show that

$$\|u_\epsilon\|_{H^s(\mathbb{R}^n)} \le \|u\|_{H^s(\mathbb{R}^n)} \quad \text{and} \quad \|u_\epsilon - u\|_{H^s(\mathbb{R}^n)} \to 0 \text{ as } \epsilon \to 0\,.$$

3.4 Calderón Extension Theorem

Existence of extension operator,

$$E \,:\, W^s(\Omega) \ni u \to Eu \in W^s(\mathbb{R}^n)\,, \quad s \ge 0\,,$$

provides a crucial argument in proving equivalence of $W^s(\Omega)$ and $H^s(\Omega)$ spaces. This section contains a simplified version of presentation in Appendix 1 from [18]. The bulk of the work will be done for the case of a Lipschitz hypograph domain with the partition of unity argument for an arbitrary Lipschitz domain presented in the end of the section.

We start with the analysis of the symmetric extension. Let

$$\Omega := \{x = (x', x_n) \in \mathbb{R}^n \,:\, x_n < \zeta(x')\,, x' \in \mathbb{R}^{n-1}\}$$

where ζ is a Lipschitz function,

$$|\zeta(x') - \zeta(y')| \le M\,|x' - y'| \quad x', y' \in \mathbb{R}^{n-1}\,.$$

For $u \in W^s(\Omega) \cap C_0^\infty(\overline{\Omega})$, define the symmetric extension by:

$$(E_0 u)(x) := \begin{cases} u(x) & x \in \Omega \\ u(\underbrace{x', 2\zeta(x') - x_n}_{=:\tilde{x}}) & x \in \mathbb{R}^n - \Omega\,. \end{cases} \tag{3.8}$$

THEOREM 3.4.1

Operator E_0 admits a unique extension to a continuous operator:

$$E_0 \,:\, W^s(\Omega) \ni u \to E_0 u \in W^s(\mathbb{R}^n)\,, \quad \text{for } s \in [0,1]\,.$$

PROOF Transformation $x \to \tilde{x}$ has a unit jacobian. This implies that

$$\|E_0 u\|^2_{L^2(\mathbb{R}^n)} = \|u\|^2_{L^2(\Omega)} + \|E_0 u\|^2_{L^2(\mathbb{R}^n-\Omega)} = 2\|u\|^2_{L^2(\Omega)}\,.$$

In turn, for $x \in \mathbb{R}^n - \overline{\Omega}$, the chain formula for differentiation,

$$\begin{cases} \dfrac{\partial(E_0 u)}{\partial x_j}(x) = \dfrac{\partial u}{\partial x_j}(\tilde{x}) + 2\dfrac{\partial u}{\partial x_n}(\tilde{x})\,\dfrac{\partial \zeta}{\partial x_j}(x') \quad & 1 \le j \le n-1 \\ \dfrac{\partial(E_u u)}{\partial x_n}(x) = -\dfrac{\partial u}{\partial x_n}(\tilde{x})\,, \end{cases}$$

implies the estimates:

$$\begin{aligned} &\|\frac{\partial(E_0 u)}{\partial x_j}\|_{L^2(\mathbb{R}^n-\Omega)} \le \|\frac{\partial u}{\partial x_j}\|_{L^2(\Omega)} + 2M\|\frac{\partial u}{\partial x_n}\|_{L^2(\Omega)}, \quad \text{for } j = 1,\ldots,n-1\,, \\ &\|\frac{\partial(E_0 u)}{\partial x_n}\|_{L^2(\mathbb{R}^n-\Omega)} = \|\frac{\partial u}{\partial x_n}\|_{L^2(\Omega)}\,. \end{aligned}$$

It remains to analyze the Sloboditskij seminorm. Let $\mu \in (0,1)$. The seminorm can be split into four integrals:

$$|E_0 u|^2_{\mu,\mathbb{R}^n} = |u|^2_{\mu,\Omega} + I_2 + I_3 + I_4\,.$$

We start with I_2,

$$I_2 = \int_{x_n > \zeta(x')} \int_{y_n > \zeta(y')} \frac{|u(\tilde{x}) - u(\tilde{y})|^2}{|x-y|^{n+2\mu}}\,dx\,dy\,.$$

As for the single integral norms, we intend to change variables from x, y to $\tilde{x}, \tilde{y}$, but we need first to estimate the denominator. We have:

$$|\tilde{x}_n - \tilde{y}_n| \le 2\underbrace{|\zeta(x') - \zeta(y')|}_{\le M|x'-y'|} + |x_n - y_n| \le \sqrt{1+4M^2}\,\underbrace{\left(|x'-y'|^2 + |x_n-y_n|^2\right)^{\frac{1}{2}}}_{=|x-y|}$$

so,

$$|\tilde{x} - \tilde{y}| = \left(|x'-y'|^2 + (\tilde{x}_n - \tilde{y}_n)^2\right)^{\frac{1}{2}} \le \left(|x-y|^2 + (1+4M^2)|x-y|^2\right)^{\frac{1}{2}} \le \underbrace{\sqrt{2+4M^2}}_{=:C}\,|x-y|\,.$$

This implies now a bound for I_2,

$$I_2 = \int_{\tilde{x}_n < \zeta(x')} \int_{\tilde{y}_n < \zeta(y')} \frac{|u(\tilde{x}) - u(\tilde{y})|^2}{|x-y|^{n+2\mu}}\,d\tilde{x}\,d\tilde{y} \le C^{n+2\mu} \underbrace{\int_{\tilde{x}_n < \zeta(x')} \int_{\tilde{y}_n < \zeta(y')} \frac{|u(\tilde{x}) - u(\tilde{y})|^2}{|\tilde{x}-\tilde{y}|^{n+2\mu}}\,d\tilde{x}\,d\tilde{y}}_{=|u|^2_{\mu,\Omega}}\,.$$

We proceed similarly with I_3,

$$I_3 = \int_{x_n > \zeta(x')} \int_{y_n < \zeta(y')} \frac{|u(\tilde{x}) - u(y)|^2}{|x-y|^{n+2\mu}}\,dx\,dy\,.$$

If $y_n - \tilde{x}_n < 0$ then

$$|y_n - \tilde{x}_n| = \tilde{x}_n - y_n \le x_n - y_n \, .$$

Otherwise,

$$y_n - \tilde{x}_n \le \zeta(y') + x_n - 2\zeta(x') \le \zeta(y') + \underbrace{\zeta(y') - y_n}_{\ge 0} + x_n - 2\zeta(x') = (x_n - y_n) + 2(\zeta(x') - \zeta(y')).$$

Consequently, for both cases,

$$|y_n - \tilde{x}_n| \le |x_n - y_n| + 2|\zeta(x') - \zeta(y')| \le |x_n - y_n| + 2M|x' - y'| \, .$$

Therefore,

$$|y - \tilde{x}|^2 = |y' - x'|^2 + |y_n - \tilde{x}_n|^2 \le |y' - x'|^2 + 2|x_n - y_n|^2 + 8M^2|x' - y'|^2 \le \underbrace{(1 + 8M^2)}_{=:C^2} |y - x|^2 \, .$$

The lower bound for $|x - y|$ implies then an upper bound for I_3,

$$I_3 \le C^{n+2\mu} |u|^2_{\mu,\Omega} \, .$$

Procedure to bound I_4 is fully analogous to that for I_3. ∎

We proceed now with the construction of a general extension operator based on a version of Sobolev representation formula. Define the cone:

$$K := \{y \in \mathbb{R}^n \, : \, y_n < -M|y'|\}$$

and notice that $x + K \subset \Omega$, for all $x \in \overline{\Omega}$. Take a cut-off function $\chi \in C_0^\infty([0,\infty))$, equal 1 in a neighborhood of 0. Consider now an arbitrary $u \in W^s(\Omega) \cap C_0^\infty(\overline{\Omega})$, a point ω on unit sphere S, and apply Exercise 3.4.1 to function $\rho \to u(x + \rho\omega)\chi(\rho)$ to arrive at the identity:

$$\begin{aligned} u(x) = u(x)\chi(0) &= \frac{(-1)^k}{(k-1)!} \int_0^\infty \rho^{k-1} \frac{d^k}{d\rho^k} [u(x + \rho\omega)\, \chi(\rho)] \, d\rho \\ &= \frac{(-1)^k}{(k-1)!} \sum_{l=0}^k \binom{k}{l} \int_0^\infty \chi^{(k-l)}(\rho) \underbrace{\frac{d^l}{d\rho^l} u(x + \rho\omega)}_{u^{(l)}(x+\rho\omega;\omega)} \rho^{k-1} \, d\rho \\ &= \frac{(-1)^k}{(k-1)!} \sum_{l=0}^k \binom{k}{l} \int_0^\infty \chi^{(k-l)}(\rho) u^{(l)}(x + \rho\omega; \rho\omega) \, \rho^{k-l-1} \, d\rho \end{aligned} \tag{3.9}$$

Take now a smooth function $\psi(\omega)$ defined on the unit sphere with a support contained in cone V scaled in a such a way that

$$\int_S \psi(\omega) \, dS = \frac{(-1)^k}{(k-1)!} \, .$$

Multiply both sides of identity (3.9) with $\psi(\omega)$ and integrate over unit sphere S to obtain:

$$\begin{aligned}
u(x) &= \sum_{l=0}^{k} \binom{k}{l} \int_S \psi(\omega) \int_0^\infty \chi^{(k-l)}(\rho) u^{(l)}(x+\rho\omega;\rho\omega)\, \rho^{k-l-n}\, \rho^{n-1}\, d\rho\, dS \\
&= \sum_{l=0}^{k} \binom{k}{l} \int_{\mathbb{R}^n} \psi(\frac{y}{|y|})\, \chi^{(k-l)}(|y|) \underbrace{u^{(l)}(x+y;y)}_{\sum_{|\alpha|=l} \frac{l!}{\alpha!} \partial^\alpha u(x+y) y^\alpha} |y|^{k-l-n}\, dy \\
&= \sum_{l=0}^{k} \sum_{|\alpha|=l} \int_{\mathbb{R}^n} \underbrace{\binom{k}{l} \frac{l!}{\alpha!} \psi(\frac{y}{|y|})\, \chi^{(k-l)}(|y|) y^\alpha\, |y|^{k-l-n}}_{=:\Psi_\alpha(-y)} \partial^\alpha u(x+y)\, dy \\
&= \sum_{l=0}^{k} \sum_{|\alpha|=l} \int_{\mathbb{R}^n} \Psi_\alpha(-y) \partial^\alpha u(x+y)\, dy = \sum_{l=0}^{k} \sum_{|\alpha|=l} \int_{\mathbb{R}^n} \Psi_\alpha(y) \partial^\alpha u(x-y)\, dy \\
&= \sum_{l=0}^{k} \sum_{|\alpha|=l} (\Psi_\alpha * \partial^\alpha u)(x)
\end{aligned}$$

At this point, this is just a clever representation formula for function $u(x), x \in \Omega$. Notice that the assumption on support of $\psi(\omega)$ assures that the integration is done only within domain Ω. The symmetric extension operator E_0 now comes in. Replacing derivatives $\partial^\alpha u$ with their extensions $E_0 \partial^\alpha u$, we define a general extension operator,

$$(E_k u)(x) := \sum_{l=0}^{k} \sum_{|\alpha|=l} (\Psi_\alpha * E_0(\partial^\alpha u))(x)\,. \tag{3.10}$$

THEOREM 3.4.2

Operator E_k admits a unique extension to a continuous operator:

$$E \,:\, W^s(\Omega) \ni u \to E_0 u \in W^s(\mathbb{R}^n)\,, \quad \text{for } s \in [k, k+1)\,.$$

Before we can prove our main result, we need to introduce the concept of homogeneous distributions and establish some basic facts about them.

Homogeneous functions and distributions. A function $u : \mathbb{R}^n - \{0\} \to \mathbb{C}$ is *homogeneous of degree* $a \in \mathbb{C}$ if

$$(M_t u)(x) := u(tx) = t^a u(x) \quad t > 0\,, x \in \mathbb{R}^n - \{0\}\,.$$

One checks easily that, for $u \in L^1_{loc}(\mathbb{R}^n)$,

$$\langle M_t u, \phi\rangle = t^{-n} \langle u, M_{1/t} \phi\rangle \quad t > 0,\, \phi \in \mathcal{D}(\mathbb{R}^n)\,.$$

This leads to the definition of $M_t u$ for a general distribution $u \in \mathcal{D}'(\mathbb{R}^n)$. We say now that the distribution u is *homogeneous of degree* $a \in \mathbb{C}$ on $\mathbb{R}^n$ if $M_t u = t^a u$, i.e.

$$\langle M_t u, \phi\rangle := t^{-n} \langle u, M_{1/t} \phi\rangle = t^a \langle u, \phi\rangle \quad \phi \in \mathcal{D}(\mathbb{R}^n)\,.$$

LEMMA 3.4.1

Let $a \in \mathbb{C}$ and let u be a homogeneous distribution of degree a on $\mathbb{R}^n$. Then its Fourier transform $\hat{u}$ is a homogeneous distribution of degree $-a-n$ on $\mathbb{R}^n$.

PROOF Let $t > 0$. One easily checks that

$$\mathcal{F}M_t\phi = t^{-n}M_{1/t}\mathcal{F}\phi \quad \phi \in \mathcal{D}(\mathbb{R}^n)\,.$$

We have thus,

$$\begin{aligned}\langle M_t\hat{u}, \phi\rangle &= t^{-n}\langle \mathcal{F}u, M_{1/t}\phi\rangle = t^{-n}\langle u, \mathcal{F}M_{1/t}\phi\rangle \\ &= t^{-n}\langle u, t^n M_t\mathcal{F}\phi\rangle = t^{-n}\langle M_{1/t}u, \mathcal{F}\phi\rangle \\ &= \langle t^{-n-a}u, \mathcal{F}\phi\rangle = t^{-n-a}\langle \mathcal{F}u, \phi\rangle\,.\end{aligned}$$

∎

Principal value integral. Let $K \in C^\infty(\mathbb{R}^n - \{0\})$ be now a homogeneous function of degree $-n$ with zero average over the unit sphere,

$$\int_{|\omega|=1} K(\omega)\,d\omega = 0\,.$$

The *principal value* of K, denoted p.v. K, is defined as

$$\langle \text{p.v.}K, \phi\rangle := \lim_{\epsilon\searrow 0}\int_{|x|>\epsilon} K(x)\phi(x)\,dx \quad \phi \in \mathcal{D}(\mathbb{R}^n)\,.$$

The following results hold.

LEMMA 3.4.2

(i) The $u :=$ p.v. K is a well-defined tempered, homogeneous distribution of degree $-n$.

(ii) Fourier transform of u is a regular distribution generated by a homogeneous function $\hat{u} \in C^\infty(\mathbb{R}^n - \{0\})$ of degree 0. In particular, $\hat{u}$ is (pointwise) bounded.

(iii) Convolution $u * \psi = (\text{p.v.}K) * \psi$ has the following continuity properties:

$$\begin{aligned}\|u * \psi\|_{H^s(\mathbb{R}^n)} &\leq C\|\psi\|_{H^s(\mathbb{R}^n)} && s \in \mathbb{R} \\ |u * \psi|_\mu &\leq C|\psi|_\mu && \mu \in (0,1)\end{aligned}$$

where $|\cdot|_\mu = |\cdot|_{\mu,\mathbb{R}^n}$ is the Sloboditskij seminorm.

PROOF

(i) Let $\operatorname{supp}\phi \subset B(0,R)$. Properties of function K imply that

$$\int_{\epsilon<|x|<R} K(x)\,dx = \int_\epsilon^R \int_{|\omega|=1} K(\rho\omega)\,\rho^{n-1}\,d\rho\,d\omega = \int_\epsilon^R \frac{d\rho}{\rho}\int_{|\omega|=1} K(\omega)\,d\omega = 0\,.$$

This implies that

$$\langle \text{p.v.}K, \phi\rangle = \lim_{\epsilon\searrow 0}\int_{|x|>\epsilon} K(x)(\phi(x)-\phi(0))\,dx\,.$$

To show that the limit exists is thus sufficient to show that the integrand is an L^1 function. This follows from the fact that every test function is Lipschitz continuous. Indeed,

$$|K(x)(\phi(x)-\phi(0))| \le C|K(x)|\,|x| \le C\rho^{-n+1}|K(\omega)|\,, \quad |\omega| = 1\,,$$

with the right hand side being summable over $\operatorname{supp}\phi$. Consequently,

$$\langle \text{p.v.}K, \phi\rangle = \int_{\mathbb{R}^n} K(x)(\phi(x)-\phi(0))\,dx\,. \tag{3.11}$$

The algebraic decay properties of function K and formula (3.11) imply now easily that the p.v. K can be extended to fast decaying test functions and it is continuous on $\mathcal{S}(\mathbb{R}^n)$, see Exercise 3.4.2. Finally, we easily check that p.v.K is a homogeneous distribution of degree $-n$:

$$\begin{aligned} M_t\langle \text{p.v.}K, \phi\rangle &= t^{-n}\langle \text{p.v.}K, M_{1/t}\phi\rangle = t^{-n}\lim_{\epsilon\searrow 0}\textstyle\int_{|y|>\epsilon} K(y)\phi(t^{-1}y)\,dy \\ &= \lim_{\epsilon\searrow 0}\int_{|z|>t\epsilon} K(tz)\phi(z)\,dz = t^{-n}\lim_{t\epsilon\searrow 0}\int_{|z|>t\epsilon} K(z)\phi(z)\,dz \\ &= t^{-n}\lim_{t\epsilon\searrow 0}\int_{|z|>t\epsilon} K(z)\phi(z)\,dz = t^{-n}\langle \text{p.w.}K, \phi\rangle\,. \end{aligned}$$

(ii) Let $\chi \in C_0^\infty([0,\infty))$ be a cut-off function, equal 1 in a neighborhood of 0. Let

$$K(x) = \underbrace{\chi(|x|)K(x)}_{=:K_1(x)} + \underbrace{(1-\chi(|x|))K(x)}_{=:K_2(x)}\,.$$

Function K_1, as a product of a test function and a tempered distribution, is a tempered distribution also, and it has a compact support. Its Fourier transform $\hat{K}_1$ is thus a regular distribution generated by a C^∞ function, see Lemma 2.6.4.

Let $\operatorname{supp}\chi \subset [0,R]$. For $|x| > R$,

$$K_2(x) = K(x) = K(|x|\omega) = |x|^{-n}K(\omega) \quad \text{where } |\omega| = 1\,.$$

As K_2 is bounded on B_R and $K(\omega)$ is bounded on the unit sphere, we have the bound,

$$|K_2(x)| \le C(1+|x|)^{-n}\,.$$

Since $\partial^\alpha K$ is a homogeneous function of degree $-n-|\alpha|$, by the same argument,

$$|\partial^\alpha K_2(x)| \le C(1+|x|)^{-n-|\alpha|}\,.$$

with constant C depending upon α. Consider now a multiindex β such that $|\beta| > |\alpha|$. We claim that

$$\partial_x^\beta[(-i2\pi x)^\alpha K_2(x)] \in L^1(\mathbb{R}^n)\,.$$

Indeed,

$$\begin{aligned}
\partial^\beta[(-i2\pi x)^\alpha K_2(x)] &= \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \partial^\gamma(-i2\pi x)^\alpha\, \partial^{\beta-\gamma} K_2(x) \\
&= \sum_{\gamma\leq\alpha,\gamma\leq\beta} \binom{\beta}{\gamma} (-i2\pi)^{|\gamma|} \underbrace{(-i2\pi x)^{\alpha-\gamma}}_{\leq C(1+|x|)^{|\alpha-\gamma|}} \quad \underbrace{\partial^{\beta-\gamma} K_2(x)}_{\leq C(1+|x|)^{-n-|\beta-\gamma|}} \\
&\leq C(1+|x|)^{-n+|\alpha|-|\beta|}
\end{aligned}$$

Consequently, by Lemma 2.5.1, $(i2\pi\xi)^\beta\partial^\alpha \hat{F}_2(\xi)$ is (absolutely) continuous on $\mathbb{R}^n$. In particular, for $|\beta| > 1$, $\xi^\beta \hat{F}_2(\xi)$ and, therefore, $\xi^\beta \hat{F}(\xi)$ as well, are L^1_{loc} functions. We have thus,

$$\begin{aligned}
\int_{\mathbb{R}^n} (t\xi)^\beta \hat{K}(t\xi)\phi(\xi)\,d\xi &= t^{-n}\int_{\mathbb{R}^n} \eta^\beta \hat{K}\phi(\frac{\eta}{t})\,d\eta \qquad\qquad (\eta = t\xi) \\
&= t^{-n+|\beta|} \underbrace{\int_{\mathbb{R}^n} \hat{K}\left(\frac{\eta}{t}\right)^\beta \phi(\frac{\eta}{t})\,d\eta}_{\langle \hat{K}, T_{1/t}\eta^\beta\phi\rangle} \\
&= t^{|\beta|}\langle \hat{K}, \eta^\beta\phi\rangle \qquad\qquad (\hat{K} \text{ is a homogeneous distribution of order } 0) \\
&= \int_{\mathbb{R}^n} (t\eta)^\beta \hat{K}(\eta)\phi(\eta)\,d\eta \\
&= \int_{\mathbb{R}^n} (t\xi)^\beta \hat{K}(\xi)\phi(\xi)\,d\xi\,.
\end{aligned}$$

Consequently,

$$(t\xi)^\beta(\hat{K}(t\xi) - \hat{K}(\xi)) = 0 \quad \text{a.e. in } \mathbb{R}^n \quad \Rightarrow \quad \hat{K}(t\xi) = \hat{K}(\xi) \quad \text{a.e. in } \mathbb{R}^n\,.$$

This implies that $\hat{K}$ is bounded and, therefore, an L^1_{loc} function.

(iii) Assume first $\psi \in \mathcal{D}(\mathbb{R}^n)$. The result is an immediate consequence of the fact that

$$\mathcal{F}(u * \psi) = \hat{u}\cdot\hat{\psi}\,,$$

representation of the Sobolev norms and Sloboditskij seminorms in the Fourier domain, and boundedness of $\hat{u}$. Final results follow from density of test functions in $H^s(\mathbb{R}^n)$.

∎

LEMMA 3.4.3

Let $K \in C^\infty(\mathbb{R}^n - \{0\})$ be a homogeneous function of degree $1-n$. Convolution $K * \psi$ has the following continuity properties:

$$\begin{aligned}
\|\partial_j(K*\psi)\|_{H^s(\mathbb{R}^n)} &\leq C\|\psi\|_{H^s(\mathbb{R}^n)} \qquad s \in \mathbb{R} \\
|\partial_j(K*\psi)|_{\mu,\mathbb{R}^n} &\leq C|\psi|_{\mu,\mathbb{R}^n} \qquad \mu \in (0,1)
\end{aligned}$$

for every $\psi \in C_0^\infty(\mathbb{R}^n)$.

PROOF As K is only weakly singular, $K \in L^1_{loc}(\mathbb{R}^n)$, and we have:

$$\begin{aligned}
\partial_j(K * \psi) &= (K * \partial_j \psi) = \int_{\mathbb{R}^n} K(x-y)\partial_j\psi(y)\, dy \\
&= \lim_{\epsilon \searrow 0} \int_{|y-x|>\epsilon} K(x-y)(\partial_j\psi)(y)\, dy \\
&= \lim_{\epsilon \searrow 0} \{ - \int_{|y-x|>\epsilon} \frac{\partial K}{\partial y_j}(x-y)\psi(y)\, dy + \underbrace{\int_{|y-x|=\epsilon} K(x-y)\psi(y) \underbrace{n_j}_{=(x_j-y_j)/\epsilon} dS}_{=:(*)} \},
\end{aligned}$$

with

$$(*) = \int_{|\omega|=1} \omega_j \underbrace{K(\epsilon\omega)}_{=\epsilon^{1-n}K(\omega)} \epsilon^{n-1}\psi(x-\epsilon\omega)\, d\omega \to \underbrace{\int_{|\omega|=1} \omega_j K(\omega)\, d\omega}_{=:a_j} \psi(x) \quad \text{as } \epsilon \to 0 .$$

Consequently,

$$\partial_j(K*\psi)(x) = a_j\psi(x) + \lim_{\epsilon \searrow 0} \int_{|y-x|>\epsilon} \frac{\partial K}{\partial y_j}(x-y)\psi(y)\, dy .$$

Function $\frac{\partial K}{\partial y_j}(y)$ is homogeneous of order $-n$. If we show that the average of $\frac{\partial K}{\partial y_j}(y)$ over the unit sphere vanishes, application of Lemma 3.4.2 will finish the proof. Towards this goal, pick an arbitrary non-negative test function $\chi \in C_0^\infty(0,\infty)$ and normalize it to satisfy the condition:

$$\int_0^\infty \chi(\rho)\, \frac{d\rho}{\rho} = 1 .$$

We have then,

$$\int_{\mathbb{R}^n} \frac{\partial K}{\partial x_j}\chi(|x|)\, dx = - \int_{\mathbb{R}^n} K(x)\chi'(|x|)\frac{x_j}{|x|}\, dx ,$$

or,

$$\int_0^\infty \int_{|\omega|=1} \frac{\partial K}{\partial x_j}(\rho\omega)\chi(\rho)\, \rho^{n-1} d\rho\, d\omega = - \int_0^\infty \int_{|\omega|=1} K(\rho\omega)\chi'(\rho)\, \omega_j \rho^{n-1} d\rho\, d\omega ,$$

or,

$$\underbrace{\int_0^\infty \chi(\rho)\frac{d\rho}{\rho}}_{=1} \int_{|\omega|=1} \frac{\partial K}{\partial x_j}(\omega)\, d\omega = - \underbrace{\int_0^\infty \chi'(\rho)\, d\rho}_{=0} \underbrace{\int_{|\omega|=1} K(\omega)\omega_j\, d\omega}_{\text{finite}}$$

which implies that

$$\int_{|\omega|=1} \frac{\partial K}{\partial x_j}\, d\omega = 0 .$$

∎

PROOF **of Theorem 3.4.2**. We start by recalling formula for functions Ψ_α in spherical coordinates. Skipping the constant factors, we have:

$$\Psi_\alpha(\rho,\omega) = \psi(-\omega)(-\omega)^\alpha\ \chi^{(k-l)}(\rho)\, \rho^{k-n}, \quad |\alpha| = l .$$

Case: $l < k$. Derivative $\chi^{(k-l)}(\rho)$ vanishes in the neighborhood of 0 and, therefore, $\Psi_\alpha \in C_0^\infty(\mathbb{R}^n)$. By Theorem 2.3.1, $\partial^\beta(\Psi_\alpha * u) = (\partial^\beta \Psi_\alpha) * u$ and, since $\partial^\beta \Psi_\alpha \in C_0^\infty(\mathbb{R}^n)$ as well, by Theorem 2.2.1,

$$\|(\partial^\beta \Psi_\alpha) * u\|_{L^2(\mathbb{R}^n)} \le \underbrace{\|\partial^\beta \Psi_\alpha\|_{L^1(\mathbb{R}^n)}}_{=:C} \|u\|_{L^2(\mathbb{R}^n)} .$$

for *any* β. This implies that

$$\|\Psi_\alpha * u\|_{H^s(\mathbb{R}^n)} \le C(s)\|u\|_{L^2(\mathbb{R}^n)}$$

for any integer $s \ge 0$ and, therefore, for any $s \in [0, \infty)$.

Case: $l = k$. We are dealing now with a (possibly singular) homogeneous function Ψ_α of degree $k - n$. Derivative $\partial^\beta \Psi_\alpha$ is also a homogeneous function of degree $k - |\beta| - n$. For $|\beta| < k$, the function is at most *weakly singular*, i.e. with singularity ρ^{1-n} that cancels out with jacobian ρ^{n-1}. Consequently, $\partial^\beta \Psi_\alpha \in L^1(\mathbb{R}^n)$ and we have the bound as above. For $|\beta| = k$, derivative $\partial^\beta \Psi_\alpha$ equals first derivative of a homogeneous function of degree $1 - n$ and, by Lemma 3.4.3 we have,

$$\|\partial^\beta \Psi_\alpha * u\|_{L^2(\mathbb{R}^n)} \le C\|u\|_{L^2(\mathbb{R}^n)} \quad \text{and} \quad \|\partial^\beta \Psi_\alpha * u\|_\mu \le C\|u\|_\mu .$$

Pulling everything together, we see that

$$\|E\|_{H^{k+\mu}(\mathbb{R}^n)} \le C \sum_{|\alpha| \le k} \|E_0(\partial^\alpha u)\|_{H^\mu(\mathbb{R}^n)} \le C \sum_{|\alpha| \le k} \|\partial^\alpha u\|_{H^\mu(\mathbb{R}^n)} \le C\|u\|_{H^{k+\mu}(\mathbb{R}^n)} ,$$

as required. ∎

Finally, we comment shortly how to extend the construction of the extension operator to an arbitrary Lipschitz domain. Let $G_j, j = 1, \ldots, J$, be an open cover of boundary Γ, and $\psi_j, j = 1, \ldots, J$, the corresponding partition of unity subordinate to the cover. Let $u \in H^s(\Omega)$ with $s = k + \mu, k \in \mathbb{N}, \mu \in (0, 1)$. We define the extension operator by

$$E_k u = E_k(\sum_{j=1}^J \psi_j u) := \sum_{j=1}^J E_k^j(\psi_j u)$$

where E_k^j denotes the extension operator for the j-th hypograph domain. We have

$$\|E_k u\|_{H^s(\mathbb{R}^n)} \le C \sum_{j=1}^J \|E_k^j(\psi_j u)\|_{H^s(\mathbb{R}^n)} \le C \sum_{j=1}^J \|\psi_j u\|_{H^s(\Omega_j)} \le C\|u\|_{H^s(\Omega)} ,$$

with the ultimate constant C dependent upon the partition of unity functions.

Exercises

Exercise 3.4.1 Let $f \in C_0^k([0, \infty)),\ k = 1, 2, \ldots$. Prove the representation formula:

$$f(0) = \frac{(-1)^k}{(k-1)!} \int_0^\infty t^{k-1} f^{(k)}(t)\, dt .$$

Hint: Use induction in k.

Exercise 3.4.2 Prove that (3.11) is a well-defined tempered distribution.

3.5 Spaces $\tilde{H}^s(\Omega)$

We begin with a construction of extension operators from Sobolev spaces defined on a hyperplane to Sobolev spaces defined on the whole space. The result logically belongs really to Section 4.1 but we will need it already in the proof of the following Hörmander's Theorem.

Let $\theta_j \in C_0^\infty(\mathbb{R}),\ j = 0, 1, \dots$ be *cut-off functions* such that

$$\theta_j(y) = \frac{y^j}{j!} \quad \text{for } |y| \le 1 \, .$$

Define extension operators:

$$\begin{aligned} &\eta_j \, : \, \mathcal{S}(\mathbb{R}^{n-1}) \to \mathcal{S}(\mathbb{R}^n) \\ &(\eta_j u)(x) := \int_{\mathbb{R}^{n-1}} \frac{\hat{u}(\xi')\, \theta_j((1+|\xi'|^2)^{\frac{1}{2}} x_n)}{(1+|\xi'|^2)^{\frac{j}{2}}} \, e^{i2\pi\xi' x'} \, d\xi' \qquad x \in \mathbb{R}^n \, . \end{aligned} \tag{3.12}$$

Definition of the cut-off functions implies that $\theta_j^{(k)}(0) = \delta_{jk}$. Consequently,

$$\partial^\alpha(\eta_j u)(x',0) = \int_{\mathbb{R}^{n-1}} (i2\pi\xi')^{\alpha'} \hat{u}(\xi') \delta_{j\alpha_n} e^{i2\pi\xi' x'} \, d\xi' = \partial^{\alpha'} u(x') \delta_{j\alpha_n} \, .$$

In other words, operators η_j satisfy the conditions:

$$\partial^\alpha(\eta_j u)(x',0) = \begin{cases} \partial^{\alpha'} u(x') & \text{for } \alpha_n = j \\ 0 & \text{otherwise} \end{cases} \tag{3.13}$$

and, for that reason, they are identified as *extension operators*.

LEMMA 3.5.1

Let $s \in \mathbb{R}$ be an arbitrary real number. Each of operators (3.12) admits a unique extension to:

$$\eta_j \, : \, H^{s-j-\frac{1}{2}}(\mathbb{R}^{n-1}) \to H^s(\mathbb{R}^n) \, .$$

PROOF We need only to show that the operators are bounded in the appropriate norms. We use the standard substitution $\xi_n = (1+|\xi'|^2)^{\frac{1}{2}} t$, to obtain:

$$\begin{aligned} \|\eta_j u\|^2_{H^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^{n-1}} \frac{|\hat{u}(\xi')|^2}{(1+|\xi'|^2)^{j+1}} \int_{-\infty}^{\infty} (1+|\xi'|^2+\xi_n^2)^s |\hat{\theta}_j((1+|\xi'|^2)^{-\frac{1}{2}}\xi_n)|^2 \, d\xi_n d\xi' \\ &= C_s \, \|u\|^2_{H^{s-j-\frac{1}{2}}(\mathbb{R}^{n-1})} \end{aligned}$$

where

$$C_s = \int_{-\infty}^{\infty} (1+t^2)^s \, |\hat{\theta}_j(t)|^2 \, dt$$

is finite for all $s \in \mathbb{R}$. Density argument finishes the proof. ∎

Consider now the hyperplane:

$$F := \{x \in \mathbb{R}^n \,:\, x_n = 0\}\,.$$

The next theorem establishes a fundamental result about the subspace $H^s_F(\mathbb{R}^n)$ of distributions in $H^s(\mathbb{R}^n)$ with compact support in hyperplane F, for any $s \in \mathbb{R}$.

THEOREM 3.5.1 (Hörmander)

The following two scenarios hold:

(i) For $s \in [-\frac{1}{2}, \infty)$, space $H^s_F(\mathbb{R}^n)$ is trivial, $H^s_F(\mathbb{R}^n) = \{0\}$.

(ii) For $s \in (\infty, -\frac{1}{2})$, any $u \in H^s_F(\mathbb{R}^n)$ must be of the form:

$$u = \sum_{j=0}^{J} v_j \otimes \delta^{(j)} \quad \text{with } v_j \in H^{s+j+\frac{1}{2}}(\mathbb{R}^{n-1}) \tag{3.14}$$

where $J = \text{entier}(-s - \frac{1}{2})$ and $\delta^{(j)}$ denotes the j-th derivative of Dirac's delta.

PROOF **Step 1:** We begin by showing that distributions (3.14) belong to $H^s_F(\mathbb{R}^n)$.

$$\begin{aligned}
&\langle \mathcal{F}_{x\to\xi}(v_j(x') \otimes \delta^{(j)}(x_n)), \phi(\xi)\rangle \\
&\quad = \langle v_j(x') \otimes \delta^{(j)}(x_n), (\mathcal{F}_{\xi\to x}\phi)(x)\rangle \\
&\quad = \langle v_j(x') \otimes \delta^{(j)}(x_n), \hat{\phi}(x)\rangle \\
&\quad = \langle v_j(x'), \langle \delta^{(j)}(x_n), \hat{\phi}(x', x_n)\rangle_{\mathbb{R}}\rangle_{\mathbb{R}^{n-1}} \qquad \text{(definition of tensor product of distributions)} \\
&\quad = \langle v_j(x'), \underbrace{\frac{\partial^j \hat{\phi}}{\partial x_n^j}(x', 0)}_{=\mathcal{F}_{\xi\to x}((-i2\pi\xi_n)^j\phi)(x',0)} \rangle_{\mathbb{R}^{n-1}} = \ldots
\end{aligned}$$

But,

$$\begin{aligned}
\mathcal{F}_{\xi\to x}((-i2\pi\xi_n)^j\phi(\xi))(x', 0) &= \int_{\mathbb{R}^n} (-i2\pi\xi_n)^j \phi(\xi', \xi_n) e^{-i2\pi(\xi' x' + \xi_n 0)} \, d\xi' \, d\xi_n \\
&= \int_{\mathbb{R}} (-i2\pi\xi_n)^j (\mathcal{F}_{\xi'\to x'}\phi)(x', \xi_n) \, d\xi_n\,.
\end{aligned}$$

Continuing, commuttativity of tensor product implies:

$$\ldots \langle v_j(x'), \langle (-i2\pi\xi_n)^j, \hat{\phi}\rangle_{\mathbb{R}}\rangle_{\mathbb{R}^{n-1}} = \langle (-i2\pi\xi_n)^j, \langle v_j(x'), \hat{\phi}\rangle_{\mathbb{R}^{n-1}}\rangle_{\mathbb{R}}$$
$$= \langle (-i2\pi\xi_n)^j, \langle \hat{v}_j(\xi'), \phi\rangle_{\mathbb{R}^{n-1}}\rangle_{\mathbb{R}} \,.$$

We are ready to compute the Sobolev norms. With the substitution:

$$\xi_n = (1+|\xi'|^2)^{\frac{1}{2}} t, \quad 1+|\xi'|^2+\xi_n^2 = (1+|\xi'|^2)(1+t^2)$$

we have:

$$\begin{aligned}
\|v_j \otimes \delta^{(j)}\|^2_{H^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{v}_j(\xi')|^2 \, |2\pi\xi_n|^{2j} \, d\xi \\
&= \int_{\mathbb{R}^{n-1}} |\hat{v}_j(\xi')|^2 \int_{\mathbb{R}} (1+|\xi'|^2)^s \, (1+t^2)^s \, |2\pi(1+|\xi'|^2)^{\frac{1}{2}} t|^{2j} \, (1+|\xi'|^2)^{\frac{1}{2}} \, dt \, d\xi' \\
&= \int_{\mathbb{R}^{n-1}} |\hat{v}_j(\xi')|^2 \, (1+|\xi'|^2)^{s+j+\frac{1}{2}} \, d\xi' \underbrace{\int_{\mathbb{R}} (1+t^2)^s \, |2\pi t|^{2j} \, dt}_{=:C_{j,s}} \,.
\end{aligned}$$

Finally,

$$C_{j,s} < \infty \quad \text{for} \quad s+j < -\frac{1}{2} \qquad \text{and} \qquad C_{j,s} \to \infty \quad \text{for} \quad s+j \to -\frac{1}{2} \,.$$

Step 2: Let $u \in H^s_F(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$ be a test function such that $\frac{\partial^j \phi}{\partial x_n^j}(x',0) = 0$ for all $0 \le j \le -(s+\frac{1}{2})$. We will show that $\langle u, \phi\rangle = 0$. Towards this goal, define:

$$\phi_\pm(x) = \begin{cases} \phi(x) & x \in \mathbb{R}^n_\pm \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 3.5.1 shows that $\phi_\pm \in \tilde{H}^{-s}(\mathbb{R}^n_\pm)$. There exists thus a sequence of test functions $\phi^\pm_m \to \phi^\pm$ in $H^{-s}(\mathbb{R}^n)$. Consequently,

$$\mathcal{D}(\mathbb{R}^n - F) \ni \phi_m := \phi^-_m + \phi^+_m \to \phi^+ + \phi^- = \phi \quad \text{a.e.} \,.$$

This implies that

$$\langle u, \phi\rangle = \lim_{m\to\infty} \underbrace{\langle u, \phi_m\rangle}_{=0} = 0 \,.$$

In particular, if $u \in H^s_F(\mathbb{R}^n)$ for $s > -\frac{1}{2}$ (no conditions on ϕ then), u vanishes.

Step 3: Let $k \in \mathbb{N}$ and s be such that $-k-\frac{3}{2} < s < -k-\frac{1}{2}$. Let $u \in H^s_F(\mathbb{R}^n)$ and $0 \le j \le k$. Let η_j be the continuous extension operators discussed in Theorem 4.1.3. Define:

$$v_j \in \mathcal{D}'(\mathbb{R}^{n-1}), \quad \langle v_j, \phi\rangle := (-1)^j \langle u, \eta_j \phi\rangle \quad \phi \in \mathcal{D}(\mathbb{R}^{n-1}) \,.$$

We have,

$$|\langle v_j, \phi\rangle| \le \|u\|_{H^s(\mathbb{R}^n)} \, \|\eta_j \phi\|_{H^{-s}(\mathbb{R}^n)} \le C \|u\|_{H^s(\mathbb{R}^n)} \|\phi\|_{H^{-s-j-\frac{1}{2}}(\mathbb{R}^{n-1})} \,,$$

i.e.

$$\|v_j\|_{H^{s+j+\frac{1}{2}}(\mathbb{R}^{n-1})} \leq C\|u\|_{H^s(\mathbb{R}^n)}\,.$$

At the same time,

$$\begin{aligned}\langle u - \sum_{j=0}^{k} v_j \otimes \delta^{(j)}, \phi\rangle &= \langle u, \phi\rangle - \underbrace{\sum_{j=1}^{k} (-1)^j \langle v_j, \underbrace{\frac{\partial^j \phi}{\partial x_n^j}(\cdot, 0)}_{=:\psi_j}\rangle_{\mathbb{R}^n}}_{=\langle u, \eta_j \psi_j\rangle} \\ &= \langle u, \phi - \sum_{j=0}^{k} \eta_j \psi_j\rangle = 0 \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n)\,,\end{aligned}$$

by Step 2 result.

Step 4: Tricky case: $s = -k - \frac{1}{2}$. If $u \in H_F^{-k-\frac{1}{2}}(\mathbb{R}^n)$ then $u \in H_F^s(\mathbb{R}^n)$ for $s = -k - \frac{1}{2} - \epsilon$ as well, for any $\epsilon > 0$. By Step 3 result, there exist functions

$$\begin{aligned} &v_j \in H^{j-k}(\mathbb{R}^{n-1}) && 0 \leq j \leq k-1 \\ &v_k \in H^{-\epsilon}(\mathbb{R}^{n-1}) && \forall \epsilon > 0 \end{aligned}$$

such that $u = \sum_{j=0}^{k} v_j \otimes \delta^{(j)}$. It is sufficient to show that contribution $v_k = 0$. In particular, this will imply the first part of the theorem for $s = -\frac{1}{2}$. We have:

$$v_k \otimes \delta^{(k)} = u - \sum_{j=0}^{k-1} v_j \otimes \delta^{(j)} \in H^{-k-\frac{1}{2}}(\mathbb{R}^n)\,.$$

and

$$\|v_k \otimes \delta^{(k)}\|_{H^{-k-\frac{1}{2}-\epsilon}(\mathbb{R}^n)} = C_{k,k-\frac{1}{2}-\epsilon}\|v_k\|^2_{H^{-\epsilon}(\mathbb{R}^{n-1})}\,.$$

With $\epsilon \to 0$,

$$\|v_k \otimes \delta^{(k)}\|_{H^{-k-\frac{1}{2}-\epsilon}(\mathbb{R}^n)} \to \|v_k \otimes \delta^{(k)}\|_{H^{-k-\frac{1}{2}}(\mathbb{R}^n)} \qquad \text{and} \qquad C_{k,k-\frac{1}{2}-\epsilon} \to \infty$$

so $\|v_k\|^2_{H^{-\epsilon}(\mathbb{R}^{n-1})}$ must converge to zero. This implies:

$$|\langle v_k, \phi\rangle| \leq \underbrace{\|v_k\|_{H^{-\epsilon}(\mathbb{R}^n)}}_{\to 0}\,\underbrace{\|\phi\|_{H^{\epsilon}(\mathbb{R}^n)}}_{\text{bounded}}\,,$$

so, in the limit, we get $\langle v_k, \phi\rangle = 0$, for any test function ϕ. ∎

Invariance of Sobolev spaces under multiplication. Let $u \in H^s(\Omega)$. Under what assumptions on a function ψ, product ψu is in space $H^s(\Omega)$ as well ? Let us start with $s = 1$. For any distribution $u \in \mathcal{D}'(\Omega)$ and $\psi \in C^\infty(\Omega)$, we have the same formula for differentiating product ψu as in the classical calculus:

$$\frac{\partial}{\partial x_j}(\psi u) = \frac{\partial \psi}{\partial x_j} u + \psi \frac{\partial u}{\partial x_j} \quad \text{in } \mathcal{D}'(\Omega) \tag{3.15}$$

where the derivatives are understood in the sense of distributions. Indeed, for any $\phi \in \mathcal{D}(\Omega)$, we have:

$$
\begin{aligned}
\langle \frac{\partial}{\partial x_j}(\psi u), \phi\rangle &= -\langle \psi u, \frac{\partial \phi}{\partial x_j}\rangle && \text{(definition of distributional derivative)}\\
&= -\langle u, \psi \frac{\partial \phi}{\partial x_j}\rangle && \text{(definition of product of a } C^\infty \text{ function with a distribution)}\\
&= -\langle u, \frac{\partial}{\partial x_j}(\psi\phi) - \frac{\partial \psi}{\partial x_j}\phi\rangle \\
&= \langle \frac{\partial u}{\partial x_j}, \psi\phi\rangle + \langle u, \frac{\partial \psi}{\partial x_j}\phi\rangle && \text{(definition of distributional derivative)}\\
&= \langle \psi \frac{\partial u}{\partial \xi_j} + \frac{\partial \psi}{\partial x_j} u, \phi\rangle && \text{(definition of product of a } C^\infty \text{ function with a distribution.)}
\end{aligned}
$$

If we assume additionally $\psi \in C_0^\infty(\overline{\Omega})$, then both ψ and its derivatives are bounded and, therefore, $u, \frac{\partial u}{\partial x_j} \in L^2(\Omega)$ imply that $\psi u, \frac{\partial \psi}{\partial x_j} u, \psi \frac{\partial u}{\partial x_j} \in L^2(\Omega)$ as well. Consequently, equality (3.15) is satisfied in the L^2 sense as well. But this still leaves us with the assumption that ψ is infinitely differentiable.

We can reduce the assumption to $\psi \in C_0^1(\overline{\Omega})$ by proceeding slightly differently. We first note that the integration by parts formula:

$$\int_\Omega \frac{\partial u}{\partial x_j}\, \phi \, dx = -\int_\Omega u \, \frac{\partial \phi}{\partial x_j}\, dx$$

holds for any $u \in H^1(\Omega)$ and $\phi \in H^1(\Omega)$ with supp $\phi \subset \Omega$. Indeed, this is a consequence of the definition of distributional derivative and Lemma 3.3.2. We can now revisit the derivation above by replacing the duality pairings with the integral and use the same reasoning to establish the final result for $\psi \in C_0^1(\overline{\Omega})$. In fact, we can still do better by recalling the famous result of Rademacher [1919] that every uniformly Lipshitz continuous function $\psi \in C^{0,1}$ is differentiable a.e. with the derivatives uniformly bounded. This is sufficient to reproduce the same steps in the context of Lebesgue integrals. In conclusion, formula:

$$\frac{\partial}{\partial x_j}(\psi u) = \frac{\partial \psi}{\partial x_j} u + \psi \frac{\partial u}{\partial x_j} \quad \text{in } L^2(\Omega) \tag{3.16}$$

holds for any $\psi \in C^{0,1}(\Omega)$ and $u \in H^1(\Omega)$ and implies that the product remains in $H^1(\Omega)$. By induction, the conclusion generalizes to integer $s = k$ and $\psi \in C^{k-1,1}(\Omega)$. In the end, we obtain the following result.

LEMMA 3.5.2

Let $\psi \in C^{k-1,1}(\mathbb{R}^n)$. There exists a constant $C = C(k)$ such that

$$\|\psi u\|_{H^s(\Omega)} \le C \|\psi\|_{W^{k,\infty}(\mathbb{R}^n)} \, \|u\|_{H^s(\Omega)} \,,$$

for any $u \in H^s(\Omega)$, and $s \in [-k, k]$.

PROOF Prove the result first for $\Omega = \mathbb{R}^n$. Use duality to establish the result for negative $s = -k$ and interpolate between $-k$ and k for real s. For general Ω the result is a consequence of the definition of $H^s(\Omega)$. Let $U \in H^s(\mathbb{R}^n)$ be an extension of u. We have:

$$\|\psi u\|_{H^s(\Omega)} \le \|\psi U\|_{H^s(\mathbb{R}^n)} \le C \|\psi\|_{W^{k,\infty}(\mathbb{R}^n)} \, \|U\|_{H^s(\mathbb{R}^n)} \,,$$

and it remains to take the infimum with respect to U to get the final result. Note that we have assumed that function ψ is defined on the whole space to avoid technicalities related to the existence of a sufficiently regular extension to $\mathbb{R}^n$. ∎

Invariance of Sobolev spaces under change of variables. Let

$$T : \mathbb{R}^n \ni \xi \to x(\xi) \in \mathbb{R}^n$$

be a sufficiently regular bijective map. Under what assumptions on map T, spaces $H^s(\mathbb{R}^n)$ get mapped onto space $H^s(\mathbb{R}^n)$? In context of what we will need, we shall restrict ourselves only to maps with unit jacobian. For *functions* $u \in H^s(\mathbb{R}^n),\ s > 0$, we define $\hat{u}$ to be the composition of transformation T and function u,

$$\hat{u}(\xi) := u(x(\xi)), \quad \xi \in \mathbb{R}^n \,.$$

For distributions $u \in H^{-s}(\mathbb{R}^n),\ s > 0$, we define $\hat{u}$ by duality:

$$\langle \hat{u}, \hat{\phi} \rangle := \langle u, \phi \rangle \,.$$

LEMMA 3.5.3

Let T be such that (components of) T and T^{-1} are in $C^{k-1,1}(\mathbb{R}^n)$ with a unit jacobian. Then, for any $s \in [-k, k]$, $u \in H^s(\mathbb{R}^n)$ iff $\hat{u} \in H^s(\mathbb{R}^n)$ with equivalent norms.

PROOF Standard change of variables and density of $C_0^\infty(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ imply that the map

$$L^2(\mathbb{R}^n) \ni u \to \hat{u} \in L^2(\mathbb{R}^n)$$

is actually an isometry. The chain formula:

$$\frac{\partial \hat{u}}{\partial \xi_i} = \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial \xi_i} \qquad \text{(summation convention at work)}$$

and boundedness of derivatives $\frac{\partial x_j}{\partial \xi_i}$ for Lipschitz T imply that the map

$$H^1(\mathbb{R}^n) \ni u \to \hat{u} \in H^1(\mathbb{R}^n)$$

is bounded. By the same argument,

$$H^k(\mathbb{R}^n) \ni u \to \hat{u} \in H^k(\mathbb{R}^n)$$

is bounded for $T \in C^{k-1,1}$. Use the interpolation argument then to establish the boundedness for positive fractional s, and then the duality argument for negative s. We have essentially repeated arguments from the proof of Lemma 3.5.2. ∎

Note that, in case of a map T with a non-unit jacobian, the map:

$$H^s(\mathbb{R}^n) \ni u \to \hat{u} \in H^s(\mathbb{R}^n)$$

is bounded for a smaller range of $s \in [-k+1, k]$. This is due to the presence of the jacobian $jac \in C^{k-2,1}(\mathbb{R}^n)$ in the duality argument.

We are ready now to extend the result of Theorem 3.5.1 to a class of sufficiently regular domains.

THEOREM 3.5.2

Let Ω be a $C^{k-1,1}$ domain, $k = 1, 2, \ldots$, with boundary Γ. Let $u \in H^s_\Gamma(\mathbb{R}^n)$ for $s \in [-\frac{1}{2}, k]$. Then $u = 0$.

PROOF Let $G_j, j = 0, \ldots, J,$ be now maps like in the proof of Theorem 3.3.2, and ψ_j, $j = 0, \ldots, J,$ be the corresponding partition of unity subordinate to maps G_j. By Lemma 3.5.2, distribution $\psi_j u \in H^s(G_j)$ with a compact support in G_j. Its extension by zero lives in $H^s_{\Gamma_j}(\mathbb{R}^n)$ where Γ_j is the boundary of the corresponding hypograph domain. By Lemma 3.5.3, the corresponding distribution $\widehat{\psi_j u}$ lives in $H^s(\mathbb{R}^n)$ with a support in the hyperplane $F = \mathbb{R}^{n-1}$. By Theorem 3.5.1, $\widehat{\psi_j u} = 0$ and, therefore, $\psi_j u$, and $u = \sum_j \psi_j u$ must be zero as well. ∎

COROLLARY 3.5.1

Let Ω be a $C^{k-1,k}$ domain, $k = 1, 2, \ldots$. Let $s \in [-\frac{1}{2}, k]$. The restriction map:

$$H^s_{\bar{\Omega}}(\mathbb{R}^n) = \tilde{H}^s_\Omega(\mathbb{R}^n) \ni u \to u|_\Omega \in H^s(\Omega)\,,$$

is injective and, therefore, can be used to identify space $\tilde{H}^s_\Omega(\mathbb{R}^n)$ with a subspace of $H^s(\Omega)$, denoted by $\tilde{H}^s(\Omega)$. For $s \in [-\frac{1}{2}, \frac{1}{2}]$, spaces $H^s(\Omega), \tilde{H}^{-s}(\Omega)$ are dual to each other.

We shall spend the rest of this section discussing the $\tilde{H}^s(\Omega)$ spaces.

LEMMA 3.5.4

Let $s \in (0, \frac{1}{2})$. There exists a constant $C = C(s)$ such that

$$\int_0^\infty x^{-2s} |u(x)|^2 \, dx \le C(s) \int_0^\infty \int_0^\infty \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}} \, dxdy \qquad (3.17)$$

for every $u \in C_0^\infty([0, \infty))$. The same inequality holds for the range of $s \in (\frac{1}{2}, 1)$ under additional assumption that $u(0) = 0$. For $s = \frac{1}{2} \pm \epsilon$, $C(s) = O(\frac{1}{\epsilon^2})$.

PROOF We begin by noticing that the right-hand side is finite for the whole range of $s \in (0, 1)$.

Let supp $u \subset [0, L]$. Introducing change of variables:

$$\begin{cases} x - y = \xi \\ x + y = \eta \end{cases} \qquad \Rightarrow \qquad \begin{cases} x = (\xi + \eta)/2 \\ y = (\eta - \xi)/2 \end{cases}$$

we can bound the integral

$$\int_0^L \int_0^L \frac{|u(x) - u(y)|^2}{|x - y|^{1+2s}} \, dx \, dy$$

by

$$\frac{1}{2} \int_0^{\sqrt{2}L} \int_{-\eta}^{\eta} \frac{|u(\frac{\xi+\eta}{2}) - u(\frac{\eta-\xi}{2})|^2}{|\xi|^{1+2s}} \, d\xi d\eta \, .$$

As u is Lipschitz continuous (explain, why?),

$$|u(\frac{\xi + \eta}{2}) - u(\frac{\eta - \xi}{2})| \leq C|\frac{\xi + \eta}{2} - \frac{\eta - \xi}{2}| = C|\xi| \, ,$$

and we can bound the integral by:

$$\frac{1}{2} C^2 \int_0^{\sqrt{2}L} \int_{-\eta}^{\eta} |\xi|^{1-2s} \, d\xi d\eta = C^2 \int_0^{\sqrt{2}L} \int_0^{\eta} \xi^{1-2s} \, d\xi \, d\eta = \frac{C^2}{2(1-s)} \int_0^{\sqrt{2}L} \eta^{2(1-s)} \, d\eta < \infty \, .$$

First version of the proof. We will present first the proof from [16], Theorem 1.4.4.4, see also [18], Lemma 3.31. We have:

$$\begin{aligned} u(x) &= u(x) - \frac{1}{x} \int_0^x u(y) \, dy + \frac{1}{x} \int_0^x u(y) \, dy \\ &= \underbrace{\frac{1}{x} \int_0^x (u(x) - u(y)) \, dy}_{=:v(x)} + \underbrace{\frac{1}{x} \int_0^x u(y) \, dy}_{=:w(x)} \end{aligned}$$

Cauchy-Schwarz inequality implies that

$$|v(x)|^2 \leq \frac{1}{x} \int_0^x |u(x) - u(y)|^2 \, dy \, .$$

Consequently,

$$\begin{aligned} \int_0^\infty x^{-2s} |v(x)|^2 \, dx &\leq \int_0^\infty x^{-1-2s} \int_0^x |u(x) - u(y)|^2 \, dy dx \\ &= \int_0^\infty \int_y^\infty x^{-1-2s} |u(x) - u(y)|^2 \, dx dy \qquad \text{(Fubini)} \\ &\leq \int_0^\infty \int_0^\infty \frac{|u(x) - u(y)|^2}{|x - y|^{1+2s}} \, dx dy \, , \end{aligned}$$

so the weighted norm of $v(x)$ is under control for the whole range of $s \in (0, 1)$. We will show now that the weighted norm of component $w(x)$ can be estimated by the weighted norm of $v(x)$ and, in turn, by the Sloboditskij seminorm as well. It is a bit tricky to see that

$$w(x) = \frac{1}{x} \int_0^x u(y) \, dy = -\int_x^\infty \frac{v(y)}{y} \, dy \, .$$

Indeed, notice that both sides vanish at infinity and, upon differentiating both sides with respect to x, we get:

$$-\frac{1}{x^2}\int_0^x u(y)\,dy + \frac{1}{x}u(x) = \frac{v(x)}{x}$$

which easily follows from the very definition of $v(x)$.

Case: $s \in (0, \frac{1}{2})$. Apply the second Hardy inequality (2.4),

$$\begin{aligned}\int_0^\infty x^{-2s}\Big|\int_x^\infty \frac{v(y)}{y}\,dy\Big|^2\,dx &= \int_0^\infty \Big|x^{\frac{1-2s}{2}}\int_x^\infty v(y)\,\frac{dy}{y}\Big|^2\,\frac{dx}{x}\\ &\le \frac{1}{(\frac{1}{2}-s)^2}\int_0^\infty |y^{\frac{1-2s}{2}}v(y)|^2\,\frac{dy}{y} \qquad (\alpha = \tfrac{1-2s}{2})\\ &= \frac{1}{(\frac{1}{2}-s)^2}\int_0^\infty y^{2s}|v(y)|^2\,dy\end{aligned}$$

Case: $s \in (\frac{1}{2}, 1)$. With $u(0) = 0$, $w(0) = 0$ as well and, therefore,

$$w(x) = w(x) - w(0) = -\int_x^\infty \frac{v(y)}{y}\,dy - \Big(-\int_0^\infty \frac{v(y)}{y}\,dy\Big) = -\int_0^x \frac{v(y)}{y}\,dy\,.$$

Apply the first Hardy inequality (2.3),

$$\begin{aligned}\int_0^\infty x^{-2s}\Big|\int_0^x \frac{v(y)}{y}\,dy\Big|^2\,dx &= \int_0^\infty \Big|x^{-\frac{2s-1}{2}}\int_0^x v(y)\,\frac{dy}{y}\Big|^2\,\frac{dx}{x}\\ &\le \frac{1}{(s-\frac{1}{2})^2}\int_0^\infty |y^{-\frac{2s-1}{2}}v(y)|^2\,\frac{dy}{y} \qquad (\alpha = \tfrac{2s-1}{2})\\ &= \frac{1}{(s-\frac{1}{2})^2}\int_0^\infty y^{-2s}|v(y)|^2\,dy\end{aligned}$$

Second version of the proof. Recall that the proof of Hardy's inequalities was based on the use of Integral Minkowski Inequality. It is not a surprise then that the theorem can be proved directly by means of the Minkowski inequality.

Case: $s \in (0, \frac{1}{2})$. We demonstrate the bound for the weighted norm of function $w(x)$. The rest of the proof remains the same. We have,

$$\int_0^\infty x^{-2s}|w(x)|^2\,dx = \int_0^\infty x^{-2s-2}\Big|\int_0^x u(t)\,dt\Big|^2\,dx = \ldots$$

Let $U \in C_0^\infty(\mathbb{R})$ by a symmetric extension of u. Introducing the Fourier transform of U, we transform the inner integral to:

$$\begin{aligned}\int_0^x u(t)\,dt &= \int_0^x\int_{\mathbb{R}} e^{2\pi i\omega t}\hat{U}(\omega)\,d\omega\,dt\\ &= \int_{\mathbb{R}} \underbrace{\int_0^x e^{2\pi i\omega t}\,dt}_{\frac{1}{2\pi i\omega}e^{2\pi i\omega t}|_0^x = \frac{1}{2\pi i\omega}(e^{2\pi i\omega x}-1)} \hat{U}(\omega)\,d\omega \qquad \text{(Fubini)}\\ &= \int_{\mathbb{R}} \frac{1}{2\pi i\omega}(e^{2\pi i\omega x}-1)\hat{U}(\omega)\,d\omega\,.\end{aligned}$$

Continuing,

$$
\begin{aligned}
\ldots &= \int_0^\infty x^{-2s-2} \left| \int_{\mathbb{R}} \frac{1}{2\pi i \omega} (e^{2\pi i \omega x} - 1) \hat{U}(\omega)\, d\omega \right|^2 dx \\
&= \int_0^\infty x^{-2s-2} \left| \int_{\mathbb{R}} \frac{1}{2\pi i t} (e^{2\pi i t} - 1) \hat{U}(x^{-1}t)\, dt \right|^2 dx && \text{(change of variable: } \omega x = t) \\
&\le \left[\int_{\mathbb{R}} \left(\int_0^\infty x^{-2s-2} \frac{|e^{2\pi i t} - 1|^2}{4\pi^2 t^2} |\hat{U}(x^{-1}t)|^2 \, dx \right)^{\frac{1}{2}} dt \right]^2 && \text{(Integral Minkowski inequality)} \\
&= \left[\int_{\mathbb{R}} \frac{|e^{2\pi i t} - 1|}{2\pi t} \left(\int_0^\infty t^{-2s-2} \omega^{2s+2} |\hat{U}(\omega)|^2 t\omega^{-2} \, d\omega \right)^{\frac{1}{2}} dt \right]^2 && \text{(change of variable: } x^{-1}t = \omega,\ dx = -t\omega^{-2}\, d\omega) \\
&= \left[\int_{\mathbb{R}} \frac{|e^{2\pi i t} - 1|}{2\pi t} t^{-s-\frac{1}{2}} \left(\int_0^\infty \omega^{2s} |\hat{U}(\omega)|^2 \, d\omega \right)^{\frac{1}{2}} dt \right]^2 \\
&= \underbrace{\left[\frac{1}{2\pi} \int_{\mathbb{R}} \frac{|e^{2\pi i t} - 1|}{t} t^{-2s-\frac{1}{2}} \, dt \right]^2}_{\text{finite, of order } 1/(\frac{1}{2}-s)^2} \int_0^\infty \omega^{2s} |\hat{U}(\omega)|^2 \, d\omega
\end{aligned}
$$

By Lemma 3.2.1, the Bessel seminorm is equivalent to the Sloboditskij seminorm,

$$
\int_0^\infty \omega^{2s} |\hat{U}(\omega)|^2 \, d\omega = a_\mu^{-1} \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{|U(x) - U(y)|^2}{|x-y|^{1+2s}} \, dy dx \,,
$$

and,

$$
\begin{aligned}
\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{|U(x) - U(y)|^2}{|x-y|^{1+2s}} \, dy dx &= \int_{-\infty}^0 \int_{-\infty}^0 \ldots + \int_{-\infty}^0 \int_0^\infty \ldots + \int_0^\infty \int_{-\infty}^0 \ldots + \int_0^\infty \int_0^\infty \ldots \\
&\le 4 \int_0^\infty \int_0^\infty \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}} \, dy dx \,.
\end{aligned}
$$

Case: $s \in (\frac{1}{2}, 1)$. We proceed now directly without splitting function $u(x)$.

$$
\int_0^\infty x^{-2s} |u(x)|^2 \, dx = \int_0^\infty x^{-2s} |u(x) - \underbrace{u(0)}_{=0}|^2 \, dx = \int_0^\infty x^{-2s} \left| \int_0^x u'(t) \, dt \right|^2 dx \,.
$$

We transform now the inner integral,

$$
\begin{aligned}
\int_0^x u'(t) \, dt &= \int_0^x \int_{\mathbb{R}} e^{2\pi i \omega t} \underbrace{\widehat{U'}(\omega)}_{2\pi i \omega \hat{U}(\omega)} \, d\omega \, dt \\
&= \int_{\mathbb{R}} \underbrace{\int_0^x e^{2\pi i \omega t} \, dt}_{\frac{e^{2\pi i \omega t}}{2\pi i \omega}|_0^x} \, 2\pi i \omega \, \hat{U}(\omega) \, d\omega && \text{(Fubini)} \\
&= \int_{\mathbb{R}} (e^{2\pi i \omega x} - 1) \hat{U}(\omega) \, d\omega
\end{aligned}
$$

Continuing,

$$\begin{aligned}
&\int_0^\infty x^{-2s} \Big| \int_{\mathbb{R}} (e^{2\pi i\omega x} - 1)\hat{U}(\omega)\, d\omega \Big|^2 dx \\
&= \int_0^\infty x^{-2s-2} \Big| \int_{\mathbb{R}} (e^{2\pi i t} - 1)\hat{U}(x^{-1}t)\, dt \Big|^2 dx \qquad (\omega x = t,\, d\omega = x^{-1} dt) \\
&\le \left[\int_{\mathbb{R}} \left(\int_0^\infty x^{-2s-2} |e^{2\pi i t} - 1|^2\, |\hat{U}(x^{-1}t)|^2\, dx \right)^{\frac{1}{2}} dt \right]^2 \\
&= \left[\int_{\mathbb{R}} |e^{2\pi i t} - 1| \left(\int_0^\infty t^{-2s-2} \omega^{2s+2} |\hat{U}(\omega)|^2\, t\omega^{-2}\, d\omega \right)^{\frac{1}{2}} dt \right]^2 \quad (x^{-1}t = \omega,\, dx = -t\omega^{-2}\, d\omega) \\
&= \left[\int_{\mathbb{R}} |e^{2\pi i t} - 1| t^{-s-\frac{1}{2}} \left(\int_0^\infty \omega^{2s} |\hat{U}(\omega)|^2\, d\omega \right)^{\frac{1}{2}} dt \right]^2 \\
&= \underbrace{\left[\int_{\mathbb{R}} |e^{2\pi i t} - 1|\, t^{-s-\frac{1}{2}}\, dt \right]^2}_{=C(s)} \int_0^\infty \omega^{2s} |\hat{U}(\omega)|^2\, d\omega
\end{aligned}$$

For $t \to 0$, $e^{2\pi i t} - 1 \sim 2\pi t$ and the integrand is of order $t\, t^{-s-\frac{1}{2}} = t^{-s+\frac{1}{2}}$ which is integrable for $s < \frac{3}{2}$. For large t, $|e^{2\pi i t} - 1| \le 2$ and factor $t^{-s-\frac{1}{2}}$ is integrable for $s > \frac{1}{2}$. Note that

$$\int_1^\infty t^{-s-\frac{1}{2}}\, dt = \frac{1}{s - \frac{1}{2}},$$

i.e. the blow up at $s \to \frac{1}{2}_+$ is the same as for $s \to \frac{1}{2}_-$. ∎

Recall that $d(x, F)$ denote the distance of point x from a closed set F,

$$d(x, F) := \min_{y \in F} d(x, y),$$

where $d(x, y)$ stands for the Euclidean distance. The following lemma is a generalization of Lemma 3.5.4. to multidimensional case.

LEMMA 3.5.5

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, and $s \in (0, \frac{1}{2})$. There exists a constant $C = C(s)$ such that

$$\int_\Omega d(x, \Gamma)^{-2s} |u(x)|^2\, dx \le C(s) \|u\|^2_{H^s(\Omega)}, \tag{3.18}$$

for every $u \in C_0^\infty(\overline{\Omega})$. The same inequality holds for the range $s \in (\frac{1}{2}, 1)$ under additional assumption that $u = 0$ on Γ.

PROOF **Case:** Ω is a Lipschitz hypograph. Let $x \in \Omega$ and $y \in \Gamma$ as depicted in Fig. 3.3. We have:

$$|\zeta(x') - x_n| = |y_n - x_n + \zeta(x') - \zeta(y')| \le |x_n - y_n| + M|x' - y'| \le \sqrt{1 + M^2} |x - y|$$

where M is Lipschitz constant for function $\zeta(x')$ defining the boundary. Consequently,

$$d(x,\Gamma) \geq \frac{\zeta(x') - x_n}{\sqrt{1+M^2}}$$

and, therefore,

$$\begin{aligned}
\int_\Omega d(x,\Gamma)^{-2s}|u(x)|^2\,dx &\leq (1+M^2)^s \int_{x_n<\zeta(x')} (\zeta(x')-x_n)^{-2s}\,|u(x)|^2\,dx \\
&= (1+M^2)^s \int_{\mathbb{R}^{n-1}} \int_0^\infty t^{-2s}|u(x',\zeta(x')-t)|^2\,dt\,dx' \\
&\leq C \int_{\mathbb{R}^{n-1}} \int_0^\infty \int_0^\infty \frac{|u(x',\zeta(x')-t)-u(x',\zeta(x')-s)|^2}{|t-s|^{1+2s}}\,dt\,ds\,dx' && \text{(Lemma 3.5.4)} \\
&= C \int_{\mathbb{R}^{n-1}} \int_{y<\zeta(x')} \int_{z<\zeta(x')} \frac{|u(x',y)-u(x',z)|^2}{|\underbrace{y-z}_{=:h}|^{1+2s}}\,dy\,dz\,dx' \\
&\leq C \int_{-\infty}^\infty \int_{\mathbb{R}^n} \frac{|U(x',x_n+h)-u(x)|^2}{|h|^{1+2s}}\,dx\,dh && (U\in C_0^\infty(\mathbb{R}^n),\ U|_\Omega = u) \\
&= C \int_{\mathbb{R}^n} |\hat{U}(\xi)|^2 \underbrace{\int_{-\infty}^\infty \frac{|e^{i2\pi\xi_n h}-1|^2}{|h|^{1+2s}}\,dh}_{\text{same integral as in the proof of Lemma 3.2.1}}\,d\xi \\
&= C \int_{\mathbb{R}^n} |\xi_n|^{2s}\,|\hat{U}(\xi)|^2\,d\xi \leq C\|U\|^2_{H^s(\mathbb{R}^n)}
\end{aligned}$$

Taking infimum with respect to extensions U finishes the proof.

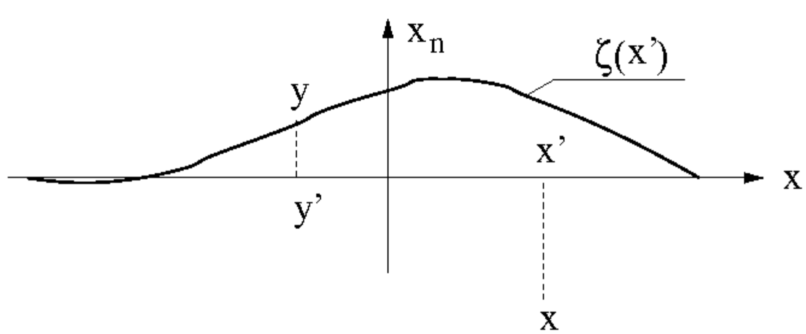

Figure 3.3
Notation for the proof of Lemma 3.5.5.

Case: Ω is a Lipschitz domain. Use the partition of unity argument.

∎

Spaces $H_0^s(\Omega)$. Let $s \in \mathbb{R}$. We define one more Sobolev space:

$$H_0^s(\Omega) := \overline{C_0^\infty(\Omega)}^{H^s(\Omega)} . \tag{3.19}$$

The space can be viewed as a prototype of energy space for the case of homogeneous (essential) boundary conditions.

The following theorem is our final characterization of spaces $\tilde{H}^s(\Omega)$ for Lipschitz domains.

THEOREM 3.5.3

Let Ω be a $C^{k-1,1}$ domain, and $s \in [-\frac{1}{2}, k]$. The following properties hold:

(i) $\tilde{H}^s(\Omega) \subset H_0^s(\Omega)$, and

(ii) $\tilde{H}^s(\Omega) = H_0^s(\Omega)$, except for $s = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots \le k$. In particular, for $s \in (-\frac{1}{2}, \frac{1}{2})$,

$$\tilde{H}^s(\Omega) = H_0^s(\Omega) = H^s(\Omega) .$$

PROOF (i) follows immediately from definitions and the continuity of the restriction operator:

$$R \,:\, H^s(\mathbb{R}^n) \ni U \to U|_\Omega \in H^s(\Omega) .$$

(ii) **Case:** $s = k \in \mathbb{N}$, $u \in C_0^\infty(\Omega)$. Let $\tilde{u}$ denote the zero extension of u. We have:

$$\begin{aligned} \|u\|^2_{\tilde{H}^k(\Omega)} &= \|\tilde{u}\|^2_{H^k(\mathbb{R}^n)} \sim \sum_{|\alpha| \le k} \|\partial^\alpha \tilde{u}\|^2_{L^2(\mathbb{R}^n)} \qquad (\partial^\alpha \tilde{u} = \widetilde{\partial^\alpha u}) \\ &= \sum_{|\alpha|\le k} \|\partial^\alpha u\|^2_{L^2(\Omega)} = \|u\|^2_{W^k(\Omega)} \sim \|u\|^2_{H^k(\Omega)} . \end{aligned}$$

Consequently,

$$\overline{C_0^\infty(\Omega)}^{H^s(\mathbb{R}^n)} = \overline{C_0^\infty(\Omega)}^{H^s(\Omega)} .$$

Case: $s = k + \mu$, $k \in \mathbb{N}, \mu \in (0,1)$. Let again $u \in C_0^\infty(\Omega)$.

$$\|u\|^2_{\tilde{H}^s(\Omega)} = \|\tilde{u}\|^2_{H^s(\mathbb{R}^n)} \sim \sum_{|\alpha|\le k} \|\partial^\alpha \tilde{u}\|^2_{L^2(\mathbb{R}^n)} + a_\mu^{-1} \sum_{|\alpha| = k} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\partial^\alpha \tilde{u}(x) - \partial^\alpha \tilde{u}(y)|^2}{|x-y|^{2\mu+n}} \, dx \, dy$$

The double integral can be broken into four parts with the last contribution vanishing:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \ldots = \int_\Omega \int_\Omega \ldots + \int_\Omega \int_{\mathbb{R}^n - \Omega} \ldots + \int_{\mathbb{R}^n-\Omega} \int_\Omega \ldots + \underbrace{\int_{\mathbb{R}^n-\Omega} \int_{\mathbb{R}^n-\Omega} \ldots}_{=0} .$$

The second integral can be reduced to a weighted L^2-norm:

$$\int_\Omega \int_{\mathbb{R}^n-\Omega} \frac{|\partial^\alpha u(x)|^2}{|x-y|^{2\mu+n}} \, dy \, dx = \int_\Omega |\partial^\alpha u(x)|^2 \underbrace{\int_{\mathbb{R}^n-\Omega} \frac{dy}{|x-y|^{2\mu+n}}}_{w_\mu(x)} \, dx .$$

It remains to use Lemma 3.5.5 to estimate the weight. Introduce an auxiliary spherical system of coordinates centered at x. Let $\hat{x}$ be a point on the unit sphere, and let

$$r(\hat{x}) := \sup\{r \,:\, [x, x + r\hat{x}) \subset \Omega\}\,,$$

see Fig. 3.4 for notation.

$$\int_{\mathbb{R}^n - \Omega} \frac{dy}{|x-y|^{2\mu+n}} \leq \int_{|\hat{x}|=1} \underbrace{\int_{r(\hat{x})}^{\infty} \frac{1}{r^{2\mu+n}}\, r^{n-1} dr}_{=\frac{1}{2\mu}\frac{1}{r(\hat{x})^{2\mu}}}\, dS \leq \frac{|S|}{2\mu}\, d(x,\Gamma)^{-2\mu}$$

where $|S|$ is the measure of the unit sphere. Consequently,

$$\|u\|^s_{\tilde{H}^s(\Omega)} \lesssim \|u\|^2_{W^k(\Omega)} + \frac{a_\mu^{-1}}{\mu} \sum_{|\alpha|=k} \int_\Omega d(x,\Gamma)^{-2\mu}\, |\partial^\alpha u(x)|^2 \leq C(\mu) \|u\|^2_{H^s(\Omega)}\,, \tag{3.20}$$

for $\mu \neq 0$. We shall return to a careful analysis of constant $C(\mu)$ shortly.

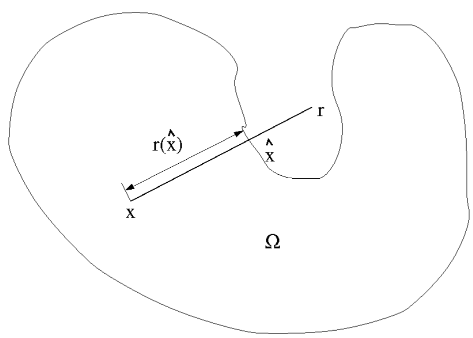

Figure 3.4
Local spherical coordinates used in the proof of Theorem 3.5.3.

Case: $s \in (-\frac{1}{2}, 0]$. For $s \in [0, \frac{1}{2})$,

$$\tilde{H}^s(\Omega) = H^s_0(\Omega) = H^s(\Omega)\,.$$

The last inequality follows from Theorem 3.5.2 and Mazur Separation Theorem [20], Lemma 5.13.1. Indeed, consider an arbitrary $w \in (H^s(\Omega))' = \tilde{H}^{-s}(\Omega)$ such that

$$\langle w, \phi \rangle = 0 \quad \forall \phi \in C_0^\infty(\Omega)\,.$$

Orthogonality of w to test functions implies that $\operatorname{supp} w \subset \Gamma$. By Theorem 3.5.2 then $w = 0$. In other words, the orthogonal component of $\overline{C_0^\infty(\Omega)}^{H^s(\Omega)}$ is trivial. Mazur's Theorem implies that $\overline{C_0^\infty(\Omega)}^{H^s(\Omega)}$ must coincide with the whole space $H^s(\Omega)$. The final result follows now from duality,

$$H^{-s}(\Omega) = (\tilde{H}^s(\Omega))' = (H^s(\Omega))' = \tilde{H}^{-s}(\Omega)\,.$$

$\blacksquare$

REMARK 3.5.1 Let the assumptions of Theorem 3.5.3 hold. Space $\tilde{H}^s(\Omega)$ can be characterized as a subspace of $H^s(\Omega)$ functions admitting zero extensions,

$$\tilde{H}^s(\Omega) = \{u \in H^s(\Omega) \, : \, \tilde{u} \in H^s(\mathbb{R}^n)\} \tag{3.21}$$

where $\tilde{u}$ denotes the extension of u by zero. For functions u, i.e. for $s \geq 0$, the notion of the zero extension is clear,

$$\tilde{u}(x) := \begin{cases} u(x) & x \in \Omega \\ 0 & x \in \mathbb{R}^n - \Omega \end{cases}$$

The delicate point of the statement above concerns the negative range of s. What do we mean by the zero extension of a distribution (functional) ? From the very definition of $\tilde{H}^s(\Omega)$ follows that

$$\langle u, \phi \rangle = 0 \quad \forall \phi \in C_0^\infty(\mathbb{R}^n - \overline{\Omega}) \, ,$$

and, therefore, we may mean by ***a zero extension*** of u any $U \in H^s(\mathbb{R}^n)$ with the support in $\overline{\Omega}$ such that $U|_\Omega = u$. A difference of two such extensions must have a support in $\Gamma = \partial\Omega$ and, therefore, by Theorem 3.5.2, it must be zero. This makes the zero extension unique. $\blacksquare$

REMARK 3.5.2 Let us talk about the behavior of embedding constant $C(\mu)$ in (3.20) as $\mu \to 0$. Constant a_μ^{-1}/μ present in (3.20) is of order one as $\mu \to 0$. However, constant in proof of Lemma 3.5.5 blows up at $\mu \to 0$. This can be traced all the way to the proof of Lemma 3.5.4 where we estimate the weighted L^2 norm of function $v(x)$ by the Sloboditskij norm. The blow up of $C(\mu)$ at zero is not expected as for $\mu \to 0$ we converge to the L^2-norm, and we expect the constant to converge to one. Fortunately, the interpolation argument discussed in Corollary 3.6.1 establishes what we expect, i.e., the embedding constant indeed converges to one as $\mu \to 0$. In the end, this simply points out to a deficiency of argument in proof of Lemma 3.5.4. $\blacksquare$

Exercises

Exercise 3.5.1 (Exercise 3.22 in [18]) Consider half space: $\mathbb{R}^n_- := \{x \in \mathbb{R}^n \, : \, x_n < 0\}$. Let u be a restriction of a function from $C_0^\infty(\mathbb{R}^n)$ to $\mathbb{R}^n_-$ and let U denote its extension by zero:

$$U(x) := \begin{cases} u(x) & \text{for } x_n < 0 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Demonstrate that

$$|\hat{U}(\xi)| \leq C_k (1 + |\xi'|)^{-k} (1 + |\xi_n|)^{-1}$$

for every $k > 0$.

(ii) Conclude that $U \in H^s_{\overline{\mathbb{R}^n_-}}(\mathbb{R}^n)$ for $s < \frac{1}{2}$.

(iii) Show additionally that

$$\partial_n^k u(x',0) = 0 \quad \text{for } 0 \leq k \leq j \qquad \Rightarrow \qquad U \in H^s_{\overline{\mathbb{R}^n_-}}(\mathbb{R}^n) \quad \text{for } s < j + \frac{3}{2}\,.$$

Exercise 3.5.2 Prove that

(i) $u = \ln|\ln|x|| \in H^{\frac{1}{2}}(-\frac{1}{2}, \frac{1}{2})$.

(ii) $u' = \frac{1}{x \ln|x|} \in H^{-\frac{1}{2}}(-\frac{1}{2}, \frac{1}{2})$.

(iii) $u = \ln|\ln|x|| \in H^{\frac{1}{2}}(0, \frac{1}{2})$ but it is *not* in $\tilde{H}^{\frac{1}{2}}(0, \frac{1}{2})$.

(ii) $u' = \frac{1}{x \ln|x|} \in H^{-\frac{1}{2}}(0, \frac{1}{2})$ but it is *not* in $\tilde{H}^{-\frac{1}{2}}(0, \frac{1}{2})$.

Hint: Use Exercise 3.2.2, Trace Theorem and Lemma 3.5.5.

3.6 Real Interpolation Method

Let X_0, X_1 be two normed subspaces of a common vector space X. We say that spaces X_0, X_1 are *compatible*. We equip the corresponding spaces $X_0 \cap X_1$ and $X_0 + X_1$ with the norms:

$$\|u\|_{X_0\cap X_1} := (\|u\|^2_{X_0} + \|u\|^2_{X_1})^{1/2} \approx \|u\|_{X_0} + \|u\|_{X_1}$$
$$\|u\|_{X_0+X_1} := \inf\{(\|u_0\|^2_{X_0} + \|u_1\|^2_{X_1})^{1/2} \,:\, u = u_0 + u_1, \quad u_0 \in X_0,\, u_1 \in X_1\}\,.$$

We have the obvious (continuous) embeddings:

$$X_0 \cap X_1 \hookrightarrow X_j \hookrightarrow X_0 + X_1 \quad j = 1, 2\,.$$

The interpolation problem consists of constructing a family of normed spaces:

$$X_{\theta,q} = (X_0, X_1)_{\theta,q} \quad 0 < \theta < 1,\, 1 \le q \le \infty$$

such that

- We have the embedding:

$$X_0 \cap X_1 \hookrightarrow X_{\theta,q} \hookrightarrow X_0 + X_1 \quad j = 1, 2\,. \tag{3.22}$$

- Space $X_{\theta,q}$ has the following *interpolation property*: for another *compatible* spaces Y_0, Y_1, and *compatible operators*: $A_0 \in \mathcal{L}(X_0, Y_0),\ A_1 \in \mathcal{L}(X_1, Y_1)$, i.e.,

$$A_0 u = A_1 u \quad \forall u \in X_0 \cap X_1\,,$$

  there exists *a unique* operator $A_\theta : X_{\theta,q} \to Y_{\theta,q}$ such that

$$A_\theta = A_0 u = A_1 u \quad \forall u \in X_0 \cap X_1 \quad \text{and}$$
$$\|A_\theta\| \le \|A_0\|^{1-\theta}\, \|A_1\|^\theta\,.$$

3.6.1 Real Interpolation (the K-) Method

We begin by introducing the so-called K-functional. Let $t > 0, u \in X_0 + X_1$. We define

$$K(t,u) := \inf\{(\|u_0\|^2_{X_0} + t^2\|u_1\|^2_{X_1})^{1/2} \,:\, u = u_0 + u_1, \quad u_0 \in X_0,\, u_1 \in X_1\}\,.$$

For a fixed $t > 0$, the K-functional is an equivalent norm on $X_0 + X_1$,

$$\min\{1, t\}\|u\|_{X_0+X_1} \le K(t,u) \le \max\{1,t\}\|u\|_{X_0+X_1}\,.$$

For a fixed u, $K(t,u)$ is (weakly) increasing in t. Moreover, see Exercise 3.6.1,

$$\min\{1, \frac{t}{s}\}K(s,u) \le K(t,u) \le \max\{1, \frac{t}{s}\}K(s,u)\,.$$

For a class of functions $f(t), t > 0$, we introduce now a weighted L^q-norm,

$$\|f\|_{\theta,q} := \begin{cases} \left(\int_0^\infty |t^{-\theta} f(t)|^q \, \frac{dt}{t}\right)^{1/q} & 1 \le q < \infty \\ \text{ess sup}_{t>0} |t^{-\theta} f(t)| & q = \infty \,. \end{cases} \tag{3.23}$$

Note that the weighted norm satisfies the *dilatation property*:

$$\|t \to f(at)\|_{\theta,q} = a^\theta \|f\|_{\theta,q} \,.$$

Finally, we define the normed space,

$$K_{\theta,q}(X) := \{u \in X_0 + X_1 \, : \, \|K(\cdot, u)\|_{\theta,q} < \infty\} \tag{3.24}$$

with the norm,

$$\|u\|_{K_{\theta,q}(X)} = N_{\theta,q} \, \|K(\cdot, u)\|_{\theta,q}$$

where $N_{\theta,q}$ is a normalizing factor to be specified later.

THEOREM 3.6.1

The following inequalities hold:

(i) $X_0 \cap X_1 \subset K_{\theta,q}(X)$, *and*

$$\|u\|_{K_{\theta,q}(X)} \le \|u\|_{X_0}^{1-\theta} \, \|u\|_{X_1}^\theta \le \|u\|_{X_0 \cap X_1} \qquad u \in X_0 \cap X_1 \,.$$

(ii) $K_{\theta,q}(X) \subset X_0 + X_1$, *and*

$$K(t, u) \le t^\theta \|u\|_{K_{\theta,q}(X)} \quad \text{and} \quad \|u\|_{X_0+X_1} \le \|u\|_{K_{\theta,q}(X)} \qquad u \in K_{\theta,q}(X) \,.$$

provided we use the normalizing factor:

$$N_{\theta,q} = \| \min\{1, \cdot\}\|_{\theta,q}^{-1} = \begin{cases} [q\theta(1-\theta)]^{1-q} & 1 \le q < \infty \,, \\ 1 & q = \infty \,. \end{cases} \tag{3.25}$$

PROOF For $u = 0$ both results are clear. Assume $u \ne 0$.

(i) We have:

$$K(t, u) \le \min\{\|u\|_{X_0}, t\|u\|_{X_1}\} = \|u\|_{X_0} \, \min\{1, at\}$$

with $a := \|u\|_{X_1} / \|u\|_{X_0}$. Consequently,

$$\|K(\cdot, u)\|_{\theta,q} \le \|u\|_{X_0} a^\theta \| \min\{1, \cdot\}\|_{\theta,q} = \|u\|_{X_0}^{1-\theta} \, \|u\|_{X_1}^\theta / N_{\theta,q}$$

because of the dilatation property. This implies the first inequality:

$$\|u\|_{K_{\theta,q}(X)} \le \|u\|_{X_0}^{1-\theta} \, \|u\|_{X_1}^\theta \,.$$

Take now $p = (1-\theta)^{-1}$. The Young's inequality implies:

$$\|u\|_{X_0}^{1-\theta}\,\|u\|_{X_1}^{\theta} \le \frac{1}{p}\left(\|u\|_{X_0}^{1-\theta}\right)^p + \frac{1}{p^*}\left(\|u\|_{X_1}^{\theta}\right)^{p^*} = (1-\theta)\|u\|_{X_0} + \theta\|u\|_{X_1}$$
$$\le [(1-\theta)^2+\theta^2]^{\frac{1}{2}}\,\|u\|_{X_0\cap X_1} \le \|u\|_{X_0\cap X_1}\,.$$

(ii) It follows from the inequality: $\min\{1,\frac{s}{t}\}K(t,u) \le K(s,u)$ (see Exercise 3.6.1) and the dilatation property that :

$$t^{-\theta}\|\min\{1,\cdot\}\|_{\theta,q}\,K(t,u) \le \|K(\cdot,u\|_{\theta,q}$$

which in turn implies $K(t,u) \le t^{\theta}\|u\|_{X_{\theta,q}}$. To prove the last inequality, recall that

$$\min\{1,t\}\|u\|_{X_0+X_1} \le K(t,u)\,.$$

Computing the (θ,q)-norm of both sides and utilizing the norm homogeneity, we obtain:

$$\|u\|_{X_0+X_1}\frac{1}{N_{\theta,q}} \le \|K(\cdot,u)\|_{\theta,q}\,.$$

∎

REMARK 3.6.1 With the normalizing factor used in Theorem 3.6.1, embeddings in (3.22) enjoy unit continuity constants. As we do not need the embedding constants to be equal one, any other normalizing factor will the job as well. We will shortly discuss the interpolation of weighted L^2-spaces where another normalizing factor turns out to be more natural. The two normalizing factors do not bound each other ***uniformly in*** θ. Consequently, in general, we do not claim uniform bounds for the embedding constants in (3.22) as well. ∎

THEOREM 3.6.2

Let $A_0 : X_0 \to Y_0$ and $A_1 : X_1 \to Y_1$ be two compatible continuous operators.

There exists then a unique continuous operator $A_\theta : K_{\theta,q}(X) \to K_{\theta,q}(Y)$ such that

$$A_\theta = A_0 \text{ on } X_0 \quad \text{and} \quad A_\theta = A_1 \text{ on } X_1\,.$$

Moreover, if $\|A_j u\|_{Y_j} \le M_j\|u\|_{X_j}$, $j=0,1$, then

$$\|A_\theta u\|_{K_{\theta,q}(Y)} \le M_0^{1-\theta}M_1^{\theta}\,\|u\|_{K_{\theta,q}(X)} \qquad u \in K_{\theta,q}(X)\,.$$

PROOF Let $u \in X_0 + X_1$. Compatibility of the two operators implies

$$A_\theta u = A_\theta(u_0+u_1) = A_\theta u_0 + A_\theta u_1 = A_0u_0 + A_1u_1$$

and the value is independent of the decomposition $u = u_0 + u_1$. Indeed, all decompositions of u are of the form:

$$u = \underbrace{u_0 + v}_{\in X_0} + \underbrace{u_1 - v}_{\in X_1} \qquad v \in X_0 \cap X_1\,.$$

But $A_0 v = A_1 v$ and, therefore, the value defining $A_\theta u$ is independent of v. Let now $u \in K_{\theta,q}(X)$. We have:

$$\begin{aligned} K(t, A_\theta u; Y) &\le (\|A_0 u_0\|_{Y_0}^2 + t^2 \|A_1 u_1\|_{Y_1}^2)^{\frac{1}{2}} \\ &\le (M_0^2 \|u_0\|_{X_0}^2 + t^2 M_1^2 \|u_1\|_{X_1}^2)^{\frac{1}{2}} \\ &\le M_0 (\|u_0\|_{X_0}^2 + (at)^2 \|u_1\|_{X_1}^2)^{\frac{1}{2}} \end{aligned}$$

where $a = M_1/M_0$. Consequently, $K(t, Au, ; Y) \le M_0 K(at, u; X)$ and, by the dilatation property,

$$\|A_\theta u\|_{K_{\theta,q}(Y)} \le M_0 a^\theta \|u\|_{K_{\theta,q}(X)} = M_0^{1-\theta} M_1^\theta \|u\|_{K_{\theta,q}(X)} .$$

This proves, in particular, that A_θ takes $K_{\theta,q}(X)$ into $K_{\theta,q}(Y)$. ∎

Note that in Theorem 3.6.2, we can use any normalization factors as long as they are used for both spaces X and Y.

3.6.2 Interpolation of Weighted L^2 Spaces

The K-method can be used to interpolate between weighted L^2 spaces. The result will be useful in controlling the equivalence constants between spaces $H^s(\Omega)$ and $\tilde{H}^s(\Omega)$ for $s \in (-\frac{1}{2}, \frac{1}{2})$. The results in this section are reproduced from [2], pp. 114-116. In the rest of this section we work with the Hilbert spaces only using $q = 2$, and drop symbol q from notation.

Let $\Omega \in \mathbb{R}^n$ be a Lipschitz domain, and $w_0(x), w_1(x)$ denote two positive weights defined on Ω.

THEOREM 3.6.3

Let $\theta \in (0,1)$. Let $L^2_{w_0}(\Omega)$ and $L^2_{w_1}(\Omega)$ denote weighted L^2-spaces with weights w_0, w_1. Interpolation between the two weighted spaces yields a weighted space,

$$(L^2_{w_0}(\Omega), L^2_{w_1}(\Omega))_\theta = L^2_{w_\theta}(\Omega) ,$$

with weight w_θ given by:

$$w_\theta = w_0^{1-\theta} w_1^\theta .$$

The norm resulting from the interpolation is the weighted norm,

$$\|f\|_\theta^2 = c_\theta \int_\Omega |f(x)|^2 w_\theta(x)\, dx$$

scaled with factor

$$c_\theta = \frac{\pi}{2 \sin \pi\theta} .$$

PROOF Let $f \in L^2_{w_0}(\Omega) + L^2_{w_1}(\Omega)$, i.e., $f = f_0 + f_1$, $f_0 \in L^2_{w_0}(\Omega)$, $f_1 \in L^2_{w_1}(\Omega)$. Let $t \in (0, \infty)$.

We have,

$$K^2(t,f) := \inf_{\phi_0+\phi_1=f} \left(\int_\Omega |\phi_0(x)|^2 w_0(x)\,dx + t^2 \int_\Omega |\phi_1(x)|^2 w_1(x)\,dx \right)$$
$$= \int_\Omega \inf_{z_0+z_1=f(x)} (|z_0|^2 w_0(x) + t^2 |z_1|^2 w_1(x))\,dx$$
$$= \int_\Omega |f(x)|^2 w_0(x) \inf_{z_0+z_1=1} (|z_0|^2 + t^2 \frac{w_1}{w_0} |z_1|^2)\,dx$$
$$= \int_\Omega |f(x)|^2 w_0(x) F(t^2 \frac{w_1}{w_0})\,dx$$

where

$$F(s) := \inf_{z_0+z_1=1} (|z_0|^2 + s|z_1|^2) = \frac{s}{1+s}\,,$$

see Exercise 3.6.2. Switching the order of taking the infimum and integration is legal since the ultimate (after the switch) integrand is measurable, see Exercise 3.6.3. Consequently,

$$K^2(t,f) = \int_\Omega |f(x)|^2 \underbrace{\frac{t^2 w_0(x) w_1(x)}{w_0(x) + t^2 w_1(x)}}_{=:w_t(x)}\,dx\,.$$

We can compute now the norm of the interpolant,

$$\|f\|_\theta^2 = \int_0^\infty t^{-2\theta} \int_\Omega |f|^2 w_t\,dx\,\frac{dt}{t}$$
$$= \int_\Omega |f|^2 \int_0^\infty t^{-1-2\theta} w_0 F(t^2 \frac{w_1}{w_0})\,dt\,dx$$
$$= \int_\Omega |f|^2 w_0^{1-\theta} w_1^\theta \int_0^\infty s^{-1-2\theta} F(s^2)\,ds\,dx \qquad \text{(change of variable } t^2 \tfrac{w_1}{w_0} = s^2)$$
$$= c_\theta \int_\Omega |f|^2 w_0^{1-\theta} w_1^\theta\,dx$$

where

$$c_\theta := \int_0^\infty s^{-1-2\theta} F(s)\,ds = \int_0^\infty \frac{s^{1-2\theta}}{1+s^2}\,ds = \frac{\pi}{2\sin\pi\theta}\,,$$

see Exercise 3.6.4. ∎

3.6.3 Interpolation of Sobolev Spaces

This section is reproduced from [18], pp.329-330.

THEOREM 3.6.4

Let s_0, s_1 be arbitrary real numbers, and $\theta \in (0,1)$. We have:

$$(H^{s_0}(\mathbb{R}^n), H^{s_1}(\mathbb{R}^n))_\theta = H^s(\mathbb{R}^n) \quad \text{with } s = (1-\theta)s_0 + \theta s_1\,.$$

The K_θ norm equals the Sobolev norm if we use the normalization factor:

$$N_\theta = \left(\frac{2 \sin \pi\theta}{\pi} \right)^{1/2} .$$

PROOF The proof follows exactly the same lines as in proof of Theorem 3.6.3. We are again dealing with weighted spaces but, this time, in the frequency domain. Let $f = f_0 + f_1$, $f_0 \in H^{s_0}(\mathbb{R}^n)$, $f_1 \in H^{s_1}(\mathbb{R}^n)$. Let $t \in (0, \infty)$. We have,

$$\begin{aligned} K_2^2(t, f) &:= \inf_{\phi_0 + \phi_1 = f} \int_{\mathbb{R}^n} [(1 + |\xi|^2)^{s_0} |\hat{\phi}_0(\xi)|^2 + t^2 (1 + |\xi|^2)^{s_1} |\hat{\phi}_1(\xi)|^2] \, d\xi \\ &= \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^{s_0} F(t^2 (1 + |\xi|^2)^{s_1 - s_0}) \, d\xi \\ &= \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \frac{t^2 (1 + |\xi|^2)^{s_0 + s_1}}{(1 + |\xi|^2)^{s_0} + t^2 (1 + |\xi|^2)^{s_1}} \, d\xi \end{aligned}$$

Upon integrating in t, we obtain the final result,

$$\|f\|_\theta^2 = c_\theta \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^{(1-\theta)s_0 + \theta s_1} \, d\xi .$$

where c_θ is the same constant as in Theorem 3.6.3. ∎

Having established the interpolation result for the Sobolev spaces in $\mathbb{R}^n$, we proceed with spaces $H^s(\Omega)$. Let $u \in H^s(\Omega)$, and let $U \in H^s(\mathbb{R}^n)$ denote the minimum energy extension of u. Let $U = U_0 + U_1$ where $U_j \in H^{s_j}(\mathbb{R}^n)$, $j = 0, 1$. Set $u_j = U_j|_\Omega$, $j = 0, 1$. We have an obvious inequality,

$$\begin{aligned} K^2(t, u; H^{s_0}(\Omega), H^{s_1}(\Omega)) &\le \|u_0\|^2_{H^{s_0}(\Omega)} + t^2 \|u_1\|^2_{H^{s_1}(\Omega)} \\ &\le \|U_0\|^2_{H^{s_0}(\mathbb{R}^n)} + t^2 \|U_1\|^2_{H^{s_1}(\mathbb{R}^n)} . \end{aligned}$$

Passing to infimum with respect to U_0, U_1, $U_0 + U_1 = U$ on the right-hand side, we obtain,

$$K^2(t, u; H^{s_0}(\Omega), H^{s_1}(\Omega)) \le K^2(t, U; H^{s_0}(\mathbb{R}^n), H^{s_1}(\mathbb{R}^n)) .$$

Consequently,

$$\begin{aligned} \|u\|_{K(\theta,2;H^{s_0}(\Omega),H^{s_1}(\Omega))} &\le \|U\|_{K(\theta,2;H^{s_0}(\mathbb{R}^n),H^{s_1}(\mathbb{R}^n))} \\ &= \|U\|_{H^s(\mathbb{R}^n)} \qquad \text{(Theorem 3.6.4)} \\ &= \|u\|_{H^s(\Omega)} . \end{aligned}$$

We have arrived at the interpolation result for Sobolev spaces defined on a domain Ω.

THEOREM 3.6.5

Let s_0, s_1 be arbitrary real numbers, and $\theta \in (0, 1)$. We have:

$$H^s(\Omega) \subset (H^{s_0}(\Omega), H^{s_1}(\Omega))_\theta \quad \text{with } s = (1 - \theta)s_0 + \theta s_1 .$$

The K_θ norm is bounded by the Sobolev norm provided we use the normalization factor:

$$N_\theta = \left(\frac{2\sin\pi\theta}{\pi}\right)^{1/2}.$$

COROLLARY 3.6.1

Let $\mu \in (0, \frac{1}{2})$. We know that, for a Lipschitz domain Ω, Sobolev space $H^\mu(\Omega)$ is embedded in the weighted space $L^2_w(\Omega)$ where weight $w = d^{-2\mu}$ with d denoting distance from boundary $\partial\Omega$. Let C_μ denote the embedding constant. The result trivially holds for $\mu = 0$ with the corresponding embedding constant $C_0 = 1$. Let $0 < s < \mu$. By Theorem 3.6.5, Interpolation of Sobolev spaces yields a superspace of $H^s(\Omega)$ and, by Theorem 3.6.3, interpolation of the weighted spaces yields the weighted L^2_w space with weight $w = d^{-2s}$. Note that both theorems use the same normalizing factor. Consequently, space $H^s(\Omega)$ is embedded in the weighted L^2_w space with the embedding constant estimated by:

$$C^\theta_\mu = C^{s/\mu}_\mu .$$

The constant converges to one as $s \to 0$.

Exercises

Exercise 3.6.1 Prove the inequality:

$$\min\{1, \frac{t}{s}\}K(s,u) \le K(t,u) \le \max\{1, \frac{t}{s}\}K(s,u).$$

Exercise 3.6.2 Let $s > 0$. Prove that

$$\inf_{z_0+z_1=1}(|z_0|^2 + s|z_1|^2) = \frac{s}{1+s}$$

where z_0, z_1 are complex numbers.

Exercise 3.6.3 Explain in detail why switching the order of taking the infimum over functions ϕ_0, ϕ_1 and integration over Ω in the proof of Theorem 3.6.3 is legal.

Exercise 3.6.4 [18], Exercise B.5. Use contour integration to show that ,

$$\int_0^\infty \frac{s^{1-2\theta}}{1+s^2}\,ds = \frac{\pi}{2\sin\pi\theta}.$$

Hint: Use contour shown in Fig.3.5.

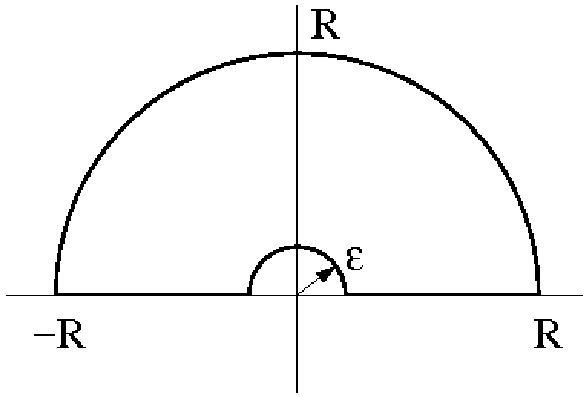

Figure 3.5
Contour for integration of $\frac{z^{1-2\theta}}{1+z^2}$.

3.7 Embedding Theorems

There are many embedding results for Sobolev spaces, see [1] for a comprehensive review. Following [18], we reproduce perhaps the two most important results. The first one identifies minimal conditions for which elements of Sobolev space represent continuous functions. Remember that elements of L^p spaces and Sobolev spaces in particular, are equivalence classes of functions that are equal to each other a.e.. The embedding theorem states that, under appropriate conditions, there exists a representative that is a Hölder continuous function and Hölder's exponent is controlled by the Sobolev norm. The second theorem reproduced in this section is the famous result of Rellich showing that, for a bounded domain, Sobolev space $H^{s_1}(\Omega)$ is compactly embedded in space $H^{s_2}(\Omega)$ where $s_2 < s_1$. The result is crucial for studying PDEs and Mikhlin compact perturbation argument in discrete stability analysis.

THEOREM 3.7.1 (Sobolev Embedding Theorem)

Let $\mu \in (0,1)$ and $u \in H^{\frac{n}{2}+\mu}(\mathbb{R}^n)$. There exists a Hölder continuous representative of u, denoted with the same symbol, such that

$$
\begin{aligned}
&|u(x)| \le C\|u\|_{H^{\frac{n}{2}+\mu}(\mathbb{R}^n)} && \forall x \in \mathbb{R}^n \\
&|u(x) - u(y)| \le C\|u\|_{H^{\frac{n}{2}+\mu}(\mathbb{R}^n)}\, |x-y| && \forall x, y \in \mathbb{R}^n
\end{aligned}
$$

with constant C independent of u.

PROOF **Step 1:** $u \in \mathcal{S}(\mathbb{R}^n)$. We have,

$$\begin{aligned}|u(x)| &= |\int_{\mathbb{R}^n} \hat{u}(\xi) e^{i2\pi x\xi}\, d\xi| \le \int_{\mathbb{R}^n} |\hat{u}(\xi)|\, d\xi \\ &= \int_{\mathbb{R}^n} (1+|\xi|^2)^{-\frac{1}{2}(\frac{n}{2}+\mu)} (1+|\xi|^2)^{\frac{1}{2}(\frac{n}{2}+\mu)} |\hat{u}(\xi)|\, d\xi \\ &\le \underbrace{\left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-(\frac{n}{2}+\mu)}\right)^{\frac{1}{2}}}_{=:C} \|u\|_{H^{\frac{n}{2}+\mu}(\mathbb{R}^n)}\end{aligned}$$

with

$$C^2 = |S| \int_0^\infty (1+r^2)^{-(\frac{n}{2}+\mu)}\, r^{n-1} dr < \infty$$

where $|S|$ denotes the measure of unit sphere S in $\mathbb{R}^n$.

Step 2: $u \in H^{\frac{n}{2}+\mu}(\mathbb{R}^n)$. By the density of $\mathcal{S}(\mathbb{R}^n)$ in $H^{\frac{n}{2}+\mu}(\mathbb{R}^n)$, there exists a sequence $u_j \in \mathcal{S}(\mathbb{R}^n)$ converging to u in $H^{\frac{n}{2}+\mu}(\mathbb{R}^n)$. By Step 1 result,

$$|u_j(x) - u_k(x)| \le C\|u_j - u_k\|_{H^{\frac{n}{2}+\mu}(\mathbb{R}^n)}$$

which implies that $u_j(x)$ is Cauchy in $\mathbb{R}$. Let $U(x) := \lim_{j\to\infty} u_j(x)$. The estimate:

$$|U(x) - U(y)| \le |U(x) - u_j(x)| + |u_j(x) - u_j(y)| + |u_j(y) - U(y)|$$

and the inequality above imply that $U(x)$ is uniformly continuous in $\mathbb{R}^n$. Let $\phi \in \mathcal{D}(\mathbb{R}^n)$.

$$\int_{\mathbb{R}^n} U\phi = \lim_{j\to\infty} \int_{\mathbb{R}^n} u_j\phi = \int_{\mathbb{R}^n} u\phi$$

and Lemma 2.4.1 imply that $u = U$ a.e. in $\mathbb{R}^n$ and

$$|U(x)| = \lim_{j\to\infty} |u_j(x)| \le C \lim_{j\to\infty} \|u_j\|_{H^{\frac{n}{2}+\mu}(\mathbb{R}^n)} = C\|u\|_{H^{\frac{n}{2}+\mu}(\mathbb{R}^n)}.$$

Step 3: Consider $\delta_h u(x) := u(x+h) - u(x)$. We have,

$$|u(x+h) - u(x)| \le \int_{\mathbb{R}^n} |\widehat{\delta_h u}(\xi)|\, d\xi \le \underbrace{\left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-\frac{n}{2}-\mu} |e^{i2\pi\xi h} - 1|^2\, d\xi\right)^{\frac{1}{2}}}_{=:M_\mu(h)} \|u\|_{H^{\frac{n}{2}+\mu}(\mathbb{R}^n)}.$$

$M_\mu(h)$ is bounded uniformly in h, see calculations ins Step 1. For $|h| \ge 1$, the

Let now $0 < |h| < 1$. By the Mean-Value Theorem, $|e^{i2\pi\xi h} - 1| \leq 2\pi|\xi \cdot h|$ and, therefore,

$$
\begin{aligned}
M_\mu^2(h) &\leq 4\pi^2 \int_{|\xi|<\frac{1}{|h|}} (1+|\xi|^2)^{-\frac{n}{2}-\mu} |\xi \cdot h|^2 \, d\xi + \int_{|\xi|>\frac{1}{|h|}} (1+|\xi|)^{-\frac{n}{2}-\mu} \, d\xi \\
&\leq 4\pi^2 |h|^2 \int_{|\omega|=1} \int_0^{\frac{1}{|h|}} (1+r^2)^{-\frac{n}{2}-\mu} \, r^2 r^{n-1} dr \, d\omega + 4 \int_{|\omega|=1} \int_{\frac{1}{|h|}}^{\infty} (1+r^2)^{-\frac{n}{2}-\mu} \, r^{n-1} dr \, d\omega \\
&\leq 4\pi^2 |h|^2 |S| \left[\int_0^1 (\underbrace{1+r^2}_{\geq 1})^{-\frac{n}{2}-\mu} \, r^{n+1} dr + \int_1^{\frac{1}{|h|}} (\underbrace{1+r^2}_{\geq r^2})^{-\frac{n}{2}-\mu} \, r^{n+1} dr \right] + 4|S| \int_{\frac{1}{|h|}}^{\infty} (\underbrace{1+r^2}_{\geq r^2})^{-\frac{n}{2}-\mu} \, r^{n-1} dr \\
&\leq 4\pi^2 |h|^2 |S| \left[\frac{r^{n+2}}{n+2} \Big|_0^1 + \frac{r^{-2\mu+2}}{2(1-\mu)} \Big|_1^{\frac{1}{|h|}} \right] + 4|S| \frac{r^{-2\mu}}{-2\mu} \Big|_{\frac{1}{|h|}}^{\infty} \\
&\leq C|h|^2 [1 + \frac{|h|^{2\mu-2}}{2(1-\mu)}] + C\frac{1}{\mu} |h|^{2\mu} \\
&\leq C|h|^{2\mu}
\end{aligned}
$$

where the ultimate constant C blows up (linearly) for $\mu \to 0, 1$. ∎

Let X, Y be normed spaces. Recall that a linear map $T : X \to Y$ is *compact* iff it maps bounded sets into precompact sets, i.e.,

$$
D \text{ bounded in } X \quad \Rightarrow \quad \overline{T(D)} \text{ compact in } Y \, .
$$

If X, Y are Banach, X is reflexive, and T is linear then the condition above is equivalent to

$$
x_n \rightharpoonup x \quad \Rightarrow \quad Ax_n \to Ax \, ,
$$

i.e., weak convergence in X implies strong convergence in Y, see Prop. 5.15.1 in [20]. Finally, we say that an embedding $X \hookrightarrow Y$ is *compact*, denoted $X \overset{c}{\hookrightarrow} Y$, if the identity map is compact.

THEOREM 3.7.2 (Rellich Theorem)

Let $-\infty < s < t < \infty$. The following compact embeddings hold:

(i) for any <u>*compact*</u> *set $K \subset \mathbb{R}^n$,*

$$
H_K^t(\mathbb{R}^n) \overset{c}{\hookrightarrow} H_K^s(\mathbb{R}^n) \, ,
$$

(ii) for any <u>*bounded*</u> *domain $\Omega \subset \mathbb{R}^n$,*

$$
H^t(\Omega) \overset{c}{\hookrightarrow} H^s(\Omega) \, .
$$

PROOF (i) Recall the Bolzano-Weiestrass Theorem stating that, in a metric space, a set is compact iff it is sequentially compact. It is sufficient thus to show that from any bounded sequence $u_j \in H_K^t(\mathbb{R}^n)$, we can extract a subsequence converging in $H_K^t(\mathbb{R}^n)$.

Step 1: Choose a cutoff function $\chi \in \mathcal{D}(\mathbb{R}^n)$ such that $\chi = 1$ on set K. We have:

$$
\hat{u}_j(\xi) = \widehat{\chi u_j}(\xi) = \int_{\mathbb{R}^n} \hat{\chi}(\xi - \eta) \hat{u}_j(\eta) \, d\eta \, .
$$

Applying Peetre's inequaity (Exercise 3.7.1), we obtain,

$$(1+|\xi|^2)^{\frac{1}{2}} \leq \int_{\mathbb{R}^n} (1+|\xi|^2)^{\frac{1}{2}} |\hat{\chi}(\xi-\eta)|\, |\hat{u}_j(\eta)|\, d\eta \leq 2^{\frac{|t|}{2}} \int_{\mathbb{R}^n} (1+|\xi\eta|^2)^{\frac{|t|}{2}} (1+|\eta|^2)^{\frac{t}{2}} |\hat{\chi}(\xi\eta)|\, |\hat{u}_j(\eta)|\, d\eta\,.$$

In turn, Cachy-Schwarz inequality leads to:

$$\begin{aligned}(1+|\xi|^2)^t\, |\hat{u}_j(\xi)|^2 &\leq 2^{|t|} \int_{\mathbb{R}^n} (1+|\xi-\eta|^2)^{|t|}\, |\hat{\chi}(\xi-\eta)|^2\, d\eta \int_{\mathbb{R}^n} (1+|\eta|^2)^t\, |\hat{u}_j(\eta)|^2\, d\eta \\ &= 2^{|t|}\, \|\chi\|^2_{H^{|t|}(\mathbb{R}^n)}\, \|u_j\|^2_{H^t(\mathbb{R}^n)}\,.\end{aligned}$$

Step 2: By standard properties of Fourier transform,

$$\partial^\alpha \hat{u}_j = \partial^\alpha(\widehat{\chi u_j}) = (\partial^\alpha \hat{\chi}) * \hat{u}_j = \hat{\chi}_\alpha * \hat{u}_h$$

where $\hat{\chi}_\alpha(x)(-i2\pi x)^\alpha \chi(x)$. The above and Step 1 resullt imply then:

$$(1+|\xi|^2)^t\, |\partial^\alpha \hat{u}_j(\xi)|^2 \leq 2^{|t|} \|\chi_\alpha\|^2_{H^{|t|}(\mathbb{R}^n)}\, \|u_j\|^2_{H^t(\mathbb{R}^n)}\,.$$

Control of the derivatives implies that $\hat{u}_j$ are uniformly bounded and equicontinuous over any compact subset of $\mathbb{R}^n$. Take now any sequence of compact sets

$$K_1 \subset K_2 \subset \ldots, \qquad \bigcup_j K_j = \mathbb{R}^n\,.$$

Arzelà-Ascoli Theorem ([20], Theorem 4.9.3) and the diagonal choice method lead to the conclusion that we can extract a subsequence of $\hat{u}_j$, denoted with the same symbol, such that $\hat{u}_j$ converges uniformly on *any* compact subset of $\mathbb{R}^n$.

Step 3: We claim now that u_j is Cauchy in $H^s_K(\mathbb{R}^n)$. Take an arbitrary $\epsilon > 0$. Next, choose a sufficientlty large R such that

$$\begin{aligned}\int_{|\xi|>R} (1+|\xi|^2)^s\, |\hat{u}_j(\xi)-\hat{u}_k(\xi)|^2\, d\xi &\leq (1+R^2)^{s-t} \int_{|\xi|>R} (1+|\xi|^2)^t\, |\hat{u}_j(\xi)-\hat{u}_k(\xi)|^2\, d\xi \\ &\leq (1+R^2)^{s-t} \int_{\mathbb{R}^n} (1+|\xi|^2)^t\, |\hat{u}_j(\xi)-\hat{u}_k(\xi)|^2\, d\xi \\ &\leq \frac{2(\|u_j\|^2_{H^t(\mathbb{R}^n)} + \|u_k\|_{H^t(\mathbb{R}^n)})}{(1+R^2)^{t-s}} < \frac{\epsilon}{2}\,.\end{aligned}$$

Use Step 2 result to choose sufficiently large N such that:

$$\int_{|\xi|\leq R} (1+|\xi|^2)^s\, |\hat{u}_j(\xi)-\hat{u}_k(\xi)|^2\, d\xi < \frac{\epsilon}{2}$$

for every $j,k \geq N$.

(ii) Let $u_j \in H^t(\Omega)$ be a bounded sequence. Let $U_j \in H^t(\mathbb{R}^n)$ be the corresponding minimum energy extensions, i.e. $\|u_j\|_{H^t(\Omega)} = \|U_j\|_{H^t(\mathbb{R}^n)}$. Lemma 3.5.2 implies that sequence χU_j is bounded in $H^t_K(\mathbb{R}^n)$ where $K = \text{supp}\chi$. By part(i) of this theorem, there exists a subsequence χU_j converging to some $U \in H^s_K(\mathbb{R}^n)$. This in turn implies that

$$u_j = (\chi U_j)|_\Omega \to U|_\Omega \quad \text{in } H^s(\Omega)\,.$$

∎

Exercises

Exercise 3.7.1 Prove Peetre's inequality:

$$(1+|\xi|^2)^s \leq 2^{|s|}(1+|\xi-\eta|)^{|s|}\,(1+|\eta|^2)^s\,,$$

for any $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. *Hint:* Proceed in the order: $s = 0$, $s = 1$, $s > 0$, $s < 0$.

4

Trace Theorems

4.1 Trace Theorems

4.1.1 3D Differential Complex and Exact Sequence.

We are finally ready to discuss the energy spaces introduced in our opening Section 1.1. We start by recalling the definitions. Let $\Omega \subset \mathbb{R}^3$ be an arbitrary domain. We have

$$H^s(\text{grad},\Omega) := \{u \in (H^s(\Omega))^3 \,:\, \boldsymbol{\nabla} u \in (H^s(\Omega))^3\} \qquad \|u\|^2_{H^s(\text{grad},\Omega)} := \|u\|^2_{H^s(\Omega)} + \|\boldsymbol{\nabla} u\|^2_{H^s(\Omega)}$$

$$H^s(\text{curl},\Omega) := \{E \in (H^s(\Omega))^3 \,:\, \boldsymbol{\nabla} \times E \in (H^s(\Omega))^3\} \qquad \|E\|^2_{H^s(\text{curl},\Omega)} := \|E\|^2_{H^s(\Omega)} + \|\boldsymbol{\nabla} \times E\|^2_{H^s(\Omega)}$$

$$H^s(\text{div},\Omega) := \{v \in (H^s(\Omega))^3 \,:\, \boldsymbol{\nabla} \cdot v \in H^s(\Omega)\} \qquad \|v\|^2_{H^s(\text{div},\Omega)} := \|v\|^2_{H^s(\Omega)} + \|\boldsymbol{\nabla} \cdot v\|^2_{H^s(\Omega)}$$

where, as usual, the derivatives are understood in the sense of distributions. From the practical point of view, we are interested in Lipschitz and polyhedral domains only which will limit the range for regularity parameter s to $(-\frac{1}{2}, \frac{1}{2})$.

Spaces $H^s(\text{grad},\Omega)$, $H^s(\text{curl},\Omega)$, $H^s(\text{div},\Omega)$ and $H^s(\Omega)$, along with operators of grad, curl and div, form the so-called *differential complex*:

$$\mathbb{R}(\mathbb{C}) \xrightarrow{\text{id}} H^s(\text{grad},\Omega) \xrightarrow{\boldsymbol{\nabla}} H^s(\text{curl},\Omega) \xrightarrow{\boldsymbol{\nabla}\times} H^s(\text{div},\Omega) \xrightarrow{\boldsymbol{\nabla}\cdot} H^s(\Omega) \xrightarrow{0} \{0\} \tag{4.1}$$

which means that the range of every involved operator is contained in the null space of the next operator in the sequence. In simple terms, gradient of a constant, curl of a gradient, and div of a curl, are all equal zero. Notice that all operators are well defined.

REMARK 4.1.1 For Lipschitz domains,

$$H^s(\text{grad},\Omega) := \{u \in H^s(\Omega) \,:\, \boldsymbol{\nabla} u \in (H^s(\Omega))^n\} = H^{1+s}(\Omega)\,. \tag{4.2}$$

The result is immediate for $\mathbb{R}^n$, see Exercise 4.1.1. For a general Lipschitz domain, it is a consequence of the existence of a bounded extension operator from $H^s(\text{grad},\Omega)$ to $H^s(\text{grad},\mathbb{R}^n)$, see [17] and comp. Theorem 3.2.1. For $s \geq 0$, both spaces coincide with $W^{1+s}(\Omega)$. ▮

For a *bounded* domain Ω "without holes"* we arrive at the structure of an *exact sequence*, i.e. the range of

*Topologically equivalent to a ball.

each operator in the sequence, is *equal* to the null space of the next one. In other words,

$$u \in H^s(\mathrm{grad}, \Omega)\,,\ \nabla u = 0 \quad \Leftrightarrow \quad \exists c \in \mathbb{R}(\mathbb{C})\,:\, u = c$$

$$E \in H^s(\mathrm{curl}, \Omega)\,,\ \nabla \times E = 0 \quad \Leftrightarrow \quad \exists u \in H^s(\mathrm{grad}, \Omega)\,:\, E = \nabla u$$

$$v \in H^s(\mathrm{div}, \Omega)\,,\ \nabla \cdot v = 0 \quad \Leftrightarrow \quad \exists E \in H^s(\mathrm{curl}, \Omega)\,:\, v = \nabla \times E$$

$$q \in H^s(\Omega) \quad \Leftrightarrow \quad \exists v \in H^s(\mathrm{div}, \Omega)\,:\, q = \nabla \cdot v$$

Usually, we simplify the notation and cut off the first and the last elements of the sequence,

$$H^s(\mathrm{grad}, \Omega) \xrightarrow{\nabla} H^s(\mathrm{curl}, \Omega) \xrightarrow{\nabla \times} H^s(\mathrm{div}, \Omega) \xrightarrow{\nabla \cdot} H^s(\Omega)$$

remembering that the div operator is a surjection and the nullspace of grad operator are constants.

2D differential complexes and exact sequences. The 3D differential complex (exact sequence) gives rise to a couple of two-dimensional sequences:

$$H^s(\mathrm{grad}, \Omega) \xrightarrow{\nabla} H^s(\mathrm{curl}, \Omega) \xrightarrow{\mathrm{curl}} H^s(\Omega)$$

and,

$$H^s(\mathrm{grad}, \Omega) \xrightarrow{\nabla \times} H^s(\mathrm{curl}, \Omega) \xrightarrow{\mathrm{div}} H^s(\Omega)$$

where

$$\mathrm{curl}\, E := E_{2,1} - E_{1,2} \quad \text{and} \quad \nabla \times u := (u_{,2}, -u_{,1},)\,.$$

The 2D sequences are easily obtained from the 3D sequence by considering functions $E = (E_1, E_2, 0)$ (first sequence) or $E = (0, 0, u)$ (second sequence) with all components depending upon x_1, x_2 only.

Traces for energy spaces $H^s(\Omega)$ and $H^s(\mathrm{div}, \Omega)$ will be discussed in n space dimensions. Traces for $H^s(\mathrm{curl}, \Omega)$ will be discussed in three space dimensions and we will comment on the two-dimensional case. We will proceed in three steps. In the first step we define the trace spaces and establish the trace theorems for the half-space domain,

$$\mathbb{R}^n_- := \{x = (\underbrace{x_1, \ldots, x_{n-1}}_{=:x'}, x_n) \in \mathbb{R}^n \,:\, x_n < 0\}$$

In the second step, we generalize the definitions and prove the trace theorems for *a piecewice smooth* hypograph, see Fig. 4.1,

$$\Omega := \{x = (x', x_n) \in \mathbb{R}^n \,:\, x_n < \zeta(x')\}$$

where $\zeta(x'),\ x' \in \mathbb{R}^{n-1}$, is a continuous, piece-wise smooth function. Finally, in the last step, we generalize the results to an arbitrary (curvilinear) polyhedron using the partition of unity technique.

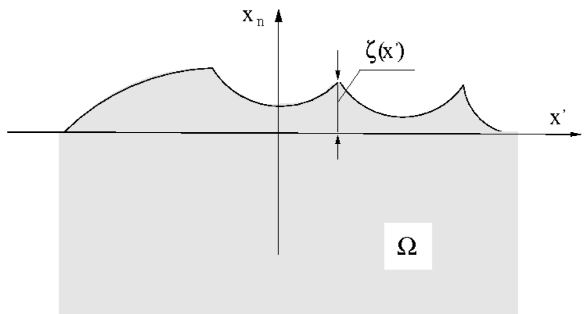

Figure 4.1
Piece-wise smooth hypograph

4.1.2 Density of Test Functions in the Energy Spaces

Critical in the proofs presented in this chapter is the density of test functions $C_0^\infty(\overline{\Omega})$ in all energy spaces forming the differential complex. We showed in Section 3.1 that test functions $\mathcal{D}(\mathbb{R}^n)$ are dense in $H^s(\mathbb{R}^n)$, for any $s \in \mathbb{R}$. As restrictions of functions (distributions) from $H^s(\mathbb{R}^n)$ to domain Ω constitute space $H^s(\Omega)$, restrictions of test functions $\mathcal{D}(\mathbb{R}^n)$ to Ω are automatically dense in $H^s(\Omega)$. The result is also true for the remaining energy spaces and we will show now an alternate reasoning that applies to all of them.

Consider first the case of the whole space, $\Omega = \mathbb{R}^n$. Let ψ_ϵ be the function used in Theorem 2.3.2. We begin with a generalization of Theorem 2.3.2.

LEMMA 4.1.1

Let $u \in H^s(\mathbb{R}^n), s \in \mathbb{R}$. Then

$$\|\psi_\epsilon * u - u\|_{H^s(\mathbb{R}^n)} \to 0 \quad \text{as } \epsilon \to 0\,.$$

PROOF We have:

$$\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{\psi_\epsilon * u - u}|^2 \, d\xi = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{\psi}_\epsilon - 1|^2 \, |\hat{u}(\xi)|^2 \, d\xi$$

where

$$\hat{\psi}_\epsilon(\xi) = \int_{\mathbb{R}^n} e^{-i2\pi\xi x} \epsilon^{-n} \psi(\epsilon^{-1}x) \, dx = \int_{\mathbb{R}^n} e^{-2\pi\epsilon\xi y} \psi(y) \, dy = \hat{\psi}(\epsilon\xi)\,.$$

As $\hat{\psi}(0) = 1$, factor $\hat{\psi}_\epsilon - 1$ converges pointwise to 0. It is also bounded by 2. Function $2(1 + |\xi|^2)^s |\hat{u}(\xi)|^2$ provides thus an integrable dominating function and, by the Lebesgue Dominated Convergence Theorem, the integral converges to zero. ∎

LEMMA 4.1.2

We have,

$$\|\text{grad}(\psi_\epsilon * u - u)\|_{H^s(\mathbb{R}^n)} \to 0 \qquad \text{as } \epsilon \to 0\,,$$
$$\|\text{div}(\psi_\epsilon * v - v)\|_{H^s(\mathbb{R}^n)} \to 0 \qquad \text{as } \epsilon \to 0\,,$$
$$\|\text{curl}(\psi_\epsilon * E - E)\|_{H^s(\mathbb{R}^3)} \to 0 \qquad \text{as } \epsilon \to 0\,,$$

for any $u \in H^s(\text{grad}, \mathbb{R}^n)$, $v \in H^s(\text{div}, \mathbb{R}^n)$, *and* $E \in H^s(\text{curl}, \mathbb{R}^3)$.

PROOF It is sufficient to show that

$$\text{grad}\,(\psi_\epsilon * u) = \psi_\epsilon * \text{grad}\,u\,,$$
$$\text{div}\,(\psi_\epsilon * v) = \psi_\epsilon * \text{div}\,v\,,$$
$$\text{curl}\,(\psi_\epsilon * E) = \psi_\epsilon * \text{curl}\,E\,.$$

The result follows then from Lemma 4.1.1. For example, we have for the div operator,

$$\begin{aligned}
\langle \text{div}\,(\psi_\epsilon * v), \phi\rangle &= -\langle \psi_\epsilon * v, \boldsymbol{\nabla}\phi\rangle && \text{(definition of distributional divergence)}\\
&= -\langle v, \check{\psi}_\epsilon * \boldsymbol{\nabla}\phi\rangle && \text{(definition of convolution of a distribution with a smooth test function)}\\
&= -\langle v, \boldsymbol{\nabla}(\check{\psi}_\epsilon * \phi)\rangle && \text{(Theorem 2.3.1)}\\
&= \langle \text{div}\,v, \check{\psi}_\epsilon * \phi\rangle && \text{(definition of distributional divergence)}\\
&= \langle \psi_\epsilon * \text{div}\,v, \phi\rangle && \text{(definition of convolution of a distribution with a smooth test function)}\,.
\end{aligned}$$

∎

The smoothing by convolution provides thus a constructive way to approximate functions from energy spaces with C^∞ functions. Let χ_j^ϵ be the approximation of indicator function χ_j of ball $\bar{B}(0,j)$ from Theorem 2.3.3, and let $v \in H^s(\text{div}, \mathbb{R}^n)$. By Lemma 3.2.2, $\chi_j^\epsilon v \to v$ in $H^s(\mathbb{R}^n)$ as $j \to \infty$. Similarly, we have,

$$\text{div}\,(\chi_j^\epsilon v) = (\boldsymbol{\nabla}\chi_j^\epsilon)\cdot v + \chi_j^\epsilon\,\text{div}\,v\,.$$

By Lemma 3.2.2 again, term $\chi_j^\epsilon\,\text{div}\,v$ converges to $\text{div}\,v$ in $H^s(\mathbb{R}^n)$ with $j \to \infty$, and the first term converges to zero, comp. Remark 3.2.1. By the same argument, identity:

$$\text{curl}\,(\chi_j^\epsilon E) = (\boldsymbol{\nabla}\chi_j^\epsilon)\times E + \chi_j^\epsilon\,\text{curl}\,E\,,$$

implies that, for any $E \in H^s(\text{curl}, \mathbb{R}^n)$, $\chi_j^\epsilon E$ converges to E in $H^s(\text{curl}, \mathbb{R}^n)$ norm. The same argument holds for space $H^s(\text{grad}, \mathbb{R}^n)$.

Combining the truncation results with Lemma 4.1.2 and Lemma **??**, we obtain the final density results:

$$\overline{(\mathcal{D}(\mathbb{R}^n))^n}^{H^s(\text{grad},\mathbb{R}^n)} = H^s(\text{grad}, \mathbb{R}^n) \quad \text{and} \quad \overline{(\mathcal{D}(\mathbb{R}^n))^n}^{H^s(\text{div},\mathbb{R}^n)} = H^s(\text{div}, \mathbb{R}^n)\,,$$

$$\overline{(\mathcal{D}(\mathbb{R}^3))^3}^{H^s(\text{curl},\mathbb{R}^3)} = H^s(\text{curl}, \mathbb{R}^3)\,.$$

We are ready now to discuss the case of an arbitrary domain Ω. The next result builds upon the reasoning similar to that in proof of Lemma 3.3.3.

LEMMA 4.1.3

Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open set, and $s \in \mathbb{R}$. Then

$$H^s(\text{grad}, \Omega) \cap C^\infty(\Omega) \text{ is dense in } H^s(\text{grad}, \Omega)\,,$$

$$H^s(\text{div}, \Omega) \cap (C^\infty(\Omega))^n \text{ is dense in } H^s(\text{div}, \Omega)\,, \text{ and}$$

$$H^s(\text{curl}, \Omega) \cap (C^\infty(\Omega))^3 \text{ is dense in } H^s(\text{curl}, \Omega)\,.$$

PROOF We will prove the result for the $H^s(\text{div}, \Omega)$. The remaining two cases are fully analogous. Let

$$G_j := \{x \in \Omega \,:\, d(x, \Gamma) < \frac{1}{j},\, |x| < j\} \quad j = 1, 2, \ldots$$

be an infinite open cover of Ω and ψ_j a C^∞ partition of unity subordinate to G_j. Let $u \in H^s(\Omega)$, and $U \in \mathcal{D}'(\mathbb{R}^n)$ be any extension of u, i.e. $U|_\Omega = u$. Consider the zero extension $\widetilde{\psi_j u}$ defined by

$$\widetilde{\psi_j u} := \tilde{\psi}_j U$$

where $\tilde{\psi}_j$ is the zero extension of partition of unity function ψ_j. It is easy to check that $\widetilde{\psi_j u}$ is independent of extension U. In particular, we can take an extension $U \in H^s(\mathbb{R}^n)$ to obtain:

$$\|\widetilde{\psi_j u}\|_{H^s(\mathbb{R}^n)} = \|\tilde{\psi}_j U\|_{H^s(\mathbb{R}^n)} \le C(\psi_j)\|U\|_{H^s(\mathbb{R}^n)}\,.$$

Taking minimum with respect to all extensions $U \in H^s(\mathbb{R}^n)$ on the right-hand side, we get

$$\|\widetilde{\psi_j u}\|_{H^s(\mathbb{R}^n)} \le C(\psi_j)\|u\|_{H^s(\Omega)}\,.$$

Let now $v \in H^s(\text{div}, \Omega)$, and let $V \in (H^s(\mathbb{R}^n))^n$ be an extension of v, and $W \in H^s(\mathbb{R}^n)$ be an extension of div v. Let $\phi \in \mathcal{D}(\mathbb{R}^n)$. We have,

$$\begin{aligned}
\langle \partial_i \widetilde{\psi_j v_i}, \phi \rangle = -\langle \widetilde{\psi_j v_i}, \partial_i \phi \rangle &= -\langle \tilde{\psi}_j V_i, \partial_i \phi \rangle \\
&= -\langle V_i, \underbrace{\tilde{\psi}_j \partial_i \phi}_{\partial_i(\tilde{\psi}_j \phi) - \partial_i \tilde{\psi}_j \phi} \rangle \\
&= \langle \partial_i V_i, \tilde{\psi}_j \phi \rangle + \langle \partial_i \tilde{\psi}_j V_i, \phi \rangle \\
&= \langle W, \tilde{\psi}_j \phi \rangle + \langle \partial_i \tilde{\psi}_j V_i, \phi \rangle \quad \text{(Both } \partial_i V_i \text{and } W \text{are extensions of div } u) \\
&= \langle \tilde{\psi}_j W, \phi \rangle + \langle \widetilde{\partial_i \psi_j} V_i, \phi \rangle\,,
\end{aligned}$$

i.e.,

$$\partial_i (\widetilde{\psi_j v_i}) = \widetilde{\psi_j (\text{div } v)} + (\widetilde{\partial_i \psi_j}) v_i\,.$$

This implies that $\partial_i(\widetilde{\psi_j v_i}) \in H^s(\mathbb{R}^n)$ and,

$$\|\partial_i(\widetilde{\psi_j v_i})\|_{H^s(\mathbb{R}^n)} \le C(\psi_j)\|\text{div } v\|_{H^s(\Omega)} + C(\partial_i \psi_j)\|v\|_{(H^s(\Omega))^n}\,.$$

Take now an arbitrary $\epsilon > 0$. Let $W^j \in (\mathcal{D}(\mathbb{R}^n))^n$ be such that

$$\|\widetilde{\psi_j v} - W^j\|_{H^s(\text{div},\mathbb{R}^n)} \leq \frac{\epsilon}{2j} \quad \text{and} \quad \text{supp}\, W^j \subset \frac{1}{j}\text{neighborhood of supp}\, \widetilde{\psi_j v}\,.$$

Let $w = (\sum_j W^j)|_\Omega$. By the same reasoning as in the proof of Lemma 3.3.3, the sum in j is locally finite in Ω which implies that w is a C^∞ function. Finally,

$$\|v - w\|_{H(\text{div},\Omega)} = \|\sum_j \psi_j v - \sum_j W^j|_\Omega\|_{H(\text{div},\Omega)} \leq \sum_j \|\widetilde{\psi_j v} - W^j\|_{H(\text{div},\mathbb{R}^n)} \leq \epsilon\,.$$

∎

The proof of our final result is essentially a reproduction of arguments from Theorem 3.3.2.

THEOREM 4.1.1

Let $\Omega \subset \mathbb{R}^n$ be a C^0 domain[†]*. Then*

$$C_0^\infty(\overline{\Omega}) \text{ is dense in } H^s(\text{grad},\Omega)\,,$$

$$(C_0^\infty(\overline{\Omega}))^n \text{ is dense in } H^s(\text{div},\Omega)\,, \text{and,}$$

$$(C_0^\infty(\overline{\Omega}))^3 \text{ is dense in } H^s(\text{curl},\Omega)\,.$$

PROOF We will again prove the result only for the $H(div)$ case. The other two proofs are fully analogous.

Case: Ω is a C^0 hypograph.

Let $v \in H^s(\text{div},\Omega)$. By Lemma 4.1.3, we can assume additionally that v is a $C^\infty(\Omega)$ function. For any $\delta > 0$, let v^δ be the shifted function

$$v^\delta(x) := v(x', x_n - \delta), \quad x \in \Omega_\delta := \{x \in \mathbb{R}^n \,:\, x_n < \zeta(x') + \delta\}\,.$$

The operations of shifting and divergence commute,

$$\text{div}\, v^\delta = (\text{div}\, v)^\delta\,.$$

Let $\epsilon > 0$. Let $V \in H^s(\mathbb{R}^n)$ be an extension of v and $W \in (C_0^\infty(\mathbb{R}^n))^n$ such a function that $\|V - W\|_{(H^s(\mathbb{R}^n))^n} < \frac{\epsilon}{6}$. Then,

$$\|v - W|_\Omega\|_{(H^s(\Omega))^n} \leq \|V - W\|_{(H^s(\mathbb{R}^n))^n} < \frac{\epsilon}{6} \quad \text{and}$$

$$\|v^\delta - (W|_\Omega)^\delta\|_{(H^s(\Omega))^n} \leq \|V^\delta - W^\delta\|_{(H^s(\mathbb{R}^n))^n} = \|V - W\|_{(H^s(\mathbb{R}^n))^n} < \frac{\epsilon}{6}\,.$$

[†] $n = 3$ in the last case.

At the same time,

$$\|W|_\Omega - (W|_\Omega)^\delta\|_{(H^s(\Omega))^n} \le \|W - W^\delta\|_{(H^s(\mathbb{R}^n))^n} < \frac{\epsilon}{6}$$

for sufficiently small δ. Indeed, by the Lebesgue Dominated Convergence Theorem,

$$\|W - W^\delta\|^2_{(H^s(\mathbb{R}^n))^n} = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |e^{-i2\pi\xi_n\delta} - 1|^2 \, |\hat{W}(\xi)|^2 \, d\xi \to 0 \quad \text{as } \delta \to 0 \, .$$

By triangle inequality,

$$\|v - v^\delta\|_{(H^s(\Omega))^n} < \frac{\epsilon}{2} \, .$$

Exactly the same reasoning[‡] for div v implies that

$$\|\text{div}\, v^\delta - \text{div}\, v\|_{H^s(\Omega)} = \|(\text{div}\, v)^\delta - \text{div}\, v\|_{H^s(\Omega)} < \frac{\epsilon}{2}$$

so, finally,

$$\|v - v^\delta\|_{(H^s(\Omega))^n} + \|\text{div}\, v^\delta - \text{div}\, v\|_{H^s(\Omega)} < \epsilon \, .$$

Once we have established the convergence of the shifted function v^δ to v, the rest of the proof is identical with the proof of Theorem 3.3.2. We truncate $v^\delta \in H^s(\text{div}, \Omega_\delta)$ with a cut-off function χ,

$$\chi = \begin{cases} 1 \text{ on } \Omega \\ 0 \text{ on } \mathbb{R}^n - \Omega_{\delta/2} \, , \end{cases}$$

to obtain $\chi v^\delta \in H^s(\text{div}, \mathbb{R}^n)$, and use the density result for $H^s(\text{div}, \mathbb{R}^n)$ to establish existence of an approximating test function W on $\mathbb{R}^n$. Restriction $w = W|_\Omega$ provides the final approximating $(C_0^\infty(\overline{\Omega}))^n$ function in $H^s(\text{div}, \Omega)$.

Case: An arbitrary Lipschitz domain Ω.
Use the standard partition of unity argument. ∎

4.1.3 The Case of Half-Space

We begin by recalling the classical results for standard Sobolev spaces.

THEOREM 4.1.2 Trace Theorem

Let $s > \frac{1}{2}$. Define the trace operator:

$$\gamma \, : \, \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^{n-1}), \qquad (\gamma u)(x') := u(x', 0) \, .$$

There exists a unique continuous extension of operator γ, denoted with the same symbol, to

$$\gamma \, : \, H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \, .$$

[‡]Notice that an analogue Z of function W need not to be related to W, i.e., we do not need div $W = Z$.

PROOF It is sufficient to prove that the original operator γ is continuous in the Sobolev norms. We have:

$$\begin{aligned}(\gamma u)(x') &= \int_{\mathbb{R}^n} \hat{u}(\xi) e^{i2\pi(\xi'\cdot x' + \xi_n 0)}\, d\xi \\ &= \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} \hat{u}(\xi', \xi_n)\, d\xi_n \right) e^{i2\pi\xi' x'}\, d\xi' \,.\end{aligned}$$

Applying the two-dimensional Fourier transform to both sides of the equality, we get,

$$\widehat{\gamma u}(\xi') = \int_{-\infty}^{\infty} \hat{u}(\xi', \xi_n)\, d\xi_n = \int_{-\infty}^{\infty} (1+|\xi|^2)^{-\frac{s}{2}} (1+|\xi|^2)^{\frac{s}{2}} \hat{u}(\xi', \xi_n)\, d\xi_n \,.$$

Cauchy-Schwarz inequality implies now,

$$|\widehat{\gamma u}(\xi')|^2 \leq \underbrace{\int_{\infty}^{\infty} \frac{d\xi_n}{(1+|\xi'|^2+|\xi_n|^2)^s}}_{=:M_s(\xi')} \int_{-\infty}^{\infty} (1+|\xi|^2)^s\, |\hat{u}(\xi', \xi_n)|^2\, d\xi_n \,.$$

Use substitution $\xi_n = (1+|\xi'|^2)^{1/2} t$, to obtain,

$$M_s(\xi') = \frac{1}{(1+|\xi'|^2)^{s-\frac{1}{2}}} \underbrace{\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^s}}_{<\infty \text{ for } s>1/2}$$

and, consequently,

$$(1+|\xi'|^2)^{s-\frac{1}{2}} |\widehat{\gamma u}(\xi')|^2 \leq C \int_{-\infty}^{\infty} (1+|\xi|^2)^s\, |\hat{u}(\xi)|^2\, d\xi_n \,.$$

Finally, integrate wrt $\xi' \in \mathbb{R}^{n-1}$ to obtain,

$$\|\gamma u\|^2_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \leq C \|u\|^2_{H^s(\mathbb{R}^n)} \,.$$

∎

REMARK 4.1.2 If $s = \frac{1}{2} + \epsilon$ then the integral in $M_s(\xi')$ is of order $O(1/\epsilon)$. This gives the final blow up in the continuity constant for the trace operator (we need to take the square root in the inequality above) of order $O(1/\epsilon^{\frac{1}{2}})$. ∎

Construction of the extension operator has already been given in Lemma 3.5.1 (operator η_0). An alternate construction can be based on the solution of the Dirichlet problem:

$$\begin{cases} -\Delta U + U = 0 & \text{in } \mathbb{R}^n_- \\ U = u & \text{on } \mathbb{R}^{n-1} \,. \end{cases} \tag{4.3}$$

In order to derive an explicit formula for U (to investigate the continuity in fractional spaces), we first Fourier transform boundary-value problem (4.3) in x' and obtain the following ODE problem in x_n.

$$(4\pi^2|\xi'|^2 + 1)\, \hat{U} - \frac{\partial^2 \hat{U}}{\partial x_n^2} = 0 \qquad \hat{U}(0) = \hat{u} \,.$$

Selecting the exponentially decaying solution for $x_n \to -\infty$, we get,

$$\hat{U}(\xi'; x_n) = Ce^{(1+4\pi^2|\xi'|^2)^{1/2} x_n}$$

and, upon utilizing the boundary condition, we obtain,

$$\hat{U}(\xi'; x_n) = \hat{u}(\xi') e^{(1+4\pi^2|\xi'|^2)^{1/2} x_n} \qquad x_n < 0\,.$$

Let $\psi \in \mathcal{S}(\mathbb{R})$ be now any extension of exponential $e^x, x \in (-\infty, 0]$ to the whole real line. We use it to extend the minimum energy extension above to the whole space:

$$\hat{U}(\xi'; x_n) = \hat{u}(\xi')\psi((1+4\pi^2|\xi'|^2)^{1/2} x_n) \qquad x_n \in \mathbb{R}\,.$$

Fourier transforming in x_n, we get:

$$\begin{aligned} \hat{U}(\xi) &= \int_{i\infty}^{\infty} \hat{u}(\xi')\psi(\underbrace{(1+4\pi^2|\xi'|^2)^{\frac{1}{2}} x_n}_{=t}) e^{-i2\pi\xi_n x_n}\, dx_n \\ &= \hat{u}(\xi')(1+4\pi^2|\xi'|^2)^{-\frac{1}{2}} \underbrace{\int_{-\infty}^{\infty} \psi(t) e^{-i2\pi \frac{\xi_n}{(1+4\pi^2|\xi'|^2)^{\frac{1}{2}}} t}\, dt}_{\hat{\psi}(\frac{\xi_n}{(1+4\pi^2|\xi'|^2)^{\frac{1}{2}}})}\,. \end{aligned} \tag{4.4}$$

THEOREM 4.1.3 Extension Theorem

Let $u \in H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$, $s \in \mathbb{R}$. Let $U(x) = \mathcal{F}_{\xi\to x}\hat{U}(\xi)$ with $\hat{U}(\xi)$ given by (4.4). Extension operator:

$$H^{s+\frac{1}{2}}(\mathbb{R}^{n-1}) \ni u \to U \in H^{s+1}(\mathbb{R}^n) \tag{4.5}$$

is well-defined and continuous.

PROOF The estimate uses the same integration techniques as in the proof of the trace theorem. We have,

$$\begin{aligned} \|U\|^2_{H^{s+1}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (1+|\xi|^2)^{s+1} |\hat{U}(\xi)|^2\, d\xi \\ &= \int_{\mathbb{R}^{n-1}} |\hat{u}(\xi')|^2\, (1+4\pi^2|\xi'|^2)^{-1} \int_{-\infty}^{\infty} (1+|\xi|^2)^{s+1} |\hat{\psi}(\frac{\xi_n}{(1+4\pi^2|\xi'|^2)^{\frac{1}{2}}})|^2\, d\xi_n \\ &\approx \int_{\mathbb{R}^{n-1}} |\hat{u}(\xi')|^2\, (1+4\pi^2|\xi'|^2)^{-1} \int_{-\infty}^{\infty} (1+4\pi^2|\xi'|^2+\xi_n^2)^{s+1} |\hat{\psi}(\underbrace{\frac{\xi_n}{(1+4\pi^2|\xi'|^2)^{\frac{1}{2}}}}_{=t})|^2\, d\xi_n \\ &= \int_{\mathbb{R}^{n-1}} (1+4\pi^2|\xi'|^2)^{s+\frac{1}{2}}\, |\hat{u}(\xi')|^2\, d\xi' \int_{-\infty}^{\infty} (1+t^2)^{s+1} |\hat{\psi}(t)|^2\, dt \\ &\approx \int_{\mathbb{R}^{n-1}} (1+|\xi'|^2)^{s+\frac{1}{2}}\, |\hat{u}(\xi')|^2\, d\xi' \underbrace{\int_{-\infty}^{\infty} (1+t^2)^{s+1} |\hat{\psi}(t)|^2\, dt}_{<\infty \text{ for any } s}\,. \end{aligned}$$

∎

LEMMA 4.1.4

Let $w \in \mathbb{C}, \xi \in \mathbb{R}$ be arbitrary numbers. The following inequality holds:

$$|z|^2 + |\xi z + w|^2 \geq \frac{|w|^2}{1+\xi^2} \qquad \forall z \in \mathbb{C}\,. \tag{4.6}$$

PROOF An elementary calculation shows that

$$\min_x \{x^2 + (\xi x + b)^2\} = \frac{b^2}{1+\xi^2}$$

for $b \in \mathbb{R}$ and real argument x. Applying the inequality to $x = \Re z, \Im z$ and $b = \Re w, \Im w$ and summing them up, gives the result. ∎

LEMMA 4.1.5

Let $\xi_1, \ldots, \xi_n \in \mathbb{R}$, $\hat{v}_n \in \mathbb{C}$. The following inequality holds:

$$|\hat{v}_1|^2 + \ldots + |\hat{v}_n|^2 + |\xi_1 \hat{v}_1 + \ldots + \xi_n \hat{v}_n|^2 \geq \frac{1+|\xi|^2}{1+|\xi'|^2}|\hat{v}_n|^2 \tag{4.7}$$

for all $\hat{v}_1, \ldots, \hat{v}_{n-1} \in \mathbb{C}$.

PROOF We use induction in n. For $n = 1$, inequality (4.1.5) turns into identity. Assume that the inequality holds for $n-1$. We have,

$$\begin{aligned}
|\underbrace{\hat{v}_1}_{z}|^2 &+ |\hat{v}_2|^2 + \ldots + |\hat{v}_n|^2 + |\xi_1 \hat{v}_1 + \underbrace{\xi_2 \hat{v}_2 + \ldots + \xi_n \hat{v}_n}_{w}|^2 \\
&\geq |\hat{v}_2|^2 + \ldots + |\hat{v}_n|^2 + \tfrac{|\xi_2 \hat{v}_2 + \ldots + \xi_n \hat{v}_n|^2}{1+\xi_1^2} && \text{(Lemma 4.1.4)} \\
&= |\hat{v}_2|^2 + \ldots + |\hat{v}_n|^2 + |\tfrac{\xi_2}{(1+\xi_1^2)^{1/2}} \hat{v}_2 + \ldots + \tfrac{\xi_n}{(1+\xi_1^2)^{1/2}} \hat{v}_n|^2 \\
&\geq \frac{1 + \frac{\xi_2^2}{1+\xi_1^2} + \ldots + \frac{\xi_n^2}{1+\xi_1^2}}{1 + \frac{\xi_2^2}{1+\xi_1^2} + \ldots + \frac{\xi_{n-1}^2}{1+\xi_1^2}} |\hat{v}_n|^2 && \text{(induction assumption)} \\
&= \tfrac{1+|\xi|^2}{1+|\xi'|^2} |\hat{v}_n|^2\,.
\end{aligned}$$

∎

THEOREM 4.1.4 Normal Trace Theorem

Let $s > -\frac{1}{2}$. Define the normal trace operator:

$$\gamma_n \,:\, (\mathcal{D}(\mathbb{R}^n))^n \to \mathcal{D}(\mathbb{R}^{n-1}), \qquad (\gamma_n v)(x') = v_n(x', 0)\,.$$

There exists a unique continuous extension of operator γ_n, denoted with the same symbol, to

$$\gamma_n \,:\, H^s(\text{div}, \mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})\,.$$

PROOF We use the same starting point as in the proof of Trace Theorem:

$$\widehat{\gamma_n v}(\xi') = \int_{-\infty}^{\infty} \hat{v}_n(\xi', \xi_n)\, d\xi_n\,.$$

We insert now factor $(\frac{(1+|\xi|^2)^{s+1}}{(1+|\xi'|^2)})^{\frac{1}{2}}$ and apply the Cauchy-Schwarz inequality,

$$|\widehat{\gamma_n v}(\xi')|^2 \le \underbrace{\int_{-\infty}^{\infty} \frac{1+|\xi'|^2}{(1+|\xi|^2)^{s+1}}\, d\xi_n}_{=:M_s(\xi')} \int_{-\infty}^{\infty} \frac{(1+|\xi|^2)^{s+1}}{1+|\xi'|^2} |\hat{v}_n(\xi', \xi_n)|^2\, d\xi_n\,.$$

We use the same substitution, $\xi_n = (1+|\xi'|^2)^{1/2} t$, to estimate term $M_s(\xi')$,

$$M_s(\xi') = (1+|\xi'|^2)^{-(s-\frac{1}{2})} \underbrace{\int_{\infty}^{\infty} \frac{dt}{(1+t^2)^{s+1}}}_{<\infty \text{ for } s>-\frac{1}{2}}\,.$$

We multiply both sides by factor $(1+|\xi'|^2)^{s-\frac{1}{2}}$, integrate over $\mathbb{R}^{n-1}$, and use Lemma 4.1.5 to obtain

$$\begin{aligned}
\|\gamma_n v\|^2_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} &\le \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{s+1}} \int_{\mathbb{R}^{n-1}} (1+|\xi^2)^s \frac{|\xi|^2}{|\xi'|^2} |\hat{v}_n|^2\, d\xi_n\, d\xi' \\
&\le \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{s+1}} \int_{\mathbb{R}^{n-1}} (1+|\xi^2)^s (|\hat{v}_1|^2 + \ldots + |\hat{v}_n|^2 + |\xi_1 \hat{v}_1 + \ldots + \xi_n \hat{v}_n|^2)\, d\xi_n\, d\xi' \\
&= \|v\|^2_{H^s(\mathrm{div}, \mathbb{R}^n)}\,.
\end{aligned}$$

∎

Given our experience with the construction of the right inverse of trace operator γ, we base the construction of the right inverse for the normal trace on the minimum energy extension problem as well:

$$\begin{cases} -\boldsymbol{\nabla}(\mathrm{div}\, V) + V = 0 & \text{in } \mathbb{R}^n_- \\ V = v & \text{on } \mathbb{R}^{n-1}\,. \end{cases} \tag{4.8}$$

Taking the divergence of $(4.8)_1$, we learn that V satisfies the problem above iff $U = \mathrm{div}\, V$ satisfies the Neumann problem:

$$\begin{cases} -\Delta U + U = 0 & \text{in } \mathbb{R}^n_- \\ \dfrac{\partial U}{\partial x_n} = v & \text{on } \mathbb{R}^{n-1}\,. \end{cases} \tag{4.9}$$

The gradient $V = \boldsymbol{\nabla} U$ is thus the desired extension of v. As before, in order to derive an explicit formula for U, we first Fourier transform boundary-value problem (4.9) in x' and obtain the following ODE problem in x_n.

$$(4\pi^2|\xi'|^2 + 1)\,\hat{U} - \frac{\partial^2 \hat{U}}{\partial x_n^2} = 0 \qquad \frac{\partial \hat{U}}{\partial x_n} = \hat{v}\,.$$

Selecting the exponentially decaying solution for $x_n \to -\infty$, we get,

$$\hat{U}(\xi'; x_n) = C e^{(1+4\pi^2|\xi'|^2)^{1/2} x_n}$$

and, upon utilizing the boundary condition, we obtain,

$$\hat{U}(\xi'; x_n) = \hat{v}(\xi')(1 + 4\pi^2|\xi'|^2)^{-1/2} e^{(1+4\pi^2|\xi'|^2)^{1/2} x_n} \qquad x_n < 0\,.$$

Similarly to the technique used to arrive at extension (4.4), we extend the function to whole $\mathbb{R}^n$ with:

$$\hat{U}(\xi'; x_n) = \hat{v}(\xi')(1 + 4\pi^2|\xi'|^2)^{-1/2} \psi((1 + 4\pi^2|\xi'|^2)^{1/2} x_n) \qquad x_n \in \mathbb{R}$$

where $\psi \in \mathcal{S}(\mathbb{R})$ is an extension of the exponential. Notice the regularizing effect of factor $(1+4\pi^2|\xi'|^2)^{-1/2}$. If $v \in H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ then the inverse Fourier transform of $\hat{v}(\xi')(1 + 4\pi^2|\xi'|^2)^{-1/2}$ lives in $H^{s+\frac{1}{2}}(\mathbb{R}^{n-1})$. Fourier transforming in x_n, we obtain the formula for the extension in the Fourier space,

$$\hat{U}(\xi) = \hat{v}(\xi')(1 + 4\pi^2|\xi'|^2)^{-1} \hat{\psi}\left(\frac{\xi_n}{(1 + 4\pi^2|\xi'|^2)^{\frac{1}{2}}}\right)\,. \tag{4.10}$$

The only difference between (4.10) and (4.4) is the different value of exponent for the $(1 + 4\pi^2|\xi'|^2)$ factor.

THEOREM 4.1.5 Extension Theorem for Normal Trace Space

Let $s \in \mathbb{R}$ and $v \in H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$. Define $V = \boldsymbol{\nabla} U$ where Fourier transform of U is given by formula (4.10). Extension operator:

$$H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \ni v \to V \in H^s(\text{div}, \mathbb{R}^n_-)$$

is well-defined and continuous.

PROOF It is sufficient to notice that operator:

$$H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \ni v \to U \in H^{s+1}(\mathbb{R}^n)$$

is continuous. Indeed, the gradient operator takes $H^{s+1}(\mathbb{R}^n)$ into $(H^s(\mathbb{R}^n))^n$ and, in $\mathbb{R}^n_-$, div $\boldsymbol{\nabla} U = \Delta U = U \in H^{s+1}(\mathbb{R}^n_-)$ (better than needed). Note that the extension above is defined in the whole $\mathbb{R}^n$ but the relation between the Laplacian ΔU and function U can be claimed only in the half space.
∎

We now restrict ourselves to three dimensions, $n = 3$.

THEOREM 4.1.6 Tangential Trace

Let $s > -\frac{1}{2}$. Define the tangential trace operator:

$$\gamma_t \,:\, (\mathcal{D}(\mathbb{R}^3))^3 \to (\mathcal{D}(\mathbb{R}^2))^2, \qquad (\gamma_t E)(x') = (E_1(x', 0), E_2(x', 0))\,.$$

There exists a unique continuous extension of operator γ_t, denoted with the same symbol, to

$$H^s(\text{curl}, \mathbb{R}^3) \to H^{s-\frac{1}{2}}(\text{curl}, \mathbb{R}^2)\,. \tag{4.11}$$

PROOF Consider $V = \boldsymbol{\nabla} \times E$. Then div $V = 0$ and, by the Normal Trace Theorem,

$$\|\text{curl}\,(\gamma_t E)\|_{H^{s-\frac{1}{2}}(\mathbb{R}^2)} = \|\gamma_n(\boldsymbol{\nabla} \times E)\|_{H^{s-\frac{1}{2}}(\mathbb{R}^2)} = \|\gamma_n V\|_{H^{s-\frac{1}{2}}(\mathbb{R}^2)}$$
$$\lesssim \|V\|_{H^s(\text{div}\,,\mathbb{R}^3)} = \|V\|_{H^s(\mathbb{R}^3)} = \|\boldsymbol{\nabla} \times E\|_{H^s(\mathbb{R}^3)} \le \|E\|_{H^s(\text{curl},\mathbb{R}^3)}\,.$$

Next, by Lemma 4.1.4, and an elementary algebraic argument,

$$|\hat{E}_1|^2 + |\hat{E}_3| + |\xi_3\hat{E}_1 - \xi_1\hat{E}_3|^2 \ge \tfrac{\xi_3^2}{1+\xi_1^2}|\hat{E}_1|^2 + |\hat{E}_1|^2 = \tfrac{1+\xi_1^2+\xi_3^2}{1+\xi_1^2}|\hat{E}_1|^2$$
$$\ge \tfrac{1+\xi_1^2+\xi_2^2+\xi_3^2}{1+\xi_1^2+\xi_2^2}|\hat{E}_1|^2 \qquad (\tfrac{a}{b} \ge \tfrac{a+c}{b+c} \text{ for } a > b, c > 0)$$
$$= \tfrac{1+|\xi|^2}{1+|\xi'|^2}|\hat{E}_1|^2\,.$$

By the same argument,

$$|\hat{E}_2|^2 + |\hat{E}_3| + |\xi_2\hat{E}_3 - \xi_3\hat{E}_2|^2 \ge \frac{1+|\xi|^2}{1+|\xi'|^2}|\hat{E}_2|^2\,.$$

Consequently,

$$\frac{1+|\xi|^2}{1+|\xi'|^2}(|\hat{E}_1|^2 + |\hat{E}_2|^2) \lesssim |\hat{E}_1|^2 + |\hat{E}_2|^2 + |\hat{E}_3|^2 + |\xi_3\hat{E}_1 - \xi_1\hat{E}_3|^2 + |\xi_2\hat{E}_3 - \xi_3\hat{E}_2|^2 + |\xi_1\hat{E}_2 - \xi_1\hat{E}_2|^2\,.$$

Note that the last term on the right-hand side is redundant here but it was used in the estimate of curl $\gamma_t E$. The rest of the proof follows the same lines as in the proof of the Normal Trace Theorem.
∎

Extension theorem for tangential traces. We start with a heurestic similar to that for the standard and normal traces. Let $e \in H^{s-\frac{1}{2}}(\text{curl}, \mathbb{R}^2)$. As in the previous two cases, we would like to work with the minimum energy extension E that satisfies the equation:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times E) + E = 0\,, \tag{4.12}$$

with boundary conditions on the tangential traces,

$$E_i(x', 0) = e_i(x')\,, \quad x' \in \mathbb{R}^2,\ i = 1, 2\,.$$

We cannot work directly with equations (4.12). After the Fourier transform in x', we obtain a system of three second order equations hard to analyze. Instead, we notice that by taking divergence of (4.12), we learn that div $E = 0$. Recalling that

$$-\Delta E = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times E) - \boldsymbol{\nabla}(\text{div}\,E)\,,$$

we realize that solution of (4.12) must also satisfy the system of decoupled equations:

$$-\Delta E + E = 0\,. \tag{4.13}$$

Conversely, taking divergence of (4.13), we learn that

$$-\Delta(\text{div}\,E) + \text{div}\,E = 0$$

To be able to conclude that solution of (4.13) is also divergence-free (equations (4.12) and (4.13) are then equivalent), we need to impose a boundary condition for divergence,

$$\text{div } E(x',0) = 0 \quad x' \in \mathbb{R}^2 .$$

This leads to a Neumann boundary condition for E_3. After the Fourier transform in x', we get:

$$\frac{\partial \hat{E}_3}{\partial x_3}(\xi';0) = -i2\pi(\xi_1\hat{e}_1 + \xi_2\hat{e}_2) .$$

In the end, we end up with the following candidates for the extension:

$$\begin{aligned} \hat{E}_i(\xi';x_3) &= \hat{e}_i(\xi')e^{(1+4\pi^2|\xi'|^2)^{\frac{1}{2}}x_3}, \quad i=1,2 \\ \hat{E}_3(\xi';x_3) &= -i2\pi(\xi_1\hat{e}_1+\xi_2\hat{e}_2)\,(1+4\pi^2|\xi'|^2)^{-\frac{1}{2}}e^{(1+4\pi^2|\xi'|^2)^{\frac{1}{2}}x_3} \quad . \end{aligned}$$

After extension to whole space we get,

$$\begin{aligned} \hat{E}_i(\xi';x_3) &= \hat{e}_i(\xi')\psi((1+4\pi^2|\xi'|^2)^{\frac{1}{2}}x_3), \quad i=1,2 \\ \hat{E}_3(\xi';x_3) &= -i2\pi(\xi_1\hat{e}_1+\xi_2\hat{e}_2)\,(1+4\pi^2|\xi'|^2)^{-\frac{1}{2}}\psi((1+4\pi^2|\xi'|^2)^{\frac{1}{2}}x_3) \end{aligned} \tag{4.14}$$

where $\psi \in \mathcal{S}(\mathbb{R})$ is an extension of the exponential. Given our experience from Theorem 4.1.3, this is probably the most convenient form to analyze continuity properties of the extension operator. With $e_i \in H^{s-\frac{1}{2}}(\mathbb{R}^2)$, extensions E_i are clearly in $H^s(\mathbb{R}^3)$. Concerning the third component, it is sufficient to notice that factor

$$-i2\pi(\xi_1\hat{e}_1+\xi_2\hat{e}_2)\,(1+4\pi^2|\xi'|^2)^{-\frac{1}{2}}$$

in E_3 represents Fourier transform of a boundary data that lives also in $H^{s-\frac{1}{2}}(\mathbb{R}^2)$.

The third component of $\boldsymbol{\nabla}\times E$ is given by the formula:

$$(\widehat{\boldsymbol{\nabla}\times E})_3(\xi';x_3) = i2\pi(\xi_1\hat{e}_2(\xi')-\xi_2\hat{e}_1)\psi((1+4\pi^2|\xi'|^2)^{\frac{1}{2}}x_3)$$

with boundary data representing the two-dimension curl that lives again in $H^{s-\frac{1}{2}}(\mathbb{R}^2)$. The only tricky part perhaps is with the remaining two components of $\boldsymbol{\nabla}\times E$. Computing the second component of $\boldsymbol{\nabla}\times E$ in the *lower half space*, we get:

$$\frac{\partial \hat{E}_1}{\partial x_3} - i2\pi\xi_1\hat{E}_3 = \left[\hat{e}_1(1+4\pi^2(\xi_1^2+\xi_2^2)) - 4\pi^2\xi_1(\xi_1\hat{e}_1+\xi_2\hat{e}_2)\right](1+4\pi^2|\xi'|^2)^{-\frac{1}{2}}\psi((1+4\pi^2|\xi'|^2)^{\frac{1}{2}}x_3) . \tag{4.15}$$

The term in the square brackets must represent a boundary data in $H^{s-\frac{3}{2}}(\mathbb{R}^2)$ (we have the regularizing factor $(1+4\pi^2|\xi'|^2)^{-\frac{1}{2}}$) which allows only for first powers of ξ_i. Clearly, there must be some cancellations there to have a success story. This is where the assumption on the boundary curl comes in again. We have

$$\widehat{\text{curl} e} = i2\pi(\xi_2\hat{e}_2 - \xi_1\hat{e}_1) .$$

Hence,

$$\xi_1\xi_2\hat{e}_2 = \xi_2^2\hat{e}_1\frac{\xi_2}{2\pi}\widehat{\text{curl}\, e} .$$

Substituting into the bracket, we get:

$$[\cdots] = [\hat{e}_1 - 2\pi\xi_2\widehat{\text{curl}\ e}]$$

which represents the expected regularity of boundary data. Consequently, the H^s norm of (4.15) is bounded by the $H^{s-\frac{1}{2}}(\text{curl}, \mathbb{R}^2)$ norm of the boundary data. Note that (4.15) represents the curl of the extension *only* in the lower half-space. Identical reasoning works for the first component of $\boldsymbol{\nabla} \times E$. We have proved the following result.

THEOREM 4.1.7 Extension Theorem for Tangential Trace Space

Let $s \in \mathbb{R}$ and $e \in H^{s-\frac{1}{2}}(\text{curl}, \mathbb{R}^2)$. Define $E = \mathcal{F}_{\xi \to x}\hat{E}$ where partial Fourier transform $\hat{E}$ is given by formulas (4.14). Extension operator:

$$H^{s-\frac{1}{2}}(\text{curl}, \mathbb{R}^2) \ni e \to E \in H^s(\text{curl}, \mathbb{R}^3_-)$$

is well-defined and continuous.

We summarize now our findings in three space dimensions.

THEOREM 4.1.8

Let $s > -\frac{1}{2}$. There exist three continuous trace operators mapping the differential complex energy spaces <u>onto</u> the corresponding trace energy spaces defined on the boundary forming a two-dimensional differential complex, with the following commuting diagram.

$$\begin{array}{ccccc} H^{s+1}(\mathbb{R}^3_-) & \xrightarrow{\boldsymbol{\nabla}} & H^s(\text{curl}, \mathbb{R}^3_-) & \xrightarrow{\boldsymbol{\nabla}\times} & H^s(\text{div}, \mathbb{R}^3_-) \\ \downarrow \gamma & & \downarrow \gamma_t & & \downarrow \gamma_n \\ H^{s+\frac{1}{2}}(\mathbb{R}^2) & \xrightarrow{\boldsymbol{\nabla}} & H^{s-\frac{1}{2}}(\text{curl}, \mathbb{R}^2) & \xrightarrow{\text{curl}} & H^{s-\frac{1}{2}}(\mathbb{R}^2) \end{array}$$

PROOF

Commuttativity follows from the construction of the trace operators. The boundary sequence is the 2D differential complex with regularity shifted by $\frac{1}{2}$. ∎

REMARK 4.1.3 Notice that the theorem has been formulated using the language of the differential complex only. This is for a reason, the discussed spaces do *not* form an exact sequence in $\mathbb{R}^n, n = 1, 2, 3$. Let us discuss for instance the 3D sequence. Fourier transforming the curl, we learn that

$$\widehat{\boldsymbol{\nabla} \times E} = 0 \quad \Leftrightarrow \quad \frac{\hat{E}_1}{\xi_1} = \frac{\hat{E}_2}{\xi_2} = \frac{\hat{E}_3}{\xi_3}\,.$$

The common value above is identified as the (Fourier transform of) scalar potential u,

$$i2\pi\hat{u} = \frac{\hat{E}_1}{\xi_1} = \frac{\hat{E}_2}{\xi_2} = \frac{\hat{E}_3}{\xi_3}\,.$$

First of all, contrary to a bounded domain where the scalar potential is always determined only up to an additive constant, in the case of $\mathbb{R}^3$, the scalar potential is unique. Fourier transform of unity is Dirac's delta that is not a function, hence constants do not live in any Sobolev space. Does u live in $H^s(\text{grad}, \mathbb{R}^3)$ if ∇u is an element of $(H^s(\mathbb{R}^3)^3$? It is easy to construct counterexamples showing that, in general, the answer is negative, comp. Exercise 4.1.2. The only way out is thus to build additional assumptions into the definition of the $H(\text{curl})$ space, to secure that the potential is in the starting space $H^{1+s}(\mathbb{R}^3) \sim H^s(\text{grad}, \mathbb{R}^3)$. The modified definition of the energy space reads as follows:

$$\{E \in H^s(\text{curl}, \mathbb{R}^3) \,:\, \int_{\mathbb{R}^3} (1+|\xi|^2)^s \frac{1+|\xi|^2}{\xi_i^2}\, |\hat{E}_i|^2 \, d\xi < \infty\,, i = 1, 2, 3\}\,. \tag{4.16}$$

Presence of the additional, singular factor $\frac{1+|\xi|^2}{\xi_i^2}$ secures that the potential is in the right energy space. In particular, continuous Fourier transforms $\hat{E}_i$ must vanish at zero, see Exercise 4.1.3. This says that, in a certain sense, components E_i have zero average. If we still insist on having an exact sequence, the extra conditions for the $H(\text{curl})$ energy space propagate into extra, more complicated, conditions for the $H(\text{div})$ space and so on. The moral of the story is to avoid the exact sequence arguments when working in the whole space and stick with differential complexes only. ∎

4.1.4 The Case of a Piecewise Smooth Hypograph

REMARK 4.1.4 We have a terrible notational conflict in what follows. The "hat" symbol $\hat{u}$ and argument ξ have been so far exclusively reserved for the Fourier transform of function u and its argument from the Fourier (frequency) domain. Unfortunately, exactly the same two symbols have been used for *parametric finite elements* and Piola transforms known also as pullback maps. I have decided to use the symbols nevertheless. I hope, you can survive it. Please review elementary facts about curvilinear systems of coordinates ([14], Appendix 1) and derivation of Piola transforms before reading this section. ∎

We shall consider the standard map from the half-space onto the hypograph of a globally continuous and piecewise smooth function ζ,

$$T \,:\, \mathbb{R}^3_- \ni \xi = (\underbrace{\xi_1, \xi_2}_{=:\xi'}, \xi_3) \to x = (\underbrace{x_1, x_2}_{=:x'}, x_3) \in \Omega \subset \mathbb{R}^3$$

defined as:

$$x' = \xi', \quad x_3 = \xi_3 + \zeta(\xi') \tag{4.17}$$

For $\xi_3 = 0$, the map implies a parametrization of the boundary Γ of hypograph Ω. We shall make precise assumptions about function ζ shortly.

If we consider the map to be a parametrization for Ω, we have the following formulas for the basis and cobasis vectors.

$$\begin{array}{lll} a_1 = \frac{\partial x}{\partial \xi_1} = (1,0,\zeta_{,1}) & a_2 = \frac{\partial x}{\partial \xi_2} = (0,1,\zeta_{,2}) & a_3 = \frac{\partial x}{\partial \xi_3} = (0,0,1) \\ a^1 = (1,0,0) & a^2 = (0,1,0) & a^3 = (-\zeta_{,1}, -\zeta_{,2}, 1) \end{array}$$

Determinant of the Jacobian matrix $\frac{\partial x_i}{\partial \xi_j}$, the jacobian $[a_1, a_2, a_3] := (a_1 \times a_2) \cdot a_3 = 1$. Normal to the boundary equals the unit vector of a^3,

$$n = \left(-\frac{\zeta_{,1}}{\sqrt{1+\zeta_{,1}^2+\zeta_{,2}^2}}, -\frac{\zeta_{,2}}{\sqrt{1+\zeta_{,1}^2+\zeta_{,2}^2}}, \frac{1}{\sqrt{1+\zeta_{,1}^2+\zeta_{,2}^2}} \right)$$

and the boundary cobasis vectors $a_\Gamma^\alpha = a^\alpha - (a^\alpha \cdot n)n$ are given by:

$$a_\Gamma^1 = \left(\frac{1+\zeta_{,2}^2}{1+\zeta_{,1}^2+\zeta_{,2}^2}, -\frac{\zeta_{,1}\zeta_{,2}}{1+\zeta_{,1}^2+\zeta_{,2}^2}, \frac{\zeta_{,1}}{1+\zeta_{,1}^2+\zeta_{,2}^2} \right) \quad a_\Gamma^2 = \left(-\frac{\zeta_{,1}\zeta_{,2}}{1+\zeta_{,1}^2+\zeta_{,2}^2}, \frac{1+\zeta_{,1}^2}{1+\zeta_{,1}^2+\zeta_{,2}^2}, \frac{\zeta_{,2}}{1+\zeta_{,1}^2+\zeta_{,2}^2} \right)$$

We employ now the usual pullback maps (Piola transforms) with unit jacobian,

$$\begin{array}{llll} \hat{u} = u, & \hat{E}_i = E_j \frac{\partial x_j}{\partial \xi_i}, & \hat{v}_i = \frac{\partial \xi_i}{\partial x_j} v_j, & \hat{f} = f \\ u = \hat{u}, & E_i = \hat{E}_j \frac{\partial \xi_j}{\partial x_i}, & v_i = \frac{\partial x_i}{\partial \xi_j} \hat{v}_j, & f = \hat{f} \end{array}$$

or, more explicitly,

$$\begin{array}{lll} E_1 = \hat{E}_1 - \zeta_{,1}\hat{E}_3, & E_2 = \hat{E}_2 - \zeta_{,2}\hat{E}_3, & E_3 = \hat{E}_3 \\ v_1 = \hat{v}_1 & v_2 = \hat{v}_2 & v_3 = \zeta_{,1}\hat{v}_1 + \zeta_{,2}\hat{v}_2 + \hat{v}_3 \end{array}$$

Let e_i denote the Cartesian unit vectors in Ω. From

$$E = E_i e_i = \hat{E}_j \underbrace{\frac{\partial \xi_j}{\partial x_i} e_i}_{=a^j}, \qquad v = v_i e_i = \frac{\partial x_i}{\partial \xi_j} \hat{v}_j e_i = \hat{v}_j \underbrace{\frac{\partial x_i}{\partial \xi_j} e_i}_{=a_j} \tag{4.18}$$

follows that $\hat{E}_i$ can be interpreted as the *covariant* components of E, and $\hat{v}_i$ are the *contravariant* components of v in the curvilinear system of coordinates implied by the parametrization T. As usual, we do not show the compositions with parametrization T or its inverse, e.g. the inequality $\hat{u} = u$ assumes that either u stands for $u \circ T$ or $\hat{u}$ means $\hat{u} \circ T^{-1}$.

We make now precise assumptions about function $\zeta(\xi')$.

$$\begin{aligned} &\mathbb{R}^2 = \bigcup_{j=1}^m \overline{G}_j, \quad G_j \text{ are open and pairwise disjoint, } j = 1, \ldots, m \\ &\zeta|_{G_j} \in C^1(\overline{G_j}),\ j = 1, \ldots, m\,, \\ &\zeta \text{ is globally continuous.} \end{aligned} \tag{4.19}$$

Partition of $\mathbb{R}^2$ into sets G_j, $j = 1, \ldots, m$, implies a partition of $\hat{\Omega} := \mathbb{R}^3_-$ into sets $\hat{\Omega}_j := G_j \times (-\infty, 0)$. Consequently, parametrization (4.17) of the hypograph domain is also piecewise smooth, see Fig. 4.2 for illustration.

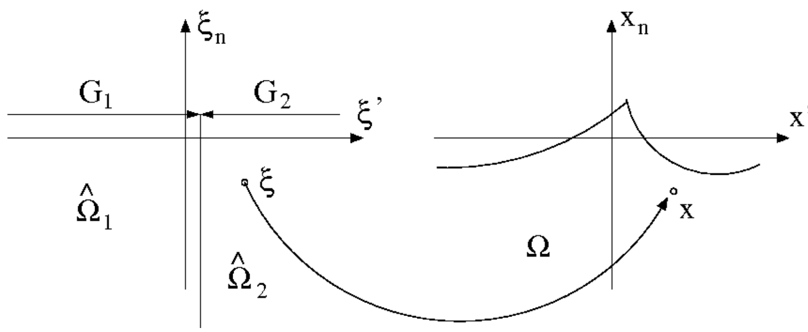

Figure 4.2
Parametrization of the hypograph domain.

The strategy for constructing traces for a piecewise smooth hypograph consists of three steps.

1. Given a function in one of the energy spaces defined on the hypograph domain Ω, we first pull it back to the parametric domain (half-space) $\hat{\Omega} = \mathbb{R}^3_-$. We will show that, with the assumed regularity on map $\zeta(\xi')$ and range $s \in (-\frac{1}{2}, \frac{1}{2})$, the Piola transforms map the energy spaces on Ω into their counterparts defined on $\hat{\Omega}$.

2. We trace the pullbacked functions to the boundary of $\hat{\Omega}$ - the hyperplane.

3. We push forward the trace to boundary Γ of the hypograph domain Ω.

The first step is a consequence of Lemma 3.5.2 and Lemma 3.5.3 where we studied invariance of Sobolev spaces under multiplication and change of variables. In the second step we utilize the just proved results on traces for the half-space domain. Finally, the preservation of boundary energy spaces necessary in the last step will simply be a consequence of definitions; the trace spaces on $\partial\Omega$ will be defined as the images of the corresponding trace spaces on the hyperplane under the boundary push forward maps.

We start by demonstrating that Piola transforms map the exact energy spaces defined on the hypograph Ω onto the energy spaces defined on the half-space $\mathbb{R}^3_-$. Notice the commuttativity properties of the pullback maps:

$$\hat{\boldsymbol{\nabla}}\hat{u} = \widehat{\boldsymbol{\nabla} u}, \quad \hat{\boldsymbol{\nabla}} \times \hat{E} = \widehat{\boldsymbol{\nabla} \times E}, \quad \hat{\boldsymbol{\nabla}} \cdot \hat{v} = \widehat{\boldsymbol{\nabla} \cdot v}\,.$$

In fact, it is exactly these commuttativity properties that have led to the definition of Piola transforms. Due to the commuttativity properties, it is sufficient to show only that all pullback maps preserve H^s spaces. Let $s \in (-\frac{1}{2}, \frac{1}{2})$. The first case is easy. Lemma 3.5.3 ($k = 1$) directly implies that

$$\|\hat{u}\|_{H^s(\mathbb{R}^n_-)} \lesssim \|u\|_{H^s(\Omega)}\,.$$

For the $H^s(\text{curl})$ case, the Piola transform involves both change of variables and multiplication with the Jacobian matrix,

$$\hat{E}_i(\xi) = E_j(x(\xi))\frac{\partial \xi_j}{\partial x_i}(x(\xi))\,.$$

While there is no problem with the change of variable (same situation as in the H^s case), the multiplication is a challenge. Jacobian is only piecewise continuous and Lemma 3.5.2 implies the result only for the trivial case of $s = 0$. This is where the tilde spaces come in. Let E be a function defined on Ω and $\hat{E}$ be the corresponding pullback function defined on $\hat{\Omega}$. Let $\hat{E}_j$ denote the restriction of $\hat{E}$ to subdomain $\hat{\Omega}_j$, and $\tilde{\hat{E}}_j$ its extension by zero to the whole space. Obviously, $E = \sum_j \tilde{\hat{E}}_j$ in $\hat{\Omega}$. Recall that

$$\|\tilde{\hat{E}}_j\|_{H^s(\mathbb{R}^3)} = \|\hat{E}_j\|_{\tilde{H}^s(\hat{\Omega}_j)} .$$

Therefore, we have,

$$\begin{aligned}
\|\hat{E}\|^2_{H^s(\hat{\Omega})} &\lesssim \sum_{j=0}^{m} \|\hat{E}\|^2_{\tilde{H}^s(\hat{\Omega}_j)} && \text{(triangle inequality)} \\
&\lesssim \sum_{j=0}^{m} \|\hat{E}\|^2_{H^s(\hat{\Omega}_j)} && (\tilde{H}^s(\hat{\Omega}_j) \sim H^s(\hat{\Omega}_j) \text{ for } s \in (-\tfrac{1}{2}, \tfrac{1}{2})) \\
&\lesssim \sum_{j=0}^{m} \|E\|^2_{H^s(\Omega_j)} && \text{(piecewise smooth parametrization)} \\
&\lesssim \|E\|^2_{H^s(\Omega)}
\end{aligned}$$

Note that, for the negative range of parameter s, Piola transforms involve multiplication of functionals with elements of the Jacobian matrix. The same result holds for the inverse transformation $\hat{E} \to E$ and the remaining Piola transforms. Consequently, for $s \in (-\frac{1}{2}, \frac{1}{2})$,

$$\|\hat{u}\|_{H^{1+s}(\mathbb{R}^3_-)} \lesssim \|u\|_{H^{1+s}(\Omega)}$$

$$\|\hat{E}\|_{H^s(\text{curl},\mathbb{R}^3_-)} \lesssim \|E\|_{H^s(\text{curl},\Omega)}$$

$$\|\hat{v}\|_{H^s(\text{div},\mathbb{R}^3_-)} \lesssim \|v\|_{H^s(\text{div},\Omega)}$$

$$\|\hat{f}\|_{H^s(\mathbb{R}^3_-)} \lesssim \|f\|_{H^s(\Omega)}$$

with same inequalities valid for the inverse transforms as well. Of course, all equivalence constants depend upon map ζ.

Traces on boundary of the hypograph. Extension of trace theorems for a half space to the hypograph domain Ω is a direct consequence of definition of trace spaces on boundary Γ of domain Ω. We will use the following notation for the *boundary jacobian*:

$$\text{jac}_\Gamma(\xi') := (1 + \zeta_{,1}^2 + \zeta_{,2}^2)^{1/2} .$$

We derive first (formally) formulas for the pullbacks of all three traces defined on hypograph boundary Γ. Given a function $u : \Gamma \to \mathbb{C}$, we denote the corresponding pullback on reference boundary by :

$$\hat{u}_\zeta \,:\, \mathbb{R}^2 \to \mathbb{C}, \qquad \hat{u}_\zeta(\xi') = u(\xi', \zeta(\xi')) . \tag{4.20}$$

It follows from (4.18) that the normal trace on Γ is given by:

$$v_n = \hat{v}_j a_j \cdot n = \frac{\hat{v}_3}{\text{jac}_\Gamma} .$$

This implies that the pullback of the normal trace is given by:

$$(\hat{v}_3)_\zeta := v_n(\xi', \zeta(\xi'))\,\mathrm{jac}_\Gamma(\xi')\,. \tag{4.21}$$

It follows again from (4.18) that the tangential trace on Γ is given by:

$$E_t = \hat{E}_\alpha a^\alpha_\Gamma\,.$$

Consequently, the pullback of the tangential trace is given by:

$$(\hat{E}_\alpha)_\zeta = E_t(\xi', \zeta(\xi')) \cdot a_\alpha\,, \qquad \alpha = 1, 2\,. \tag{4.22}$$

The main idea is now to identify the trace spaces on hypograph boundary Γ by requesting the corresponding pullbacks to be in the corresponding trace spaces for the half-space domain. Let $s \in (0,1)$. We define:

$$H^s(\Gamma) := \{u : \Gamma \to \mathbb{C}\, :\, \hat{u}_\zeta \in H^s(\mathbb{R}^{n-1})\} \tag{4.23}$$

with the norm,

$$\|u\|_{H^s(\Gamma)} := \|\hat{u}_\zeta\|_{H^s(\mathbb{R}^{n-1})}\,.$$

By construction, spaces $H^s(\Gamma)$ and $H^s(\mathbb{R}^{n-1})$ are isometric, and the pullback map is an isometric isomorphism.

Before we define the normal trace spaces, we need to extend the definition (4.21) to functionals. We do it by using the duality. Notice that

$$\int_{\mathbb{R}^{n-1}} (\hat{v}_3)_\zeta \hat{\phi}_\zeta\, d\xi' = \int_{\mathbb{R}^{n-1}} v_n(\xi', \zeta(\xi'))\hat{\phi}_\zeta(\xi')\ \mathrm{jac}_\Gamma(\xi') d\xi' = \int_\Gamma v_n \phi\, d\Gamma\,.$$

Let w be now any linear functional defined on just defined $H^s(\Gamma)$. Let $\phi \in H^s(\Gamma)$. We define the pullback $\hat{w}_\zeta$ of w by:

$$\langle \hat{w}_\zeta, \hat{\phi}_\zeta \rangle_{\mathbb{R}^{n-1}} := \langle w, \phi \rangle_\Gamma\,.$$

This identifies the normal trace space as the topological dual of space $H^s(\Gamma)$:

$$H^{-s}(\Gamma) := \{w \in (H^s(\Gamma))^*\ :\ \hat{w}_\zeta \in H^{-s}(\mathbb{R}^{n-1})\} = (H^s(\Gamma))'\,. \tag{4.24}$$

with the norm:

$$\|w\|_{H^{-s}(\Gamma)} = \sup_{v \in H^s(\Gamma)} \frac{|\langle w, v\rangle|}{\|v\|_{H^s(\Gamma)}} = \|\hat{w}_\zeta\|_{H^{-s}(\mathbb{R}^{n-1})}\,.$$

Again, by construction, spaces $H^{-s}(\Gamma)$ and $H^{-s}(\mathbb{R}^{n-1})$ are isometric, and the pullback map is an isometric isomorphism.

We show now how the trace theorems for H^s and $H^s(\mathrm{div})$ spaces in the reference domain imply automatically their counterparts in the hypograph domain. Let $u \in H^{1+s}(\Omega)$. Then $\hat{u} \in H^{1+s}(\mathbb{R}^n_-)$. Consequently, by Theorem 4.1.2, we have trace $\hat{\gamma}\hat{u} \in H^{\frac{1}{2}+s}(\mathbb{R}^{n-1})$. We define trace $\gamma u \in H^{\frac{1}{2}+s}(\Gamma)$ by requesting that the trace of the domain pullback matches the boundary pullback of the trace,

$$\widehat{(\gamma u)}_\Gamma = \hat{\gamma}\hat{u}\,.$$

For sufficiently regular functions u this simply means that

$$(\gamma u)(x) := (\hat\gamma \hat u)(\xi') \quad \text{where } x = (\xi', \zeta(\xi')) \in \Gamma \,.$$

Continuity of trace operator γ follows immediately from continuity of trace operator $\hat\gamma$ and definition of trace space on Γ,

$$\|\gamma u\|_{H^{1/2+s}(\Gamma)} = \|\hat\gamma \hat u\|_{H^{1/2+s}(\mathbb{R}^{n-1})} \lesssim \|\hat u\|_{H^{1+s}(\mathbb{R}^n_-)} \lesssim \|u\|_{H^{1+s}(\Omega)} \,.$$

Note that, by construction, surjectivity of $\hat\gamma$ implies the surjectivity of γ.

We have the same reasoning for normal traces. Let $v \in H^s(\text{div}, \Omega)$, $s \in (-\frac{1}{2}, \frac{1}{2})$. Then $\hat v \in H^s(\widehat{\text{div}}, \mathbb{R}^n_-)$ and, by Theorem 4.1.4, we have the normal trace $\hat\gamma_n \hat v \in H^{s-1/2}(\mathbb{R}^{n-1})$. Trace $\gamma_n v \in H^{s-1/2}(\Gamma)$ is identified as the unique functional that satisfies:

$$\widehat{(\gamma_n v)}_\zeta = \hat\gamma_n \, \hat v \,.$$

Continuity and surjectivity of the normal trace follows in the same way as above.

Traces for space $H^s(\text{curl}, \Omega)$ are defined in three dimensions only. Let $s \in (0, 1)$. We begin by introducing the *space of tangential test vector fields*:

$$H^s_t(\Gamma) := \{F^t : \Gamma \to \mathbb{R}^3 \ : \ \hat F^t_\zeta \in (H^s(\mathbb{R}^2))^2\} \tag{4.25}$$

where the boundary pullback is defined as:

$$H^s_t(\Gamma) \ni F^t \to \hat F^t_\zeta := (\hat F^1_\zeta, \hat F^2_\zeta) \in (H^s(\mathbb{R}^2))^2, \quad \text{such that } \hat F^\beta a_\beta \, \text{jac}_\Gamma^{-1} = F^t$$

Note that,

$$\int_\Gamma E_t F^t \, dS = \int_{\mathbb{R}^2} ((\hat E_\alpha)_\zeta a^\alpha_\Gamma) \cdot (\hat F^\beta_\zeta a_\beta) \, d\xi' = \int_{\mathbb{R}^2} (\hat E_\alpha)_\zeta \hat F^\alpha_\zeta \, d\xi' \,.$$

This suggests to define the pullback $(\hat E_t)_\zeta$ for any $E_t \in (H^s_t(\Gamma))^*$ by duality as:

$$\langle (\hat E_t)_\zeta, \hat F^t_\zeta \rangle_{\mathbb{R}^2} = \langle E_t, F^t \rangle_\Gamma \qquad \forall F^t \in H^s_t(\Gamma) \,.$$

The trace space is now defined as:

$$H^{-s}(\text{curl}_\Gamma, \Gamma) := \{E_t \in (H^s_t(\Gamma))^* \ : \ (\hat E_t)_\zeta \in H^{-s}(\text{curl}, \mathbb{R}^2)\}. \tag{4.26}$$

equipped with the norm:

$$\|E^t\|_{H^{-s}(\text{curl}_\Gamma, \Gamma)} := \|(\hat E_t)_\zeta\|_{H^{-s}(\text{curl}, \mathbb{R}^2)} \,.$$

Note that, again, spaces $H^{-s}(\text{curl}_\Gamma, \Gamma)$ and $H^{-s}(\text{curl}, \mathbb{R}^2)$ are isometrically isomorphic by construction. Now, let $E \in H^s(\text{curl}, \Omega)$, $s \in (-\frac{1}{2}, \frac{1}{2})$. As a consequence of regularity assumptions on function ζ defining the boundary, the pullback function $\hat E \in H^s(\text{curl}, \mathbb{R}^3_-)$. Let $\hat\gamma_t \hat E \in H^{s-1/2}(\text{curl}, \mathbb{R}^2)$ be the corresponding tangential trace on reference boundary $\mathbb{R}^2$. We define tangential trace $\gamma_t E$ for the hypograph domain as the unique functional in $H^{s-\frac{1}{2}}(\text{curl}_\Gamma, \Gamma)$ such that

$$\widehat{(\gamma_t E)}_\zeta = \hat\gamma_t \hat E \,.$$

The continuity of the trace on Γ is once again a direct consequence of the definitions,

$$\|\gamma_t E\|_{H^{s-1/2}(\text{curl}_\Gamma,\Gamma)} = \|\hat{\gamma}_t \hat{E}\|_{H^{s-1/2}(\text{curl},\mathbb{R}^2)} \lesssim \|\hat{E}\|_{H^s(\text{curl},\mathbb{R}^3_-)} \lesssim \|E\|_{H^s(\text{curl},\Omega)} .$$

As before, surjectivity of γ_t follows from the surjectivity of $\hat{\gamma}_t$.

REMARK 4.1.5 Note that we are not concerned with the topology of test space $H^s_t(\Gamma)$. The only reason for introducing the space is to be able to define the pullback $(\hat{E}_t)_\zeta$. We simply need a class of objects for which we can define the pullback for. What matters in the end is only the subspace of the algebraic dual $(H^s_t(\Gamma))^*$ corresponding to pullbacks from $H^{-s}(\text{curl},\mathbb{R}^2)$. We could have used more regular test functions in place of $H^s_t(\Gamma)$.

In the same spirit, boundary pullback for the normal trace could have been defined using more regular test functions than $H^s(\Gamma)$. Note that only a-posteriori $H^{-s}(\Gamma)$ is identified as a topological dual of $H^s(\Gamma)$. ▮

Commuting Boundary Diagram. For sufficiently regular functions, we define the boundary operators as follows.

- Surface gradient:
$$\boldsymbol{\nabla}_\Gamma u = (\boldsymbol{\nabla} u)_t = \frac{\partial \hat{u}}{\partial \xi_\alpha} a^\alpha_\Gamma .$$
- Surface scalar-valued curl:
$$\text{curl}_\Gamma E_t = (\boldsymbol{\nabla} \times E) \cdot n = \frac{\hat{E}_{2,1} - \hat{E}_{1,2}}{\text{jac}_\Gamma} .$$

For elements of trace spaces, we have to define the operators by duality or, more directly, by utilizing the boundary pullback operators. Let $s \in (-\frac{1}{2}, \frac{1}{2})$. The surface gradient $\boldsymbol{\nabla}_\Gamma u$ of a function $u \in H^s(\Gamma)$ is the unique element of $H^{-s}(\text{curl}_\Gamma, \Gamma)$ such that its boundary pullback coincides with the two-dimensional gradient $\boldsymbol{\nabla}_{\xi'} \hat{u}_\zeta$. Similarly, the surface $\text{curl}_\Gamma E_t$ of $E_t \in H^{-s}(\text{curl}_\Gamma, \Gamma)$ is the unique element of $H^{-s}(\Gamma)$ whose boundary pullback coincides with two-dimensional curl $\hat{E}_{2,1} - \hat{E}_{1,2}$.

We summarize our discussions for the hypograph domain with the following theorem.

THEOREM 4.1.9

Let $s \in (-\frac{1}{2}, \frac{1}{2})$ and let Ω by a piecewise smooth hypograph domain with boundary Γ. There exist three continuous trace operators mapping the differential complex energy spaces onto the corresponding trace energy spaces defined on the boundary forming a 2D differential complex, with the following commuting diagram.

$$\begin{array}{ccccc} H^{s+1}(\Omega) & \xrightarrow{\boldsymbol{\nabla}} & H^s(\text{curl},\Omega) & \xrightarrow{\boldsymbol{\nabla}\times} & H^s(\text{div},\Omega) \\ \downarrow \gamma & & \downarrow \gamma_t & & \downarrow \gamma_n \\ H^{s+\frac{1}{2}}(\Gamma) & \xrightarrow{\boldsymbol{\nabla}_\Gamma} & H^{s-\frac{1}{2}}(\text{curl}_\Gamma,\Gamma) & \xrightarrow{\text{curl}_\Gamma} & H^{s-\frac{1}{2}}(\Gamma) \end{array}$$

PROOF The proof is a direct consequence of discussed definitions of trace spaces, trace operators and boundary operators for the hypograph domain and Theorem 4.1.8. ∎

4.1.5 The Case of a Polyhedral Domain

Definition of trace operators and spaces, continuity and surjectivity of the trace operators follows now the classical construction based on the definition of a polyhedral domain and the partition of unity argument. Following the definition of a polyhedral domain, we consider a partition of unity $\{\psi_j\}$ subordinate to the open cover W_j of domain Ω with its boundary Γ, i.e.

$$\psi_j \in \mathcal{D}(W_j), \quad \sum_{j=0}^{J} \psi_j(x) = 1, \quad x \in \overline{\Omega}, \quad \text{supp}\, \psi_0 \subset \Omega\,.$$

Let $s \in (-\frac{1}{2}, \frac{1}{2})$ and Γ_j denote the boundary of hypograph Ω_j. We define,

$$\begin{aligned} H^{s+\frac{1}{2}}(\Gamma) &:= \{u : \Gamma \to \mathbb{C} \,:\, \psi_j u \in H^{s+\frac{1}{2}}(\Gamma_j), \quad j = 1, \ldots, J\} \\ \|u\|^2_{H^{s+\frac{1}{2}}(\Gamma)} &:= \sum_{j=1}^{J} \|\psi_j u\|^2_{H^{s+\frac{1}{2}}(\Gamma_j)}\,. \end{aligned} \tag{4.27}$$

For $v \in (H^{-s+\frac{1}{2}}(\Gamma))^*$, we define the product of v with partition of unity function ψ_j in the usual way,

$$\langle \psi_j v, \phi \rangle := \langle v, \psi_j \phi \rangle, \quad \phi \in H^{-s+\frac{1}{2}}(\Gamma)\,.$$

Note that the product $\psi_j \phi \in H^{-s+\frac{1}{2}}(\Gamma)$ which makes the product $\psi_j v$ well-defined. The normal trace space is now defined as follows:

$$\begin{aligned} H^{s-\frac{1}{2}}(\Gamma) &:= \{v \in (H^{-s+\frac{1}{2}}(\Gamma))^* \,:\, \psi_j v \in H^{s-\frac{1}{2}}(\Gamma_j), \quad j = 1, \ldots, J\} \\ \|v\|^2_{H^{s-\frac{1}{2}}(\Gamma)} &:= \sum_{j=1}^{J} \|\psi_j w\|^2_{H^{s-\frac{1}{2}}(\Gamma_j)}\,. \end{aligned} \tag{4.28}$$

In order to define the tangential trace space, we need to define first the space of tangential test fields,

$$H_t^{-s+\frac{1}{2}}(\Gamma) := \{F^t : \Gamma \to \mathbb{R}^3 \,:\, \psi_j F^t \in H_t^{-s+\frac{1}{2}}(\Gamma_j), \quad j = 1, \ldots, J\}\} \tag{4.29}$$

where space $H_t^{-s+\frac{1}{2}}(\Gamma_j)$ is defined by (4.25). We define now the tangential trace space as:

$$\begin{aligned} H^{s-\frac{1}{2}}(\text{curl}_\Gamma, \Gamma) &:= \{E \in (H_t^{-s+\frac{1}{2}}(\Gamma))^* \,:\, \psi_j E \in H^{s-\frac{1}{2}}(\text{curl}_\Gamma, \Gamma_j), \quad j = 1, \ldots, J\} \\ \|E\|^2_{H^{s-\frac{1}{2}}(\text{curl}_\Gamma, \Gamma)} &:= \sum_{j=1}^{J} \|\psi_j E\|^2_{H^{s-\frac{1}{2}}(\text{curl}_\Gamma, \Gamma_j)} \end{aligned} \tag{4.30}$$

where the product $\psi_j E$ is again defined by duality,

$$\langle \psi_j E, F^t \rangle := \langle E, \psi_j F^t \rangle, \quad F^t \in H_t^{-s+\frac{1}{2}}(\Gamma)\,.$$

Note that the product $\psi_j F^t$ is in the correct space by definition.

The trace theorems follow now immediately from the corresponding trace theorems for the hypograph domain.

THEOREM 4.1.10

Let $s \in (-\frac{1}{2}, \frac{1}{2})$ and let Ω by a piecewise smooth polyhedral domain with boundary Γ. There exist three continuous trace operators mapping the differential complex energy spaces onto the corresponding trace energy spaces defined on the boundary forming a 2D differential complex, with the following commuting diagram.

$$\begin{array}{ccccc} H^{s+1}(\Omega) & \xrightarrow{\nabla} & H^s(\text{curl}, \Omega) & \xrightarrow{\nabla\times} & H^s(\text{div}, \Omega) \\ \downarrow \gamma & & \downarrow \gamma_t & & \downarrow \gamma_n \\ H^{s+\frac{1}{2}}(\Gamma) & \xrightarrow{\nabla_\Gamma} & H^{s-\frac{1}{2}}(\text{curl}_\Gamma, \Gamma) & \xrightarrow{\text{curl}_\Gamma} & H^{s-\frac{1}{2}}(\Gamma) \end{array}$$

PROOF We shall prove the continuity and surjectivity of trace operator γ. The corresponding proofs for operators γ_n and γ_t are fully analogous. Let $u \in C^\infty(\overline{\Omega})$. We have,

$$\begin{aligned} \|u\|^2_{H^{s+\frac{1}{2}}(\Gamma)} &= \sum_{j=1}^{J} \|\psi_j u\|^2_{H^{s+\frac{1}{2}}(\Gamma_j)} && \text{(Definition (4.27))} \\ &\lesssim \sum_{j=1}^{J} \|\psi_j u\|^2_{H^{s+1}(\Omega_j)} && \text{(Theorem 4.1.9)} \\ &= \sum_{j=1}^{J} \|\psi_j u\|^2_{H^{s+1}(\Omega)} && \text{(change of coordinates)} \\ &\lesssim \|u\|^2_{H^{s+1}(\Omega)} && \text{(Lemma 3.5.2)}\,. \end{aligned}$$

Not that, in passing from domains Ω_j to domain Ω we change the system of coordinates but the Sobolev spaces and norms are invariant under translation and rotation of coordinates. Finally, we use the density of $C^\infty(\overline{\Omega})$ in $H^{s+1}(\Omega)$, to conclude the existence and continuity of trace operator γ defined on $H^{s+1}(\Omega)$.

To prove the surjectivity of the trace operator, consider and arbitrary $u \in H^{s+\frac{1}{2}}(\Gamma)$. By definition, $\psi_j u \in H^s(\Gamma_j)$ with a support in the shaded subset of G_j illustrated in Fig.4.3. Let $U_j \in H^{s+\frac{1}{2}}(\Omega_j)$ be an extension of $\psi_j u$ to hypograph domain Ω_j. Let χ_j be the indicator function of the shaded subset and χ_j^ϵ its smoothed version with a slightly larger support illustrated with the larger rectangle that is still contained in open set G_j. Truncate extension U_j with χ_j^ϵ and define

$$U := \sum_{j=1}^{J} \chi_j^\epsilon U_j$$

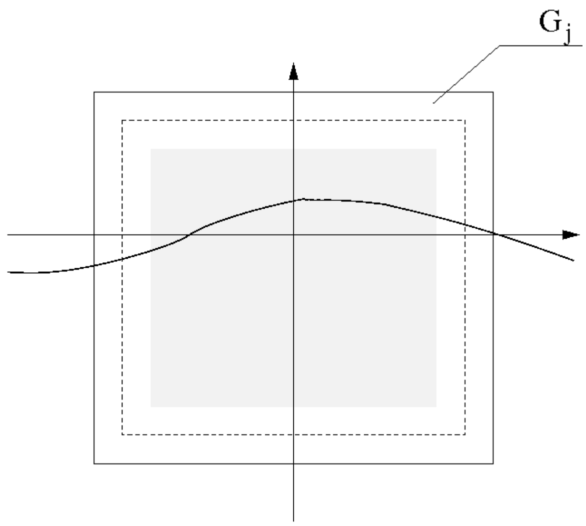

Figure 4.3
Open cover set G_j

We have:

$$\begin{aligned} \|U\|_{H^{s+\frac{1}{2}}(\Omega)} &\lesssim \sum_{j=1}^{J} \|\chi_j^{\epsilon} U_j\|_{H^{s+\frac{1}{2}}(G_j)} && \text{(triangle inequality)} \\ &\lesssim \sum_{j=1}^{J} \|U_j\|_{H^{s+\frac{1}{2}}(\Omega_j)} && \text{(Lemma 3.5.2)} \\ &\lesssim \sum_{j=1}^{J} \|\psi_j u\|_{H^s(\Gamma_j)} && \text{(Extension Theorem for } H^{s+\frac{1}{2}}(\Gamma_j)) \,. \end{aligned}$$

Square both sides to finish the argument. ∎

REMARK 4.1.6 The trace norms depend upon the partition of unity but the corresponding trace spaces do not. Critical in the proof of this fact is the existence of right-inverses of trace operators. Let $\{\psi_j\}$ and $\{\phi_i\}$ be two partitions of unity subordinate to two open covers G_j and H_i. Let $u \in H^{s+\frac{1}{2}}(\Gamma)$ and $U \in H^{s+1}(\Omega)$ be the corresponding extension discussed above. Using the continuity of trace operator defined with the second partition $\{\phi_i\}_{i=1}^{I}$ we have,

$$\sum_{i=1}^{I} \|\phi_i U\|^2_{H^s(\Gamma_i)} \lesssim \|U\|^2_{H^{s+\frac{1}{2}}(\Omega)}$$

and, so, norm (4.27) corresponding to partition $\{\phi_i\}$ is bounded by the norm corresponding to partition $\{\psi_j\}$. Analogous results hold for the normal and tangential traces. ∎

4.1.6 Characterization of functions with zero traces. Relation with spaces $H_0^s(\Omega)$

Recall that

$$H_0^s(\Omega) := \overline{C_0^\infty(\Omega)}^{H^s(\Omega)} .$$

The test functions vanish on the boundary. Given the trace theorem, should we expect, for the range $s \in (\frac{1}{2}, \frac{3}{2})$, the closure to coincide with functions that also vanish (in the sense of traces) on the boundary ? The answer is "yes".

THEOREM 4.1.11

Let Ω be a $C^{k-1,1}$ domain, and $s \in (\frac{1}{2}, k]$. We have:

$$H_0^s(\Omega) = \{u \in H^s(\Omega) \,:\, \gamma(\partial^\alpha u) = 0 \quad \text{on } \Gamma, \quad \forall\, |\alpha| < s - \frac{1}{2}\} .$$

PROOF Inclusion $\subset$ is a direct consequence of definition of $H_0^s(\Omega)$ and continuity of trace operator. We will prove the inverse inclusion for the half space $\Omega = \mathbb{R}^n_-$ and leave the rest of the proof for Exercise 4.1.4. Consider the closed subspace of $H^s(\mathbb{R}^n_-)$:

$$V := \{u \in H^s(\mathbb{R}^n_-) \,:\, \gamma(\partial^\alpha u) = 0 \quad \forall\, |\alpha| < s - \frac{1}{2}\} .$$

If $H_0^s(\mathbb{R}^n_-)$ were only a proper (closed) subspace of V then, by Mazur's Theorem or Orthogonal Decomposition Theorem, there would exist non-zero functionals $l \in V'$ that vanish on $H_0^s(\mathbb{R}^n_-)$. We will show that this is impossible. Towards this end, let $l \in V'$ such that $l(\phi) = 0 \quad \forall\, \phi \in H_0^s(\mathbb{R}^n_-)$. By Hahn-Banach Theorem, functional l admits an extension

$$L \in (H^s(\mathbb{R}^n_-))' = \tilde{H}^{-s}_\Omega(\mathbb{R}^n) = H^{-s}_{\overline{\Omega}}(\mathbb{R}^n) ,$$

As $L = l$ on V and vanishes on $\mathcal{D}(\mathbb{R}^n_-)$, the support of L is contained in $F := \mathbb{R}^{n-1}$. By Theorem 3.5.1, there exist $v_j \in H^{-s+j+\frac{1}{2}}(\mathbb{R}^{n-1})$ such that

$$L = \sum_{0 \le j \le s - \frac{1}{2}} v_j \otimes \delta^{(j)} .$$

But this implies that, for all $\phi \in V$,

$$l(\phi) = L(\phi) = \sum_{0 \le j < s - \frac{1}{2}} (-1)^j \langle v_j, \gamma(\partial_n^j \phi)\rangle = 0 .$$

Thus l vanishes on the entire V. ∎

REMARK 4.1.7 As usual, for a domain with piecewise smooth $k-1$ derivatives, the result extends to $s \in (\frac{1}{2}, k + \frac{1}{2})$. In particular, for a piecewise smooth C^0 domain,

$$H_0^s(\Omega) = \{u \in H^s(\Omega) \,:\, \gamma u = 0 \quad \text{on } \Gamma\} ,$$

for all $s \in (\frac{1}{2}, \frac{3}{2})$. In other words, test functions from a dense subset in the subspace of functions with zero trace. $\blacksquare$

The result discussed above generalizes to the remaining energy spaces.

THEOREM 4.1.12

Let Ω be a polyhedral domain, and $s \in (-\frac{1}{2}, \frac{1}{2})$. The following density results hold:

$$\{u \in H^s(\text{grad}, \Omega) \,:\, \gamma u = 0 \quad \text{on } \Gamma\} = \overline{C_0^\infty(\Omega)}^{H^s(\text{grad},\Omega)}$$

$$\{v \in H^s(\text{div}, \Omega) \,:\, \gamma_n u = 0 \quad \text{on } \Gamma\} = \overline{(C_0^\infty(\Omega))^n}^{H^s(\text{div},\Omega)}$$

$$\{v \in H^s(\text{curl}, \Omega) \,:\, \gamma_t u = 0 \quad \text{on } \Gamma\} = \overline{(C_0^\infty(\Omega))^3}^{H^s(\text{curl},\Omega)}$$

PROOF The first result has already been proved but we present here an alternate proof that applies to the remaining cases as well. We will consider again only the half-space case, $\Omega = \mathbb{R}^n_-$, leaving the rest of the proof for an exercise. We shall present the $H(\text{div})$ case, the proof of the other two cases is fully analogous. Consider a function $v \in H(\text{div}, \mathbb{R}^n_-)$ with zero normal trace $\gamma_n v = 0$, and let $\varepsilon > 0$ be an arbitrary number. By the density result from Theorem 4.1.1, for any ε_1, there exists a function $\varphi \in (C_0^\infty(\overline{\mathbb{R}^n_-}))^n$ such that

$$\|v - \varphi\|_{H^s(\text{div},\mathbb{R}^n_-)} < \varepsilon_1 \,.$$

Note that function φ may not have the zero trace. However, we can use results on traces and extension operators to modify it to enforce the zero trace condition. We define,

$$\phi = \varphi - E^{\text{div}} \gamma_n \varphi$$

where E^{div} is the extension operator discussed in Theorem 4.1.5. We have

$$\|u - \phi\|_{H^s(\text{div},\mathbb{R}^n_-)} = \|(I - E^{\text{div}}\gamma_n)(u - \varphi)\|_{H^s(\text{div},\mathbb{R}^n_-)} \le \|I - E^{\text{div}}\gamma_n\| \, \|u - \varphi\|_{H^s(\text{div},\mathbb{R}^n_-)} \,.$$

Selecting $\varepsilon_1 = \varepsilon/4\|I - E^{\text{div}}\gamma_n\|$, we can bound the term above by $\varepsilon/4$. Definition of extension operator[§] implies that ϕ is a C^∞ function but not necessarily with a compact support, though.

Next we shift function ϕ downward and extend it to the whole space by zero.

$$\phi_\delta(x', x_n) := \begin{cases} \phi(x', x_n + \delta) & x_n < -\delta \\ 0 & x_n > -\delta \,. \end{cases}$$

The zero trace condition and the localization argument used in Section 4.1.4 imply that, for the range of $s \in (-\frac{1}{2}, \frac{1}{2})$, function $\phi_\delta \in H^s(\text{div}, \mathbb{R}^n)$. For sufficiently small shift δ,

$$\|\phi - \phi_\delta\|_{H^s(\text{div},\mathbb{R}^n_-)} < \frac{\varepsilon}{4} \,.$$

[§]Solution of the Neumann problem in half space with C^∞ data on the hyperplane is a C^∞ function.

Note that ϕ_δ, in general, is not a C^∞ function. It may also not have a compact support. We first take care of the support. Truncating function ϕ_δ with the standard smooth approximation χ_R^ϵ (different ϵ!) of indicator function χ_R of ball $\bar{B}(0,R)$ (comp. proof of Theorem 4.1.1), we replace ϕ_δ with $\phi_{\delta R}$ that is still in space $H^s(\text{div}, \mathbb{R}^n)$ but now has a compact support contained in half-space $\mathbb{R}^n_-$ and, for sufficiently large radius R, it remains withing the $\varepsilon/4$ distance from function ϕ_δ.

Finally, we convolute function $\phi_{\delta R}$ with function ψ_ϵ used in Theorem 2.3.2 (yet another ϵ, sorry...). Convolution $\psi_\epsilon * \phi_{\delta R}$ is a C^∞ function and, for sufficiently small ϵ,

$$\text{supp}\, \psi_\epsilon * \phi_{\delta R} \subset \mathbb{R}^n_- \quad \text{and} \quad \|\phi_{\delta R} - \psi_\epsilon * \phi_{\delta R}\|_{H^s(\mathbb{R}^n)} < \frac{\varepsilon}{4}\,.$$

We conclude that

$$\|v - \psi_\epsilon * \phi_{\delta R}\|_{H^s(\mathbb{R}^n_-)} \le \|v-\phi\|_{H^s(\mathbb{R}^n_-)} + \|\phi-\phi_\delta\|_{H^s(\mathbb{R}^n_-)} + \|\phi_\delta - \phi_{\delta R}\|_{H^s(\mathbb{R}^n_-)} + \|\phi_{\delta R} - \psi_\epsilon * \phi_{\delta R}\|_{H^s(\mathbb{R}^n)_-} < \varepsilon\,.$$

∎

The proof seems to be perhaps somehow technical but it is quite elementary, and it recycles the already familiar arguments. The key point of the reasoning above is that, for the range $s \in (-\frac{1}{2}, \frac{1}{2})$, functions with zero trace can be extended by zero to a finite energy function defined on the whole space.

Exercises

Exercise 4.1.1 Prove equality (4.2) for $\Omega = \mathbb{R}^n$.

Exercise 4.1.2 Construct a functions u such that $\boldsymbol{\nabla} u \in (H^s(\mathbb{R}^3))^3$ but $u \notin H^s(\mathbb{R}^3)$. *Hint:* Work in the frequency domain.

Exercise 4.1.3 Show that if E belongs to energy space (4.16) and its Fourier transform is continuous, then $\hat{E}_i(0) = 0$.

Exercise 4.1.4 Finish proof of Theorem 4.1.11, first for the case of a $C^{k-1,k}$ hypograph domain, and then for a general $C^{k-1,k}$ domain.

4.2 Minimum Energy Extensions and Rotated Trace for Space $H(\textbf{curl}, \Omega)$

As we have shown, the norms used for the trace spaces depend upon the open cover of the boundary and the corresponding subordinate partition of unity functions. However, the corresponding trace spaces are unique. It is no surprise perhaps then that these norms are of little practical importance. The actual, physically meaningful norms are given by the minimum energy extensions:

$$\begin{aligned}
\|u\|^2_{H^{s+\frac{1}{2}}(\Gamma)} &= \min\{\|U\|^2_{H^{s+1}(\Omega)} \,:\, U \in H^{s+1}(\Omega),\, \gamma U = u\} \\
\|v\|^2_{H^{s-\frac{1}{2}}(\Gamma)} &= \min\{\|V\|^2_{H^{s}(\mathrm{div},\Omega)} \,:\, V \in H^{s}(\mathrm{div},\Omega),\, \gamma_n V = v\} \\
\|e\|^2_{H^{s-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)} &= \min\{\|E\|^2_{H^{s}(\mathrm{curl},\Omega)} \,:\, E \in H^{s}(\mathrm{curl},\Omega),\, \gamma_t E = e\}
\end{aligned} \tag{4.31}$$

The continuity and surjectivity of trace operators implies that the minimum energy extension norms are equivalent to trace norms defined with partition of unity functions in Section 4.1 (Exercise 4.2.2).

Recall that, for a hypograph domain, trace spaces $H^s(\Gamma)$ and $H^{-s}(\Gamma)$ form a duality pairing. It turns out that this duality pairing is preserved for a general domain when we use the minimum energy extension norms but only for $s = 0$. The first hint comes from the integration by parts formula:

$$(\mathrm{div}\, v, u) = -(v, \boldsymbol{\nabla} u) + \langle v \cdot n, u\rangle \,.$$

This gives us the representation of the boundary term in terms of the domain integrals:

$$\langle v \cdot n, u\rangle = (\mathrm{div}\, v, u) + (v, \boldsymbol{\nabla} u)\,.$$

Cauchy-Schwarz inequality implies[¶] then,

$$|\langle v \cdot n, u\rangle| \leq \|v\|_{H^{-s}(\mathrm{div},\Omega)}\, \|u\|_{H^{s}(\mathrm{grad},\Omega)}\,.$$

Taking minimum energy extensions on the right-hand side, we get,

$$|\langle v \cdot n, u\rangle| \leq \|v \cdot n\|_{H^{-s}(\Gamma)}\, \|u\|_{H^{s}(\Gamma)}$$

where the norms on the right-hand side are now the minimum energy extension norms. The inequality implies that

$$\|v \cdot n\|_{(H^{s-\frac{1}{2}}(\Gamma))'} \leq \|v \cdot n\|_{H^{-s-\frac{1}{2}}(\Gamma)} \quad \text{and} \quad \|u\|_{(H^{s+\frac{1}{2}}(\Gamma))'} \leq \|u\|_{H^{-s+\frac{1}{2}}(\Gamma)}\,. \tag{4.32}$$

The reverse inequalities hold only with multiplicative constants, Due to the equivalence of the minimum energy extension norms and trace norms defined with partition of unity functions, it is sufficient to prove it for the latter norms. Critical is the following property of Hilbert spaces.

LEMMA 4.2.1

Let V be a Hilbert space and

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_n$$

be an orthogonal decomposition of V, i.e.,

$$\|v\|^2 = \sum_{j=1}^{n} \|v_j\|^2 \quad \text{where } v = \sum_{j=1}^{n} v_j,\, v_j \in V_j,\, j = 1, \ldots, n\,.$$

[¶]Provided we equip $\tilde{H}^{-s}(\Omega) = H^{-s}(\Omega)$, for $s > 0$, with the dual norm to H^s-norm.

Then, for any $l \in V'$,

$$\|l\|_{V'}^2 = \sup_{v\in V} \frac{|l(v)|^2}{\|v\|^2} = \sum_{j=1}^{n} \sup_{v_j \in V_j} \frac{|l(v_j)|^2}{\|v_j\|^2} \,.$$

PROOF See Exercise 4.2.3. ∎

Let $v_n = v \cdot n$ denote the normal trace. We have,

$$\begin{aligned}
\sup_{u\in H^{s+\frac12}(\Gamma)} \frac{|\langle v_n, u\rangle|^2}{\|u\|^2_{H^{s+\frac12}(\Gamma)}} &= \sup_{u\in H^{s+\frac12}(\Gamma)} \frac{|\langle v_n, \sum_{j=1}^J \psi_j u\rangle|^2}{\sum_{j=1}^J \|\psi_j u\|^2_{H^{s+\frac12}(\Gamma_j)}} \\
&= \sum_{j=1}^J \sup_{u\in H^{s+\frac12}(\Gamma)} \frac{|\langle v_n, \psi_j u\rangle|^2}{\|\psi_j u\|^2_{H^{s+\frac12}(\Gamma_j)}} \qquad \text{(Lemma 4.2.1)} \\
&= \sum_{j=1}^J \sup_{u\in H^{s+\frac12}(\Gamma)} \frac{|\langle \psi_j v_n, u\rangle|^2}{\|\psi_j u\|^2_{H^{s+\frac12}(\Gamma_j)}} \\
&\geq \frac{1}{C^2} \sum_{j=1}^J \sup_{u\in H^{s+\frac12}(\Gamma)} \frac{|\langle \psi_j v_n, u\rangle|^2}{\|u\|^2_{H^{s+\frac12}(\Gamma_j)}} \quad (\|\psi_j u\|_{H^{s+\frac12}(\Gamma_j)} \leq C\|u\|_{H^{s+\frac12}(\Gamma_j)}) \\
&= \frac{1}{C^2} \sum_{j=1}^J \|\psi_j v_n\|^2_{H^{-s-\frac12}(\Gamma_j)} \qquad \text{(duality of norms on the hypograph boundary)} \\
&= \frac{1}{C^2} \|v_n\|^2_{H^{-s-\frac12}(\Gamma)}
\end{aligned}$$

We can conclude thus that, modulo a multiplicative constant, the minimum energy extension norm is bounded by the dual norm. Same result holds for trace $u \in H^{s+\frac12}(\Gamma)$.

For $s = 0$, however, the two minimum energy extension norms are dual to each other. Indeed, consider the dual norm to the minimum energy extension norm in $H^{\frac12}(\Gamma)$,

$$|\!|\!| v_n |\!|\!|_{(H^{\frac12}(\Gamma))'} := \sup_{u\in H^{\frac12}(\Gamma)} \frac{|\langle v_n, u\rangle|}{|\!|\!| u |\!|\!|_{H^{\frac12}(\Gamma)}} = \sup_{U\in H(\text{grad},\Omega)} \frac{|\langle v_n, U\rangle|}{\|U\|_{H(\text{grad},\Omega)}}$$

where the second equality inequality is an immediate consequence of the minimum energy extension norm for trace space $H^{\frac12}(\Gamma)$. Riesz Representation Theorem implies that the dual norm is equal to the $H(\text{grad}, \Omega)$ norm of the solution U of the variational problem:

$$(\boldsymbol{\nabla} U, \boldsymbol{\nabla} \Phi) + (U, \Phi) = \langle v \cdot n, \Phi\rangle \qquad \Phi \in H(\text{grad}, \Omega)$$

or, equivalently, the Neumann problem:

$$\begin{cases} -\text{div}(\boldsymbol{\nabla} U) + U = 0 & \text{in } \Omega \\ \frac{\partial U}{\partial n} = \boldsymbol{\nabla} U \cdot n = v_n & \text{on } \Gamma \,. \end{cases}$$

Taking gradient of the first equation, we realize that $V = \boldsymbol{\nabla} U$ satisfies the boundary-value problem (BVP):

$$\begin{cases} -\boldsymbol{\nabla}(\text{div}\, V) + V = 0 & \text{in } \Omega \\ V \cdot n = v_n & \text{on } \Gamma \,. \end{cases}$$

Vice-versa, by taking divergence of the grad-div equation for V, we realize that $U = \text{div}\, V$ satisfies the first BVP. The two problems are thus equivalent to each other and $\|U\|_{H(\text{grad},\Omega)} = \|V\|_{H(\text{div},\Omega)}$. The point of the story is that the dual norm leads to a Neumann BVP for Riesz Representation function U which in turn is equivalent to the Dirichlet BVP for V. Clearly, *V is the minimum energy extension of v_n* and $\|V\|_{H(\text{div},\Omega)}$ equals the dual norm $\|v_n\|_{(H^{\frac{1}{2}}(\Gamma))'}$. Note that we had already exploited this relation between functions U and V in our construction of the extension operator for trace space $H^{-s}(\mathbb{R}^{n-1})$.

REMARK 4.2.1 An interpolation argument implies that reverse inequalities to (4.32) hold with multiplicative constants converging to one, as $s \to 0$, see Exercise 4.2.4. ▮

Now comes a big point. If operators grad and div, and the corresponding traces and trace spaces are in duality,

where is a duality cousin of operator curl, energy space $H^s(\text{curl}, \Omega)$, and trace space $H^{s-\frac{1}{2}}(\text{curl}_\Gamma, \Gamma)$?

The answer comes again from the integration by parts formula. Let $s \in (-\frac{1}{2}, \frac{1}{2})$ and $E_t := \gamma_t E$, $E \in H^{-s}(\text{curl}, \Omega)$. Integration by parts formula:

$$\langle n \times E_t, F_t \rangle = (\boldsymbol{\nabla} \times E, F) - (E, \boldsymbol{\nabla} \times F) \tag{4.33}$$

identifies $n \times E_t$ as a linear and continuous functional defined on $H^{s-\frac{1}{2}}(\text{curl}_\Gamma, \Gamma)$, comp. Exercise 4.2.5. For $s = 0$, the dual norm of $n \times E_t$,

$$\begin{aligned}\|n \times E_t\|_{(H^{-1/2}(\text{curl}_\Gamma,\Gamma))'} &= \sup_{F_t \in H^{-1/2}(\text{curl}_\Gamma,\Gamma)} \frac{|\langle n \times E_t, F_t \rangle|}{\|F_t\|_{H^{-1/2}(\text{curl}_\Gamma,\Gamma)}} \\ &= \sup_{F \in H(\text{curl},\Omega)} \frac{|\langle n \times E_t, F_t \rangle|}{\|F_t\|_{H(\text{curl},\Omega)}}\end{aligned}$$

equals the $H(\text{curl}, \Omega)$ norm of solution F of the Neumann problem (Riesz operator argument):

$$\begin{cases} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times F) + F = 0 \\ \qquad n \times (\boldsymbol{\nabla} \times F) = n \times E_t \,. \end{cases} \tag{4.34}$$

Function $H = \boldsymbol{\nabla} \times F$ solves the Dirichlet problem:

$$\begin{cases} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times H) + H = 0 \\ \qquad\qquad n \times H_t = n \times E_t \,. \end{cases}$$

and $\|H\|_{H(\text{curl},\Omega)} = \|E\|_{H(\text{curl},\Omega)}$. As $n \times H_t = n \times E_t$ is equivalent to $H_t = E_t$, the rotation $n\times$ is an isometry from $H^{-\frac{1}{2}}(\text{curl}_\Gamma, \Gamma)$ into its dual. To see that rotation $n\times$ is actually a surjection, consider an arbitrary $\psi \in (H^{-\frac{1}{2}}(\text{curl}_\Gamma, \Gamma))'$ and the corresponding Neumann boundary-value problem (4.34) with ψ in place of $n \times E_t$. Take then $E = \boldsymbol{\nabla} \times H$ where H is the solution of the Neumann problem to get $n \times E_t = \psi$. By the same arguments as for the grad-div pair, the rotation is no longer an isometry for $s \neq 0$, but it is continuous with norm converging to one as $s \to 0$.

The dual space $(H^{s-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma))'$ is denoted with symbol $H^{-s-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$. Indeed, in the case of half space $\Omega=\mathbb{R}^3_-$,

$$\mathrm{div}_\Gamma(n\times E) = -\mathrm{curl}_\Gamma E\,.$$

The surface div_Γ operator can be extended to a smooth boundary with the $H^{-s-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$ defined independently and only a-posteriori identified with the dual of trace space $H^{s-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma))$. The point of this presentation has been to show that there is no need for a separate construction of such a space as its elements can be simply identified with the rotated traces of space $H^{s-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma))$. This applies to the discretization of rotated traces as well.

Exercises

Exercise 4.2.1 Justify the use of minimum in the definition of minimum energy extension norms (4.31).

Exercise 4.2.2 Show that minimum energy extension norms (4.31) are equivalent to the norms defined with partition of unity functions in Section 4.1. *Hint:* Use the fact that both trace operator and its right inverse are continuous.

Exercise 4.2.3 Prove Lemma 4.2.1.

Exercise 4.2.4 Use the real interpolation theory and results discussed in this section to show that there exist constants $C_1, C_2 \geq 1$ such that

$$\|v_n\|_{H^{s-\frac{1}{2}}(\Gamma)} \leq C_1^s \|v_n\|_{(H^{-s+\frac{1}{2}}(\Gamma))'} \quad \text{and} \quad \|u\|_{H^{s+\frac{1}{2}}(\Gamma)} \leq C_2^s \|u\|_{(H^{-s-\frac{1}{2}}(\Gamma))'}\,.$$

for any $s\in[-s_0,s_0]$, $s_0<\frac{1}{2}$.

Exercise 4.2.5 Let $\phi\in H^{s-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)$ and $\Phi\in H^s(\mathrm{curl},\Omega)$ be an extension of ϕ. Let $E\in H^{-s}(\mathrm{curl},\Omega)$. Consider (4.33):

$$\langle n\times E_t,\phi\rangle = (\boldsymbol{\nabla}\times E,\Phi) - (E,\boldsymbol{\nabla}\times\Phi)$$

with $\gamma_t\Phi=\phi$. Demonstrate that the right-hand side is invariant under the change of extension and, therefore, defines a linear functional defined on $H^{s-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)$. Show that the functional is continuous. *Hint:* You will need Theorem 4.1.12.

5

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