CSE386M/EM386M
FUNCTIONAL ANALYSIS IN THEORETICAL MECHANICS
Fall 2018, Exam 3

1. Define the following notions and provide a non-trivial example (2+2 points each).

   - Lebesgue measurable sets.
   - Essential supremum.
   - Stronger and weaker topologies.
   - Topological subspace.
   - Compact topological space.

   See the book.

2. State and prove *three* out of the following four theorems (10 points each).

   - Properties of Borel sets (Prop. 3.1.4, 3.1.5 combined)
   - Fatou's Lemma
   - Properties of compact sets.
   - The Heine-Borel Theorem in $\mathbb{R}^n$ (you may use the 1D version w/o proof).

   See the book.

3. Prove the Generalized Hölder Inequality:

$$|\int uvw| \leq \|u\|_p \, \|v\|_q \, \|w\|_r$$

   where $1 < p, q, r < \infty, 1/p + 1/q + 1/r = 1$ (10 points).

   Solution:
   In view of estimate:

$$|\int uvw| \leq \int |uvw| = \int |u| \, |v| \, |w|$$

   we can restrict ourselves to nonnegative, real-valued functions only. The inequality follows then from the original Hölder's result,

$$\int uvw \leq \left(\int u^p\right)^{1/p} \left(\int (v\,w)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$$

If $1/q + 1/r = 1 - 1/p = (p-1)/p$, then for $q' = q(p-1)/p, r' = r(p-1)/p$, we have $1/q' + 1/r' = 1$. Consequently,

$$\int v^{\frac{p}{p-1}} w^{\frac{p}{p-1}} \leq \left(\int (v^{\frac{p}{p-1}})^{q'} \right)^{\frac{1}{q'}} \left(\int (w^{\frac{p}{p-1}})^{r'} \right)^{\frac{1}{r'}} = \left(\int v^q \right)^{\frac{p}{q(p-1)}} \left(\int w^r \right)^{\frac{p}{r(p-1)}}$$

Combining the two inequalities, we get the final result.

4. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two topologies on a set X and let $I\colon (X, \mathcal{X}_1) \to (X, \mathcal{X}_2)$ be the identity function. Show that I is continuous if and only if $\mathcal{X}_1$ is stronger than $\mathcal{X}_2$; i.e., $\mathcal{X}_2 \subset \mathcal{X}_1$. (10 points)

   Solution:
   I is continuous iff the inverse image of every open set is open. But $I^{-1}(A) = A$, so this is equivalent to $\mathcal{X}_2 \subset \mathcal{X}_1$.

5. An integration exercise.

   - State the Lebesgue Dominated Convergence Theorem.
   - Let $\Omega \subset \mathbb{R}^n$ be an arbitrary *unbounded* open set, and let $f \in L^1(\Omega)$. Prove that

     $$\int_{\Omega - B(0,n)} f(x)\,dx \to 0 \quad \text{as } n \to \infty\,.$$

     where $B(0,n)$ denotes the ball centered at 0 with radius n.

   (10 points)

   Solution:

   - See the book.
   - Consider

     $$f_n(x) := \begin{cases} 0 & x \in \Omega \cap \bar{B}(0,n) \\ f(x) & x \in \Omega - B(0,n) \end{cases}$$

     Obviously, $f_n(x) \to 0$ as $n \to \infty$, and $|f_n(x)| \leq |f(x)|$, so $|f(x)|$ provides a dominating function. By the Lebesque Theorem,

     $$\int_{\Omega - B(0,n)} f(x)\,dx = \int_\Omega f_n(x)\,dx \to 0\,.$$

6. We say that a topology has been introduced in a set X through the *operation of closure*, if we have introduced operation (of taking closure)

   $$\mathcal{P}(X) \ni A \to \mathrm{cl}A \in \mathcal{P}(X) \quad \text{with} \quad A \subset \mathrm{cl}A$$

   that satisfies the following four properties:

(i) $\mathrm{cl}\emptyset = \emptyset$

(ii) $A \subset B$ implies $\mathrm{cl}A \subset \mathrm{cl}B$

(iii) $\mathrm{cl}(\mathrm{cl}A) = \mathrm{cl}A$

(iv) $\mathrm{cl}(A \cup B) = \mathrm{cl}A \cup \mathrm{cl}B$

Sets F such that $\mathrm{cl}F = F$ are identified then as *closed sets*.

(a) Prove that the closed sets defined in this way, satisfy the usual properties of closed sets (empty set and the whole space are closed, intersections of arbitrary families, and unions of finite families of closed sets are closed) (7 points).

(b) Define *open sets* $\mathcal{X}$ by taking complements of *closed sets*. Notice that the duality argument implies that family $\mathcal{X}$ satisfies the axioms for the open sets. Use then family $\mathcal{X}$ to introduce a topology (through open sets) in X. Consider next the corresponding closure operation $A \to \overline{A}$ with respect to the new topology. Prove then that the original and the new operations of taking the closure coincide with each other, i.e.,

$$\mathrm{cl}A = \overline{A}$$

for every set A (6 points).

(c) Conversely, assume that a topology was introduced by open sets $\mathcal{X}$. The corresponding operation of closure satisfies then properties listed above and can be used to introduce a (potentially different) topology and corresponding (potentially different) open sets $\mathcal{X}'$. Prove that families $\mathcal{X}$ and $\mathcal{X}'$ must be identical (7 points).

(total 20 points).

Solution:

(a) By assumption, $X \subset \mathrm{cl}X \subset X$ so $\mathrm{cl}X = X$. The empty set is closed by axiom (i). Let $F_\iota, \iota \in I$, be now an arbitrary family of sets such that $\mathrm{cl}F_\iota = F_\iota$. By assumption,

$$\mathrm{cl}\bigcap_{\iota\in I} F_\iota \supset \bigcap_{\iota\in I} F_\iota = \bigcap_{\iota\in I} \mathrm{cl}F_\iota$$

Conversely,

$$F_\iota \supset \bigcap_{\kappa\in I} F_\kappa, \quad \forall \iota \in I$$

Axiom (ii) implies that,

$$F_\iota \supset \mathrm{cl}\bigcap_{\kappa\in I} F_\kappa, \quad \forall \iota \in I$$

and, consequently,

$$\bigcap_{\iota\in I} F_\iota \supset \mathrm{cl} \bigcap_{\kappa\in I} F_\kappa$$

Finally, by induction, axiom (iv) implies that closure of a union of a finite family of sets is equal to the union of their closures.

(b) Let A be an arbitrary set. Recall that $\overline{A}$ is the smallest closed set that contains A. By axiom (iii), $\mathrm{cl}A$ is closed and it contains A, so $\overline{A} \supset \mathrm{cl}A$. On the other side, $A \subset \mathrm{cl}A$ and the fact that $\mathrm{cl}A$ is closed imply $\overline{A} \subset \overline{\mathrm{cl}A} = \mathrm{cl}A$.

(c) This is trivial. Closed sets F in both families are identified through the same property: $\overline{F} = \mathrm{cl}F = F$.