CSE386M/EM386M
FUNCTIONAL ANALYSIS IN THEORETICAL MECHANICS
Fall 2018, Final Exam, 2:00-5:00 PM, Wed, Dec 19, ACES 6.304

1. A linear algebra "sanity check". Consider $\mathbb{R}^2$ with the inner product:

$$(x, y) = x_1 y_1 + 2x_2 y_2, \quad x = (x_1, x_2),\, y = (y_1, y_2)\,. \tag{0.1}$$

- Recall definition of an inner product and (quickly) check that function (0.1) indeed satisfies the necessary properties.
- Is $\mathbb{R}^2$ with the inner product a Hilbert space ? Explain, why?
- Consider vectors $e_1 = (1, 0), e_2 = (1, 1)$ and prove that they provide a basis for $\mathbb{R}^2$.
- Determine the corresponding dual basis $e_1^*, e_2^* \in (\mathbb{R}^2)^*$.
- Define the Riesz operator R corresponding to the inner product and find its matrix representation in the bases e_i, e_j^*.
- Determine the transpose operator to R and its matrix representation with respect to the same bases. Discuss the result.

(20 points).

Answers:

- The form is bilinear, symmetric and positive definite and, therefore, it can be identified as an inner product.
- Yes, it is. Space $\mathbb{R}^n$ is complete with respect to the Euclidean metric and, due to the equivalence of any two norms, it is also complete with respect to the norm implied by our inner product.
- The two vectors are not collinear so they are linearly independent.
- Let $x \in \mathbb{R}^2$, We have:

$$x = (x_1, x_2) = (x_1 - x_2)(1, 0) + x_2(1, 1) = (x_1 - x_2)e_1 + x_2 e_2$$

  The dual basis is thus:

$$e_1^*(x) = x_1 - x_2, \quad e_2^*(x) = x_2\,.$$

- The Riesz operator R corresponding to a particular inner product (u,v) sets vector u into the linear functional $(u,\cdot)$. More precisely,

$$\langle Ru, v\rangle = (u,v)$$

Riesz operator is injective and, in the finite dimensional setting, automatically surjective as the dual space is of the same dimension as the original space. In order to determine the matrix representation of R, we consider vectors Re_j,

$$(Re_1)(y) = (e_1, y) = y_1 = (y_1 - y_2) + y_2 = e_1^*(y) + e_2^*(y)$$
$$(Re_2)(y) = (e_2, y) = y_1 + 2y_2 = (y_1 - y_2) + 3y_2 = e_1^*(y) + 3e_2^*(y)$$

The matrix representation of operator R is thus:

$$\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}.$$

- The transpose R^T goes from the bidual to the dual space. In the finite dimensional case (in fact, for any Hilbert space), the bidual is identified (canonically isomorphic) with the original space. Due the symmetry of the inner product, transpose R^T coincides with R. Indeed,

$$\langle Ru, v\rangle_{V'\times V} = (u,v)_V = (v,u)_V = \langle Rv, u\rangle_{V'\times V} = \langle u, Rv\rangle_{V''\times V'}.$$

Consequently, matrix representation of transpose R^T coincides with that of R.

2. An integration exercise.

   (a) State the Lebesgue Dominated Convergence Theorem (5 points).

   (b) Let $\gamma > 0$ be a positive constant. Prove that the integral

$$\int_{\pi/2}^{3\pi/2} \frac{e^{\gamma + n\cos\theta}}{\sqrt{(\gamma + n\cos\theta)^2 + (n\sin\theta)^2}}\, n d\theta$$

   converges to zero as $n \to \infty$ (15 points).

Answers:

   (a) See the book.

   (b) Rewrite the integral in the form,

$$\int_{\pi/2}^{3\pi/2} \frac{e^{\gamma + n\cos\theta}}{\sqrt{(\gamma/n + \cos\theta)^2 + (\sin\theta)^2}}\, d\theta$$

   For $\theta \in (\pi/2, 3\pi/2)$, the denominator converges to one, whereas the numerator converges to zero (exponential with a negative exponent), as $n \to \infty$. Consequently the integrand converges a.e. to zero. In order to apply the Lebesgue Dominated Convergence Theorem, we need to show only that the integrand is dominated by an integrable function, for all n. The numerator is bounded by e^{γ}. For the denominator, we have

$$\begin{aligned} \left(\frac{\gamma}{n} + \cos\theta\right)^2 + \sin^2\theta &= \frac{\gamma^2}{n^2} + \frac{2\gamma}{n}\cos\theta + 1 \\ &\geq \frac{\gamma^2}{n^2} - \frac{2\gamma}{n} + 1 \\ &= \left(\frac{\gamma}{n} - 1\right)^2 \end{aligned}$$

   Thus, for sufficiently large n, the denominator is bounded below by a positive number (independent of angle θ).

3. A topology problem. Let $f : X \to Y$ where X and Y are arbitrary topological spaces. Prove that f is continuous iff $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for every $B \subset Y$ (20 points).

   Recall that f is continuous iff the inverse image of every closed set is closed. Assume that f is continuous and pick an arbitrary set $B \subset Y$. The closure $\overline{B}$ is closed, so the inverse image $f^{-1}(\overline{B})$ must be closed. It also contains $f^{-1}(B)$. Since the closure of a set is the smallest closed set including the set, we have

$$\overline{f^{-1}(B)} \subset f^{-1}(\overline{B}) \text{ (closed)}$$

   Conversely, assume that the condition is satisfied. Pick any closed set $B = \overline{B} \subset Y$. Then

$$\overline{f^{-1}(B)} \subset f^{-1}(\overline{B}) = f^{-1}(B)$$

   which implies that set $f^{-1}(B)$, being equal to its closure, is closed.

4. A metric space problem. Let X be a set and $\rho_1(x,y), \rho_2(x,y)$ two metrics on X. Define:

$$d(x,y) := \max\{\rho_1(x,y), \rho_2(x,y)\}\,. \tag{0.2}$$

   - Is d also a metric on X ? Prove or disprove.
   - If the answer to the first question is positive, you have three topologies in X corresponding to the three metrics. Discuss the relative strength of the corresponding topologies (which one is stronger or weaker than others ?). *Hint:* Recall the definition of bases of neighborhoods in a metric space.

(20 points).

Answers:

- Yes, it is.

  Positive definitness: If $d(x,y) = 0$ then both $\rho_1(x,y) = \rho_2(x,y) = 0$ which implies that $x = y$.

  Symmetry: We have:

  $$\rho_i(x,y) = \rho_i(y,x), \quad i = 1,2\,.$$

  Apply $\max_{i=1,2}$ to both sides.

  Triangle inequality: Start with:

  $$\rho_i(x,y) \le \rho_i(x,z) + \rho_i(z,y) \le \max_{j=1,2}\rho_j(x,z) + \max_{j=1,2}\rho_j(z,y), \quad j = 1,2\,,$$

  and take maximum with respect to i on both sides.

- Let $B^d(x,\epsilon)$ and $B^{\rho_i}(x,\epsilon)$ denote balls corresponding to metrics d and ρ_i, resp. Inequality

  $$\rho_i(x,y) \le d(x,y), \quad i = 1,2$$

  implies that

  $$B^d(x,\epsilon) \subset B^{\rho_i}(x,\epsilon), \quad i = 1,2\,.$$

  Consequently, if $\mathcal{B}^d, \mathcal{B}^{\rho_i}$ denote the bases of neighborhoods in topologies generated by d and ρ_i, resp., then

  $$\mathcal{B}^{\rho_i} \succ \mathcal{B}^d$$

  which demonstrates that metric topogy corresponding to d is *stronger* than both topologies corresponding to metrics ρ_i. We cannot draw any general conclusion about the relative strength of metric topologies corresponding to ρ_i, $i = 1,2$.

5. Contraction Maps. Consider the following Initial-Value Problem (IVP):

$$\begin{cases} \dfrac{dq}{dt} = t\ln(q(t)), \quad t > 0 \\ q(0) = 1 \end{cases}$$

- State Banach Contractive Map Theorem (3 points).

  Answer:
  Let (X, d) be a complete metric space. Let $D \subset X$ (then (D, d) is itself a metric space, too...), and $A : D \to D$ is a contraction, i.e.

  $$d(A(f), A(g)) \leq k\, d(f, g), \quad \forall f, g \in D, \quad k < 1$$

  Then function A has a unique fixed point in set D.

- Use the theorem to prove local existence and uniqueness of solution to the IVP, i.e. that there exists an interval $(0, T)$ in which the equation is satisfied. Provide a *concrete* value of T (17 points).

  Solution:
  The problem is equivalent to the solution of the integral equation:

  $$q(t) = 1 + \int_0^t s\ln(q(s))\, ds$$

  Consider the Chebyshev space $C[0, T]$ (with unknown T at this point...) and define the map A using the right-hand side of the equation above:

  $$(Aq)(t) = 1 + \int_0^t s\ln(q(s))\, ds$$

  First of all, we need to define a set $D \subset C[0, T]$ such that map A sets the set D into itself. Assume that $q(t)$ will vary in the box:

  $$D = \{q \in C[0, T] \,:\, e^{-1} \leq q(t) \leq e, \quad 0 \leq t \leq T\} \tag{0.3}$$

  (notice that the box includes the initial value $q = 1$). Then $-1 \leq \ln q(t) \leq 1$, i.e. $|\ln q(t)| \leq 1$. Consequently,

  $$|\int_0^t s\ln q(s)\, ds| \leq \int_0^t s\, ds = \frac{1}{2}t^2$$

  so,

  $$|(Aq)(t) - 1| \leq \frac{1}{2}T^2$$

This gives two bounds for T. From the right:

$$(Aq)(t) \leq 1 + \frac{1}{2}T^2 \leq e \quad \Rightarrow \quad T \leq \sqrt{2(e-1)}\,,$$

and from the left:

$$e^{-1} \leq 1 - \frac{1}{2}T^2 \leq (Aq)(t) \quad \Rightarrow \quad T \leq \sqrt{2(1-e^{-1})}\,.$$

Now, map A must be a contraction. With flux $F(s,q) = s \ln q$,

$$|\frac{\partial F}{\partial q}| = s|\frac{1}{q}| \leq es$$

so, with q coming form box (0.3), the flux satisfies the Lipschitz condition:

$$|F(s,q_1) - F(s,q_2)| \leq es|q_s - q_2|\,.$$

This leads to the estimate;

$$|(Aq_1)(t) - (Aq_2)(t)| \leq \int_0^t es\,ds\,\|q_1 - q_2\|_{C[0,T]} \leq \frac{e}{2}T^2\,\|q_1 - q_2\|_{C[0,T]}\,.$$

Consequently, a sufficient condition for a contraction is

$$T < \sqrt{\frac{2}{e}}\,.$$

In conlusion, the IVP will have a unique solution for

$$T < \min\{\sqrt{2(e-1)}, \sqrt{2(1-e^{-1})}, \sqrt{\frac{2}{e}}\}\,.$$