CSE386M/EM386M
FUNCTIONAL ANALYSIS IN THEORETICAL MECHANICS
Fall 2018, Exam 1

1. Define the following notions and provide a non-trivial example (2+2 points each).

   - union of an arbitrary (possibly infinite) family of sets,
   - quotient set,
   - Cartesian product of two functions,
   - a *chain* in a partially ordered set,
   - cardinal number.

   See the book.

2. State and prove *three* out of the following four theorems (10 points each).

   - Characterization of a bijection.
   - Properties of the direct image.
   - Properties of countable sets.
   - Comparability of cardinal numbers.

   See the book.

3. A function $f : X \to Y$ is called *left invertible* if there exists a function $g : Y \to X$ such that $g \circ f = id_X$. Prove that f is left-invertible if and only if f is a injection. Is the left-inverse g unique ? (10 points).

   See the book.

4. Let $f : X \to Y$ be a bijection and f^{-1} its inverse. Show that:

$$\begin{aligned} (f^{-1})(B) &= f^{-1}(B) \\ \text{(direct image of } B \text{ through inverse } f^{-1}) &= \text{(inverse image of } B \text{ through } f) \end{aligned}$$

   (10 points).

   Let $x \in (f^{-1})(B)$. By definition of the direct image, there exists $y \in B$ such that $f^{-1}(y) = x$. Since $f(x) = f(f^{-1}(y)) = y$, this implies that $x \in f^{-1}(B)$. Conversely, assume that $x \in f^{-1}(B)$. By the definition of the inverse image, $y = f(x) \in B$. But f is invertible so $x = f^{-1}(y)$. Consequently, $x \in (f^{-1})(B)$.

5. Let $f : X \to Y$ be a function. Prove that, for an arbitrary set $C \subset Y$,

$$f^{-1}(\mathcal{R}(f) \cap C) = f^{-1}(C)$$

(10 points).

Use the property

$$f^{-1}(D \cap C) = f^{-1}(D) \cap f^{-1}(C)$$

with $D = \mathcal{R}(f)$. Notice that $f^{-1}(\mathcal{R}(f)) = X$.

6. Prove that if A is infinite, then $\#(A \times \{1, 2, \ldots, m\}) = \#A$. You may use the fact that if A is infinite, and B is finite then $\#(A \cup B) = \#A$. Follow the steps:

   (a) Argue why the result is true for a denumerable set A.

   Let $A = \{a_1, a_2, a_3, \ldots\}$. Define map T:

$$T(n) = \begin{cases} (a_k, 1) & \text{if } n = m(k-1) + 1 \\ (a_k, 2) & \text{if } n = m(k-1) + 2 \\ \vdots & \\ (a_k, m) & \text{if } n = m(k-1) + m = mk \end{cases}$$

   The map is a bijection from N onto $A \times \{1, 2, \ldots, m\}$.

   (b) Define a family $\mathcal{F}$ of couples (X, T_X), where $X \subset A$ is infinite and $T_X : X \times \{1, 2, \ldots, m\} \to X$ is a bijection. Prove that the following relation is a partial ordering in $\mathcal{F}$.

$$(X_1, T_{X_1}) \le (X_2, T_{X_2}) \quad \text{iff} \quad X_1 \subset X_2 \text{ and } T_{X_2} \text{ is an extension of } T_{X_1}$$

   Standard reasoning. The relation is *reflexive* since $X \subset X$ and T_X is an extension of itself. It is *antisymmetric*. Indeed, $(X_1, T_{X_1}) \le (X_2, T_{X_2})$ and $(X_2, T_{X_2}) \le (X_1, T_{X_1})$ imply that $X_1 = X_2$ and, consequently, $T_{X_1} = T_{X_2}$. Similar arguments are used to prove transitiveness.

   (c) Use the Kuratowski-Zorn lemma to conclude that $\mathcal{F}$ has a maximal element (X, T_X).

   Let $(X_\tau, T_{X_\tau}), \tau \in I$ be a chain. Define,

$$X = \bigcup_{\tau \in I} X_\tau, \quad T_X(x) = T_{X_\tau}(x), \text{ where } x \in X_\tau$$

   Then map T_X is well defined, T_X is abijection from X onto $X \times \{1, 2\}$ so, the pair (X, T_X) is an upper bound for the chain.

   Consequently, by the K-Z lemma, there exists a maximal element (X, T_X).

(d) Use the existence of the maximal element to conclude the theorem.

This is the tricky part... Two cases are possible.

Case: $X \sim A$. We have then

$$A \sim X \sim X \times \{1, 2, \ldots, m\} \sim A \times \{1, 2, \ldots, m\}$$

since $X \sim A$ implies $X \times \{1, 2, \ldots, m\} \sim A \times \{1, 2, \ldots, m\}$.

Case: $\sharp X < \sharp A$. The difference $A - X$ cannot be finite as then $X \sim A$. In this case, we are able to find a denumerable subset $Y \subset A$. By step (a), there exists a bijection T_Y from Y onto $Y \times \{1, \ldots, m\}$. Consequently, the pair

$$(X \cup Y, T_X \cup T_Y)$$

is in the family and provides a strict bound for (X, T_X), a contradiction with (X, T_X) being maximal.

Question: Where do we need step (a) ?
To assure that family $\mathcal{F}$ is non-empty, and in the last step.
(20 points).