CSE386M/EM386M
FUNCTIONAL ANALYSIS IN THEORETICAL MECHANICS
Fall 2018, Exam 2

1. Define the following notions and provide a non-trivial example (2+2 points each).

    - Limit inferior of a real-valued sequence.

    - Hamel basis.

    - Dual basis.

    - Riesz map.

    - Abstract σ-algebra.

    See the book.

2. State and prove *three* out of the following four theorems (10 points each).

    - Characterization of limit inferior.

    - Rank and nullity theorem.

    - Relation between rank of a transformation and the rank of its transpose.

    - Properties of an abstract measure.

    See the book.

3. A linear algebra "sanity check"...

    (a) Prove that vectors $\boldsymbol{a}_1 = (1, 0, 0)$, $\boldsymbol{a}_2 = (1, 0, 1)$, $\boldsymbol{a}_3 = (1, 1, 1)$ form a basis for $\mathbb{R}^3$.

    (b) Determine the dual basis $\boldsymbol{a}_i^*$.

    (c) Consider the weighted inner product in $\mathbb{R}^3$,

    $$(\boldsymbol{x}, \boldsymbol{y}) = x_1 y_1 + 2 x_2 y_2 + 3 x_3 y_3$$

    and construct the cobasis $\boldsymbol{a}^i$.

    (d) Determine matrix representation of the corresponding Riesz operator with respect basis $\boldsymbol{a}_i$ and its dual $\boldsymbol{a}_i^*$.

    (15 points).

Solution:

(a) It is sufficient to check linear independence.

$$\sum_{i=1}^{3} \alpha_i \boldsymbol{a}_i = (\alpha_1 + \alpha_2 + \alpha_3, \alpha_3, \alpha_2 + \alpha_3) = \boldsymbol{0} = (0,0,0)$$

implies

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \quad \alpha_3 = 0, \quad \alpha_2 + \alpha_3 = 0$$

which, in turn, implies that

$$\alpha_1 = \alpha_2 = \alpha_3 = 0.$$

(b) Expanding an arbitrary vector $\boldsymbol{x}$ in the basis $\boldsymbol{a}_i$,

$$\sum_{i=1}^{3} \alpha_i \boldsymbol{a}_i = (\alpha_2 + \alpha_3, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3) = \boldsymbol{x} = (x_1, x_2, x_3)$$

gives:

$$\alpha_1 = x_1 - x_3, \quad \alpha_2 = x_3 - x_2, \quad \alpha_3 = x_2$$

The components α_i represent precisely the dual basis $\boldsymbol{a}_i^*$. In terms of canonical dual basis $\boldsymbol{e}_i^*$,

$$\boldsymbol{a}_1^* = \boldsymbol{e}_1^* - \boldsymbol{e}_3^*, \quad \boldsymbol{a}_2^* = -\boldsymbol{e}_2^* + \boldsymbol{e}_3^*, \quad \boldsymbol{a}_3^* = \boldsymbol{e}_2^*.$$

(c) This can be done in many ways. Here is one. First, we determine the action of Riesz operator on canonical basis vectors $\boldsymbol{e}_i$,

$$\langle R\boldsymbol{e}_1, \boldsymbol{x} \rangle = (\boldsymbol{e}_1, \boldsymbol{x}) = x_1 = \langle \boldsymbol{e}_1^*, \boldsymbol{x} \rangle \quad \Rightarrow \quad R\boldsymbol{e}_1 = \boldsymbol{e}_1^*,$$

$$\langle R\boldsymbol{e}_2, \boldsymbol{x} \rangle = (\boldsymbol{e}_2, \boldsymbol{x}) = 2x_2 = 2\langle \boldsymbol{e}_2^*, \boldsymbol{x} \rangle \quad \Rightarrow \quad R\boldsymbol{e}_2 = 2\boldsymbol{e}_2^*,$$

$$\langle R\boldsymbol{e}_3, \boldsymbol{x} \rangle = (\boldsymbol{e}_3, \boldsymbol{x}) = 3x_3 = 3\langle \boldsymbol{e}_3^*, \boldsymbol{x} \rangle \quad \Rightarrow \quad R\boldsymbol{e}_3 = 3\boldsymbol{e}_3^*.$$

Remembering that Riesz operator sends cobasis vectors into dual basis vectors, we have:

$$\boldsymbol{a}^1 = R^{-1}\boldsymbol{a}_1^* = R^{-1}(\boldsymbol{e}_1^* - \boldsymbol{e}_3^*) = \boldsymbol{e}_1 - \tfrac{1}{3}\boldsymbol{e}_3 = (1, 0, -\tfrac{1}{3}),$$

$$\boldsymbol{a}^2 = R^{-1}\boldsymbol{a}_2^* = R^{-1}(-\boldsymbol{e}_2^* + \boldsymbol{e}_3^*) = -\tfrac{1}{2}\boldsymbol{e}_2 + \tfrac{1}{3}\boldsymbol{e}_3 = (0, -\tfrac{1}{2}, \tfrac{1}{3}),$$

$$\boldsymbol{a}^3 = R^{-1}\boldsymbol{a}_3^* = R^{-1}(\boldsymbol{e}_2^*) = \tfrac{1}{2}\boldsymbol{e}_2 = (0, \tfrac{1}{2}, 0).$$

(d) We use basis $\boldsymbol{a}_i$ for the original space $\mathbb{R}^3$ and basis $\boldsymbol{a}_i^*$ for its dual. The corresponding matrix representation of the Riesz operator R_{ij} is simply the Gram matrix for the original basis. Indeed,

$$R_{ij} = \langle \boldsymbol{a}_i^{**}, R\boldsymbol{a}_j \rangle = \langle R\boldsymbol{a}_j, \boldsymbol{a}_i \rangle = (\boldsymbol{a}_j, \boldsymbol{a}_i)$$

which gives

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 6 \end{pmatrix}.$$

4. Another linear algebra sanity check...

   (a) Demonstrate that

   $$(\boldsymbol{x}, \boldsymbol{y})_w = x_1 y_1 + 3 x_2 y_2 + 2 x_3 y_3 \qquad (0.1)$$

   is an inner product on $\mathbb{R}^3$.

   (b) Consider the map $A : \mathbb{R}^3 \to \mathbb{R}^3$,

   $$A\boldsymbol{x} = (x_1 + x_2, x_3 + x_1, x_1 + x_2) \qquad (0.2)$$

   Prove that the map is linear and write down its matrix representation with respect to the canonical basis.

   (c) Determine *adjoints* of map A with respect to the canonical inner product and inner product (0.1).

(15 points).

Solution:

   (a) Nothing to show really. The function is obviously linear in both $\boldsymbol{x}$ and $\boldsymbol{y}$ and it is symmetric. For $\boldsymbol{y} = \boldsymbol{x}$, we get;

   $$(\boldsymbol{x}, \boldsymbol{x}) = x_1^2 + 3 x_2^2 + 2 x_3^2 \geq 0$$

   Since $(\boldsymbol{x}, \boldsymbol{x})$ is the sum of non-negative contributions (no cancellation can occur), $(\boldsymbol{x}, \boldsymbol{x}) = 0$ implies that all three terms must be zero, i.e. $\boldsymbol{x} = \boldsymbol{0}$, i..e. the form is positive-definite.

   (b) Linearity is obvious as all components of $\boldsymbol{x}$ enter the definition in the first power and the summation is a linear operation. The matrix representation in the canonical basis is as follows,

   $$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

   (c) To obtain the matrix representtaion of the adjoint of A with respect to the canonical inner product, we only have to transpose the matrix above. Consequently, the adjoint map is:

   $$A^* \boldsymbol{y} = (y_1 + y_2 + y_3, y_1 + y_3, y_2)$$

The map is *self-adjoint* with respect to the canonical inner product. The adjoint with respect to inner product (0.1) can be computed directly:

$$
\begin{aligned}
(A\boldsymbol{x}, \boldsymbol{y}) &= (x_1 + x_2)y_1 + 3(x_3 + x_1)y_2 + 2(x_1 + x_2)y_3 \\
&= x_1(y_1 + 3y_2 + 2y_3) + x_2(y_1 + 2y_3) + x_3(3y_2) \\
&= x_1(y_1 + 3y_2 + 2y_3) + 3x_2[\tfrac{1}{3}(y_1 + 2y_3)] + 2x_3(\tfrac{3}{2}y_2)
\end{aligned}
$$

which gives

$$
A^*\boldsymbol{y} = (y_1 + 3y_2 + 2y_3, \frac{1}{3}(y_1 + 2y_3), \frac{3}{2}y_2)
$$

The map is *not* self-adjoint with respect to this inner product.

5. Let X, Y be two real finite dimensional vector spaces. Consider two different vector spaces:

   - space $L(X^*, Y)$ of all linear transformations from dual X^* into space Y,

   - space $B(X^*, Y^*)$ of all bilinear functionals defined on the Cartesian product $X^* \times Y^*$ of the dual spaces.

   (a) Argue why the linear transformations and bilinear functionals form vector spaces.

   (b) Select bases $\boldsymbol{e}_j$ and $\boldsymbol{g}_i$ for X and Y respectively, and recall (derive) representations for arbitrary linear transformations and bilinear maps relative to the bases and/or their dual bases. Argue why the spaces $L(X^*, Y)$ and $B(X^*, Y^*)$ are isomorphic.

   (c) Attempt to construct a *canonical*[1] isomorphism between the two spaces.

   (20 points).

   (a) This follows from the fact that a linear combination of linear maps is a linear map, and a linear combination of bilinear functionals is a bilinear map. In other words, both families are closed with respect to the vector space operations for functions.

   (b) For a linear map $A \in L(X^*, Y)$, the corresponding matrix representation A_{ij} with respect to bases $\boldsymbol{e}_j^*$ and $\boldsymbol{g}_i$ is given by the relation:

   $$
   A_{ij} = \langle \boldsymbol{g}_i^*, A\boldsymbol{e}_j^* \rangle, \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n
   $$

   where $n = \dim X$, $m = \dim Y$. The space of $m \times n$ matrices, $\mathrm{Mat}(m, n)$, is isomorphic (by construction dicussed in the book and class) to the space $L(X^*, Y)$ of linear transformations from X^* into Y.

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[1] Constructed without using any bases

Similarly, from the representation formula for bilinear functionals

$$b(\boldsymbol{x}^*, \boldsymbol{y}^*) = b(\sum_{j=1}^{n} x_j^* \boldsymbol{e}_j^*, \sum_{i=1}^{m} y_i^* \boldsymbol{g}_i^*) = \sum_{j=1}^{n} \sum_{i=1}^{m} x_j^* y_i^* \underbrace{b(\boldsymbol{e}_j^*, \boldsymbol{g}_i^*)}_{=:b_{ji}}$$

also implies that the space of bilinear functionals is isomorphic with the space of matrices $\mathrm{Mat}(m, n)$.

Consequently, the two spaces, being isomorphic with the same space, must be isomorphic with each other. In perhaps simpler terms, both spaces are of the same dimension, and all spaces of the same dimension are isomorphic with each other.

(c) This is deeper... The construction generalizes the concept of Riesz map. Given a bilinear form $b(\boldsymbol{x}^*, \boldsymbol{y}^*)$, if we fix $\boldsymbol{x}^*$, we obtain a linear functional on Y^*,

$$Y^* \ni \boldsymbol{y}^* \to b(\boldsymbol{x}^*, \boldsymbol{y}^*) \in \mathbb{R}$$

The map,
$$B \; : \; X^* \ni \boldsymbol{x}^* \to b(\boldsymbol{x}^*, \cdot) \in Y^{**}$$

is a linear map from X into Y^{**}. As, in the finite dimensional case, the bidual Y^{**} is isomorphic with space Y, we can think of map B being an operator from X^* into Y. Finally, the map,

$$B(X^*, Y^*) \ni b \to B := \{X^* \ni \boldsymbol{x}^* \to b(\boldsymbol{x}^*, \cdot) \in Y^{**} \sim Y\} \in L(X^*, Y)$$

establishes a (canonical) isomorphism between the two spaces. Linearity is straightforward. Injectivity follows from the definition of zero vectors in both spaces. Dimensions are the same, so the map must be surjective. Done.